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ASYMPTOTICS OF SOLUTIONS FOR NONLOCAL ELLIPTIC PROBLEMS IN PLANE ANGLES

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In the paper we investigate asymptotics of solutions for nonlocal elliptic problems in plane angles and in $\mathbb{R}^2 \setminus \{0\}$. These problems arise in studying nonlocal problems in bounded domains in the case where support of nonlocal terms intersects with a boundary of a domain. We obtain explicit formulas for the asymptotic coefficients in terms of eigenvectors and associated vectors of both adjoint nonlocal operators acting in spaces of distributions and formally adjoint (with respect to the Green formula) nonlocal transmission problems.

1 INTRODUCTION

T. Carleman [1] was one of the first to study nonlocal elliptic problems. Investigation of nonlocal problems with shifts mapping a boundary on itself are closely associated with paper [1]. In [2], A. V. Bitsadze and A. A. Samarskii considered the Laplace equation in a domain $G \subset \mathbb{R}^n$ with the boundary condition that connects the values of an unknown function on a manifold $\Upsilon_1 \subset \partial G$ with its values on some manifold inside G; on $\partial G \setminus \Upsilon_1$ the Dirichlet condition was set. Such a formulation is associated with further investigating nonlocal problems with shifts mapping a boundary inside a domain. One can find a detailed bibliography devoted to nonlocal elliptic problems in [3].

In the theory of nonlocal elliptic problems of this type, the most difficult case deals with the situation where support of nonlocal terms intersects with a boundary [4–8]. This leads to appearance of power singularities for solutions near some set $\mathcal{K}$. Therefore, the problem of asymptotics of solutions near this set arises. Asymptotic formulas for solutions to nonlocal elliptic problems in plane domains were first obtained by A. L. Skubachevskii in [5]. They allow one to prove a number of principally new properties (in comparison with "local" elliptic problems both in domains with angular points [9, 10] and in domains with smooth boundary). For example, smoothness of generalized solutions for nonlocal elliptic problems can be violated both near vertexes of small angles and near a smooth boundary even for arbitrarily small coefficients in nonlocal terms [5, 11].

In this paper we investigate the asymptotic behavior of solutions for nonlocal elliptic boundary-value problems in plane angles and in $\mathbb{R}^2 \setminus \{0\}$. Such problems arise as model ones when studying asymptotics of solutions for nonlocal elliptic problems in bounded domains near the set $\mathcal{K}$. We obtain explicit formulas for the asymptotic coefficients in terms of eigenvectors and associated vectors of both adjoint nonlocal operators (acting in spaces of distributions) and formally adjoint (with respect to the Green formula) nonlocal problems. Earlier adjoint nonlocal problems were studied in [12, 13].

Note that a number of statements are proved similarly to results of papers [14, 15]. In these cases we shall just give schemes of proofs.

2 STATEMENT OF NONLOCAL PROBLEMS IN PLANE ANGLES AND PRELIMINARY INFORMATION. ASYMPTOTICS OF SOLUTIONS

1. Consider the plane angle $K = \{y \in \mathbb{R}^2 : r > 0,\ b_1 < \omega < b_2\}$ with sides $\gamma_\sigma = \{y \in \mathbb{R}^2 : r > 0,\ \omega = b_\sigma\}$ $(\sigma = 1, 2)$. Here (ω, r) are the polar coordinates of a point y; $-\pi < b_1 < b_2 < \pi$.

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Denote by $\mathcal{P}(D_y)$, $B_{\sigma\mu}(D_y)$, and $B^{\mathcal{G}}_{\sigma\mu}(D_y)$ homogeneous differential operators with constant complex coefficients of orders $2m$, $m_{\sigma\mu} \leqslant 2m-1$, and $m_{\sigma\mu} \leqslant 2m-1$ respectively ($\sigma = 1,2$; $\mu = 1,\dots,m$). We shall suppose that the operator $\mathcal{P}(D_y)$ is properly elliptic and the system of operators $\{B_{\sigma\mu}(D_y)\}_{\mu=1}^m$ is normal and covers $\mathcal{P}(D_y)$ on γ_σ (see [16, Chap. 2]). We do not impose any conditions (except the restrictions on orders) on the operators $B^{\mathcal{G}}_{\sigma\mu}(D_y)$, which play further the role of nonlocal ones.

Consider the following nonlocal elliptic problem in the plane angle K:

$$\mathcal{P}(D_y)u = f(y) \quad (y \in K), \tag{2.1}$$

$$\mathcal{B}_{\sigma\mu}(D_y)u \equiv B_{\sigma\mu}(D_y)u(y)\big|_{\gamma_\sigma} + (B^{\mathcal{G}}_{\sigma\mu}(D_y)u)(\mathcal{G}_\sigma y)\big|_{\gamma_\sigma} = g_{\sigma\mu}(y) \ \ (y \in \gamma_\sigma), \quad \sigma = 1,2; \ \ \mu = 1,\dots,m. \tag{2.2}$$

The notation $(B^{\mathcal{G}}_{\sigma\mu}(D_y)u)(\mathcal{G}_\sigma y)$ means that the expression $(B^{\mathcal{G}}_{\sigma\mu}(D_{y'})u)(y')$ is taken for $y' = \mathcal{G}_\sigma y$; $\mathcal{G}_\sigma$ is the operator of rotation by the angle ω_σ and expansion by β_σ times in the plane $\{y\}$ such that $b_1 < b_1 + \omega_1 = b_2 + \omega_2 = b < b_2$, $0 < \beta_\sigma$.

For any set $G \subset \mathbb{R}^n$ ($n \geqslant 1$), denote by $C_0^\infty(G)$ the set of infinitely differentiable in $\bar G$ functions with supports belonging to G. We introduce the space $H_a^l(K)$ as a completion of the set $C_0^\infty(\bar K \setminus \{0\})$ in the norm $\|w\|_{H_a^l(K)} = \Big(\sum_{|\alpha|\leqslant l}\int_K r^{2(a-l+|\alpha|)}|D_y^\alpha w(y)|^2\,dy\Big)^{1/2}$, where $a \in \mathbb{R}$, $l \geqslant 0$ is an integer. By $H_a^{l-1/2}(\gamma')$ for $l \geqslant 1$ we denote the space of traces on a ray $\gamma' = \{y \in \mathbb{R}^2 \colon r > 0, \ \omega = b'\}$ ($b_1 \leqslant b' \leqslant b_2$) with the norm $\|\psi\|_{H_a^{l-1/2}(\gamma')} = \inf \|w\|_{H_a^l(K)}$ ($w \in H_a^l(K)\colon w\big|_{\gamma'} = \psi$).

Introduce the bounded operator corresponding to problem (2.1), (2.2):

$$\mathcal{L} = \{\mathcal{P}(D_y), \mathcal{B}_{\sigma\mu}(D_y)\}\colon H_a^{l+2m}(K) \to H_a^l(K,\gamma) = H_a^l(K) \times \prod_{\sigma=1,2}\prod_{\mu=1}^m H_a^{l+2m-m_{\sigma\mu}-1/2}(\gamma_\sigma).$$

2. Write the operators $\mathcal{P}(D_y)$, $B_{\sigma\mu}(D_y)$, and $B^{\mathcal{G}}_{\sigma\mu}(D_y)$ in polar coordinates: $\mathcal{P}(D_y) = r^{-2m}\tilde{\mathcal{P}}(\omega, D_\omega, \ rD_r)$, $B_{\sigma\mu}(D_y) = r^{-m_{\sigma\mu}}\tilde B_{\sigma\mu}(\omega, D_\omega, rD_r)$, and $B^{\mathcal{G}}_{\sigma\mu}(D_y) = r^{-m_{\sigma\mu}}\tilde B^{\mathcal{G}}_{\sigma\mu}(\omega, D_\omega, rD_r)$, where $D_\omega = -i\frac{\partial}{\partial\omega}$ and $D_r = -i\frac{\partial}{\partial r}$.

We shall denote by $\tilde w(\lambda)$ the Mellin transformation of a function $w \in C_0^\infty(\mathbb{R}_+)$:

$$\tilde w(\lambda) = (2\pi)^{-1/2}\int_0^\infty r^{-i\lambda-1}w(r)\,dr.$$

Put $\{f, g_{\sigma\mu}\} = 0$ in (2.1) and (2.2) and formally perform the Mellin transformation. Then we get

$$\tilde{\mathcal{P}}(\lambda)\tilde u(\omega,\lambda) = 0 \quad (b_1 < \omega < b_2), \tag{2.3}$$

$$\tilde{\mathcal{B}}_{\sigma\mu}(\lambda)\tilde u(\omega,\lambda) \equiv \tilde B_{\sigma\mu}(\lambda)\tilde u(\omega,\lambda)\big|_{\omega=b_\sigma} + \beta_\sigma^{-m_{\sigma\mu}+i\lambda}\tilde B^{\mathcal{G}}_{\sigma\mu}(\lambda)\tilde u(\omega+\omega_\sigma,\lambda)\big|_{\omega=b_\sigma} = 0. \tag{2.4}$$

Here (and further) for short we omit the arguments ω and D_ω in differential operators. This problem is the ordinary differential equation (2.3) with nonlocal conditions (2.4) that connect the values of a solution $\tilde u$ and its derivatives at the point $\omega = b_\sigma$ with the values of a solution $\tilde u$ and its derivatives at the internal point $\omega = b$ of the interval (b_1, b_2). The asymptotics of solutions for the nonlocal problem (2.1), (2.2) in the angle K will be described in terms of eigenvalues and corresponding Jordan chains of problem (2.3), (2.4).

Let us consider the operator-valued function corresponding to nonlocal problem (2.3), (2.4)

$$\tilde{\mathcal{L}}(\lambda) = \{\tilde{\mathcal{P}}(\lambda), \tilde{\mathcal{B}}_{\sigma\mu}(\lambda)\}\colon W^{l+2m}(b_1,b_2) \to W^l[b_1,b_2] = W^l(b_1,b_2) \times \mathbb{C}^{2m}.$$

Here $W^l(\cdot) = W_2^l(\cdot)$ is the Sobolev space of order $l \geqslant 0$ (if $l = 0$, we put $W^0(\cdot) = L_2(\cdot)$).

Now we shall recall some well-known definitions and facts (see [17]). A holomorphic at a point λ_0 vector-function $\varphi(\lambda)$ with values in $W^{l+2m}(b_1,b_2)$ is called a *root function* of the operator $\tilde{\mathcal{L}}(\lambda)$ at λ_0 if $\varphi(\lambda_0) \neq 0$ and the vector-function $\tilde{\mathcal{L}}(\lambda)\varphi(\lambda)$ is equal to zero at λ_0. If $\tilde{\mathcal{L}}(\lambda)$ has at least one root function at a point λ_0, then λ_0 is called an *eigenvalue* of $\tilde{\mathcal{L}}(\lambda)$. The multiplicity of zero for the vector-function $\tilde{\mathcal{L}}(\lambda)\varphi(\lambda)$ at the point λ_0 is called a *multiplicity of the root function* $\varphi(\lambda)$; the vector $\varphi^{(0)} = \varphi(\lambda_0)$ is called an *eigenvector* corresponding to the eigenvalue λ_0. Let $\varphi(\lambda)$ be a root function at a point λ_0 of multiplicity $\varkappa$ and $\varphi(\lambda) = \sum_{j=0}^\infty (\lambda-\lambda_0)^j\varphi^{(j)}$. Then the vectors $\varphi^{(1)},\dots,\varphi^{(\varkappa-1)}$ are called *associated with the eigenvector* φ_0, and the ordered set $\varphi^{(0)},\dots,\varphi^{(\varkappa-1)}$ is called a *Jordan chain* corresponding to the eigenvalue λ_0. The *rank* of the eigenvector $\varphi^{(0)}$ ($\operatorname{rank}\varphi^{(0)}$) is the maximum of the multiplicities of all root functions such that $\varphi(\lambda_0) = \varphi^{(0)}$.

Remark 2.1. An eigenvector and associated vectors $\varphi^{(0)}$,..., $\varphi^{(\varkappa-1)}$ of the operator $\tilde{\mathcal{L}}(\lambda)$ corresponding to an eigenvalue λ_0 satisfy the equalities

$$\sum_{q=0}^{\nu} \frac{1}{q!} \partial_\lambda^q \tilde{\mathcal{L}}(\lambda_0) \varphi^{(\nu-q)} = 0, \quad \nu = 0, \dots, \varkappa - 1. \tag{2.5}$$

Here and further, ∂_λ^q is the derivative of order q with respect to λ.

From equalities (2.5) and Lemma A.1, it follows that eigenvectors and associated vectors of the operator $\tilde{\mathcal{L}}(\lambda)$ are infinitely differentiable functions in the interval $[b_1, b_2]$.

From Lemma 2.1 of [6], it follows that all eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ are isolated. Moreover, $\dim \ker \tilde{\mathcal{L}}(\lambda_0) < \infty$ for any eigenvalue λ_0, and the ranks of all eigenvectors are finite. Suppose $J = \dim \ker \tilde{\mathcal{L}}(\lambda_0)$ and $\varphi^{(0,1)}, \dots, \varphi^{(0,J)}$ is a system of eigenvectors such that $\operatorname{rank} \varphi^{(0,1)}$ is the greatest of the ranks of all eigenvectors corresponding to the eigenvalue λ_0, and $\operatorname{rank} \varphi^{(0,j)}$ $(j = 2, \dots, J)$ is the greatest of the ranks of eigenvectors from some orthogonal supplement in $\ker \tilde{\mathcal{L}}(\lambda_0)$ to the linear manifold of the vectors $\varphi^{(0,1)}, \dots, \varphi^{(0,j-1)}$. The numbers $\varkappa_j = \operatorname{rank} \varphi^{(0,j)}$ are called *partial multiplicities* of the eigenvalue λ_0, and the sum $\varkappa_1 + \dots + \varkappa_J$ is called a (*full*) *multiplicity* of λ_0. If the vectors $\varphi^{(0,j)}, \dots, \varphi^{(\varkappa_j-1,j)}$ form a Jordan chain for every $j = 1, \dots, J$, then the set of vectors $\{\varphi^{(0,j)}, \dots, \varphi^{(\varkappa_j-1,j)} \colon \ j = 1, \dots, J\}$ is called a *canonical system of Jordan chains* corresponding to the eigenvalue λ_0.

Example 2.1. Put $b_1 = -\omega_0$ and $b_2 = \omega_0$. In the plane angle $K = \{y \in \mathbb{R}^2 \colon |\omega| < \omega_0\}$ $(0 < \omega_0 < \pi)$ with sides $\gamma_\sigma = \{y \in \mathbb{R}^2 \colon \omega = (-1)^\sigma \omega_0\}$, $\sigma = 1, 2$, we consider the nonlocal problem

$$\Delta u = f(y) \quad (y \in K), \tag{2.6}$$

$$u\big|_{\gamma_1} = 0, \quad u\big|_{\gamma_2} + bu(\mathcal{G}_2 y)\big|_{\gamma_2} = 0, \tag{2.7}$$

where $b \in \mathbb{R}$, $\mathcal{G}_2$ is the operator of rotation by the angle $-\omega_0$. The following model nonlocal eigenvalue problem corresponds to problem (2.6), (2.7):

$$\frac{d^2\varphi(\omega)}{d\omega^2} - \lambda^2 \varphi(\omega) = 0 \quad (|\omega| < \omega_0), \tag{2.8}$$

$$\varphi(-\omega_0) = 0, \quad \varphi(\omega_0) + b\varphi(0) = 0. \tag{2.9}$$

One can immediately check (see also [14, Chap. 2]) that, for $b = 0$ (that is, if problem (2.6), (2.7) is "local"), the eigenvalues of problem (2.8), (2.9) have the form $\lambda_k = i\frac{\pi k}{2\omega_0}$, $k \in \mathbb{Z} \setminus \{0\}$. The eigenvectors $\varphi_k^{(0)}(\omega) = e^{i\frac{\pi k}{2\omega_0}\omega} - e^{-i\frac{\pi k}{2\omega_0}\omega}$ correspond to these eigenvalues. Associated vectors are absent, that is, all the eigenvalues are of multiplicity 1.

Now we shall show that, for $b \neq 0$, there may be Jordan chains with a length more than 1 corresponding to eigenvalues of problem (2.8), (2.9).

(I) First we consider the case $\lambda \neq 0$. Substituting the general solution $\varphi(\omega) = c_1 e^{\lambda\omega} + c_2 e^{-\lambda\omega}$ for Eq. (2.8) into nonlocal conditions (2.9), we get

$$\begin{gathered} c_1 e^{-\lambda\omega_0} + c_2 e^{\lambda\omega_0} = 0, \\ (e^{\lambda\omega_0} + b)c_1 + (e^{-\lambda\omega_0} + b)c_2 = 0. \end{gathered} \tag{2.10}$$

Equate the determinant $D(\lambda)$ of system (2.10) with zero:

$$(e^{-\lambda\omega_0} - e^{\lambda\omega_0})(e^{\lambda\omega_0} + e^{-\lambda\omega_0} + b) = 0.$$

(1) Let us have $e^{-\lambda\omega_0} - e^{\lambda\omega_0} = 0$. Then we obtain the series of eigenvalues

$$\lambda_{1k} = i\frac{\pi k}{\omega_0}, \quad k \in \mathbb{Z} \setminus \{0\}.$$

The eigenvectors

$$\varphi_{1k}^{(0)}(\omega) = e^{i\frac{\pi k}{\omega_0}\omega} - e^{-i\frac{\pi k}{\omega_0}\omega}$$

correspond to these eigenvalues. Consider the problem of finding an associated vector $\varphi_{1k}^{(1)}(\omega)$. According to (2.5), $\varphi_{1k}^{(1)}(\omega)$ satisfies the equation

$$\frac{d^2\varphi_{1k}^{(1)}}{d\omega^2} + \frac{(\pi k)^2}{\omega_0^2}\varphi_{1k}^{(1)} - 2i\frac{\pi k}{\omega_0}\varphi_{1k}^{(0)} = 0 \quad (|\omega| < \omega_0)$$

and nonlocal conditions (2.9). Substituting the general solution $\varphi(\omega) = c_1 e^{i\frac{\pi k}{\omega_0}\omega} + c_2 e^{-i\frac{\pi k}{\omega_0}\omega} + \omega(e^{i\frac{\pi k}{\omega_0}\omega} + e^{-i\frac{\pi k}{\omega_0}\omega})$ of the last equation into nonlocal conditions (2.9), we get

$$\begin{gathered} c_1 + c_2 = 2\omega_0, \\ ((-1)^k + b)c_1 + ((-1)^k + b)c_2 = -2(-1)^k\omega_0. \end{gathered} \tag{2.11}$$

Therefore, an associated vector $\varphi_{1k}^{(1)}(\omega)$ exists if and only if

$$b = 2(-1)^{k+1}.$$

If $b = 2(-1)^{k+1}$, we can put

$$\varphi_{1k}^{(1)}(\omega) = (\omega + 2\omega_0)e^{i\frac{\pi k}{\omega_0}\omega} + \omega e^{-i\frac{\pi k}{\omega_0}\omega}.$$

Analogously, using (2.5), we find the second associated vector

$$\varphi_{1k}^{(2)}(\omega) = \left(\frac{\omega^2}{2} + 2\omega_0\omega + 2\omega_0^2\right)e^{i\frac{\pi k}{\omega_0}\omega} - \frac{\omega^2}{2}e^{-i\frac{\pi k}{\omega_0}\omega}.$$

One can directly check that the third associated vector is absent.

(2) Let us have

$$e^{\lambda\omega_0} + e^{-\lambda\omega_0} + b = 0. \tag{2.12}$$

Then we obtain the following series of eigenvalues:

$$\begin{aligned} \lambda_{2n}^{\pm} &= \frac{\ln\left(-\frac{b}{2} \pm \frac{\sqrt{b^2-4}}{2}\right)}{\omega_0} + i\frac{2\pi n}{\omega_0} && \text{for } b < -2; \\ \lambda_{2n}^{\pm} &= i\frac{\pm\arctan\frac{\sqrt{4-b^2}}{b} + 2\pi n}{\omega_0} && \text{for } -2 < b < 0; \\ \lambda_{2n}^{\pm} &= i\frac{\pm\arctan\frac{\sqrt{4-b^2}}{b} + (2n+1)\pi}{\omega_0} && \text{for } 0 < b < 2; \\ \lambda_{2n}^{\pm} &= \frac{\ln\left(\frac{b}{2} \pm \frac{\sqrt{b^2-4}}{2}\right)}{\omega_0} + i\frac{(2n+1)\pi}{\omega_0} && \text{for } b > 2, \end{aligned}$$

$n \in \mathbb{Z}$. If $|b| = 2$, then we have eigenvalues from the series $\{\lambda_{1k}\}_{k\in\mathbb{Z}\setminus\{0\}}$, which is considered above. The eigenvector

$$\varphi_{2n}^{(0)\pm}(\omega) = e^{\lambda_{2n}^{\pm}\omega} - e^{-2\lambda_{2n}^{\pm}\omega_0}e^{-\lambda_{2n}^{\pm}\omega}$$

corresponds to the eigenvalue $\lambda_{2n}^{\pm}$. Let us show that there are no associated vectors if $\lambda = \lambda_{2n}^{\pm}$. Substitute the general solution $\varphi_{2n}^{(1)\pm}(\omega) = c_1 e^{\lambda_{2n}^{\pm}\omega} + c_2 e^{-\lambda_{2n}^{\pm}\omega} + \omega(e^{\lambda_{2n}^{\pm}\omega} + e^{-2\lambda_{2n}^{\pm}\omega_0}e^{-\lambda_{2n}^{\pm}\omega})$ for the equation

$$\frac{d^2\varphi_{2n}^{(1)\pm}}{d\omega^2} - (\lambda_{2n}^{\pm})^2\varphi_{2n}^{(1)\pm} - 2\lambda_{2n}^{\pm}\varphi_{2n}^{(0)\pm} = 0 \quad (|\omega| < \omega_0)$$

into nonlocal conditions (2.9). Then we have

$$\begin{gathered} e^{-\lambda_{2n}^{\pm}\omega_0}c_1 + e^{\lambda_{2n}^{\pm}\omega_0}c_2 = 2\omega_0 e^{-\lambda_{2n}^{\pm}\omega_0}, \\ (e^{\lambda_{2n}^{\pm}\omega_0} + b)c_1 + (e^{\lambda_{2n}^{\pm}\omega_0} + b)c_2 = -\omega_0(e^{\lambda_{2n}^{\pm}\omega_0} + e^{-3\lambda_{2n}^{\pm}\omega_0}). \end{gathered} \tag{2.13}$$

The rank of the matrix of system (2.13) is equal to 1. Therefore, system (2.13) is compatible if and only if

$$\begin{vmatrix} e^{-\lambda_{2n}^{\pm}\omega_0} & 2\omega_0 e^{-\lambda_{2n}^{\pm}\omega_0} \\ e^{\lambda_{2n}^{\pm}\omega_0}+b & -\omega_0(e^{\lambda_{2n}^{\pm}\omega_0}+e^{-3\lambda_{2n}^{\pm}\omega_0}) \end{vmatrix} = 0.$$

The last equality is equivalent to the following one:

$$3e^{\lambda_{2n}^{\pm}\omega_0} + e^{-3\lambda_{2n}^{\pm}\omega_0} + 2b = 0.$$

From this, taking into account (2.12), it follows that either $e^{\lambda_{2n}^{\pm}\omega_0} = 1$, $b = -2$ or $e^{\lambda_{2n}^{\pm}\omega_0} = -1$, $b = 2$. But we now consider the case $|b| \neq 2$. Hence, there are no associated vectors if $\lambda = \lambda_{2n}^{\pm}$.

(II) The case $\lambda = 0$ is studied analogously. It turns out that $\lambda = 0$ is an eigenvalue of problem (2.8), (2.9) if and only if $b = -2$. Moreover, if $b = -2$, then for the eigenvalue $\lambda = 0$, there exist one eigenvector $\varphi_0^{(0)}(\omega) = \omega + \omega_0$ and one associated vector $\varphi_0^{(1)}(\omega) = 0$.

Thus, we have shown that problem (2.8), (2.9) has eigenvalues of multiplicity more than 1 if and only if $|b| = 2$.

3. The following result on isomorphism follows from [6, Sect. 2].

Theorem 2.1. *Suppose the line* $\operatorname{Im}\lambda = a + 1 - l - 2m$ *contains no eigenvalues of the operator* $\tilde{\mathcal{L}}(\lambda)$. *Then the nonlocal boundary-value problem* (2.1), (2.2) *has a unique solution* $u \in H_a^{l+2m}(K)$ *for any right-hand side* $\{f, g_{\sigma\mu}\} \in H_a^l(K,\ \gamma)$. *This solution is represented in the form*

$$u(\omega, r) = (2\pi)^{-1/2} \int\limits_{-\infty+ih}^{+\infty+ih} r^{i\lambda}\tilde{\mathcal{L}}^{-1}(\lambda)\{\tilde{F}(\omega,\lambda), \tilde{G}_{\sigma\mu}(\lambda)\}\, d\lambda.$$

Here $h = a + 1 - l - 2m$, *and* $\tilde{F}(\omega,\lambda)$ *and* $\tilde{G}_{\sigma\mu}(\lambda)$ *are the Mellin transformations of the functions* $r^{2m} f(\omega, r)$ *and* $r^{m_{\sigma\mu}} g_{\sigma\mu}(r)$ *respectively.*

Before we formulate a theorem concerning the asymptotic behavior of solutions for problem (2.1), (2.2), let us prove two lemmas that describe solutions of the homogeneous problem.

Lemma 2.1. *The function*

$$u(\omega, r) = r^{i\lambda_0} \sum_{q=0}^{p} \frac{1}{q!}(i \ln r)^q \varphi^{(p-q)}(\omega), \tag{2.14}$$

where $\varphi^{(s)} \in W^{l+2m}(b_1, b_2)$, $s = 0, \dots, \varkappa - 1$, *is a solution of homogeneous problem* (2.1), (2.2) *if and only if* λ_0 *is an eigenvalue of the operator* $\tilde{\mathcal{L}}(\lambda)$ *and* $\varphi^{(0)}, \dots, \varphi^{(\varkappa-1)}$ *is a Jordan chain corresponding to the eigenvalue* λ_0; $p \leqslant \varkappa - 1$.

Proof. Omitting as above the arguments ω and D_ω in differential operators, write

$$\mathcal{P}(D_y)u = r^{-2m}\tilde{\mathcal{P}}(rD_r)u = r^{-2m+i\lambda_0}\tilde{\mathcal{P}}(\lambda_0 + rD_r)\sum_{q=0}^{p}\frac{1}{q!}(i\ln r)^q\varphi^{(p-q)}$$
$$= r^{-2m+i\lambda_0}\sum_{\nu=0}^{p}\frac{1}{\nu!}\partial_\lambda^\nu\tilde{\mathcal{P}}(\lambda_0)\sum_{q=\nu}^{p}\frac{1}{(q-\nu)!}(i\ln r)^{q-\nu}\varphi^{(p-q)}. \tag{2.15}$$

Similarly,

$$B_{\sigma\mu}(D_y)u = r^{-m_{\sigma\mu}+i\lambda_0}\sum_{\nu=0}^{p}\frac{1}{\nu!}\partial_\lambda^\nu\tilde{B}_{\sigma\mu}(\lambda_0)\sum_{q=\nu}^{p}\frac{1}{(q-\nu)!}(i\ln r)^{q-\nu}\varphi^{(p-q)}. \tag{2.16}$$

Finally, consider the expression $(B^{\mathcal{G}}_{\sigma\mu}(D_y)u)(\mathcal{G}y)$:

$$(B^{\mathcal{G}}_{\sigma\mu}(D_y)u)(\mathcal{G}y) = r^{-m_{\sigma\mu}+i\lambda_0}\beta_\sigma^{-m_{\sigma\mu}+i\lambda_0}\sum_{s=0}^{p}\frac{1}{s!}\partial_\lambda^s\tilde{B}^{\mathcal{G}}_{\sigma\mu}(\lambda_0)\sum_{q=s}^{p}\frac{1}{(q-s)!}(i\ln r + i\ln\beta_\sigma)^{q-s}\varphi^{(p-q)}(\omega+\omega_\sigma). \quad (2.17)$$

Applying the binomial formula to $(i\ln r + i\ln\beta_\sigma)^{q-s}$ and using the relation

$$\beta_\sigma^{-m_{\sigma\mu}+i\lambda_0}\sum_{s=0}^{\nu}\frac{1}{s!(\nu-s)!}\partial_\lambda^s\tilde{B}^{\mathcal{G}}_{\sigma\mu}(\lambda_0)(i\ln\beta_\sigma)^{\nu-s} = \frac{1}{\nu!}\partial_\lambda^\nu(\beta_\sigma^{-m_{\sigma\mu}+i\lambda}\tilde{B}^{\mathcal{G}}_{\sigma\mu}(\lambda))|_{\lambda=\lambda_0},$$

we obtain from (2.17)

$$(B^{\mathcal{G}}_{\sigma\mu}(D_y)u)(\mathcal{G}y) = r^{-m_{\sigma\mu}+i\lambda_0}\sum_{\nu=0}^{p}\frac{1}{\nu!}\partial_\lambda^\nu(\beta_\sigma^{-m_{\sigma\mu}+i\lambda}\tilde{B}^{\mathcal{G}}_{\sigma\mu}(\lambda))\big|_{\lambda=\lambda_0}\sum_{q=\nu}^{p}\frac{1}{(q-\nu)!}(i\ln r)^{q-\nu}\varphi^{(p-q)}(\omega+\omega_\sigma). \quad (2.18)$$

Combining the summands at the same powers of $i\ln r$ in (2.15), (2.16), and (2.18), we see that the function u satisfies homogeneous problem (2.1), (2.2) if and only if

$$\sum_{h=0}^{k}\frac{1}{h!}\partial_\lambda^h\tilde{\mathcal{L}}(\lambda_0)\varphi^{(k-h)} = 0, \quad k = 0,\dots,p. \quad \square$$

Any solution of the form (2.14) for homogeneous problem (2.1), (2.2) is called a *power solution of order* p corresponding to the eigenvalue λ_0.

Repeating the proof of Lemma 1.2 of [15], from Lemma 2.1 of the present work, we derive the following statement.

Lemma 2.2. *Let* $\{\varphi^{(0,j)},\dots,\varphi^{(\varkappa_j-1,j)}: j=1,\dots,J\}$ *be a canonical system of Jordan chains of the operator* $\tilde{\mathcal{L}}(\lambda)$ *corresponding to an eigenvalue* λ_0. *Then the functions*

$$u^{(k,j)}(\omega,r) = r^{i\lambda_0}\sum_{q=0}^{k}\frac{1}{q!}(i\ln r)^q\varphi^{(k-q,j)}(\omega), \quad k=0,\dots,\varkappa_j-1, \quad j=1,\dots,J, \quad (2.19)$$

form a basis for the space of power solutions to homogeneous problem (2.1), (2.2) *corresponding to the eigenvalue* λ_0.

Similarly to Theorem 1.2 of [15], using Theorem 2.1 and Lemma 2.2 of this work, one can prove the following statement concerning the asymptotic representation of solutions for nonlocal problem (2.1), (2.2).

Theorem 2.2. *Let us have* $\{f, g_{\sigma\mu}\} \in H^l_a(K,\gamma)\cap H^l_{a_1}(K,\gamma)$, *where* $a > a_1$. *Suppose the lines* $\operatorname{Im}\lambda = a_1+1-l-2m$ *and* $\operatorname{Im}\lambda = a+1-l-2m$ *contain no eigenvalues of the operator* $\tilde{\mathcal{L}}(\lambda)$. *If* u *is a solution for problem* (2.1), (2.2) *from the space* $H^{l+2m}_a(K)$, *then*

$$u(\omega,r) = \sum_{n=1}^{N}\sum_{j=1}^{J_n}\sum_{k=0}^{\varkappa_{j,n}-1}c_n^{(k,j)}u_n^{(k,j)}(\omega,r) + u_1(\omega,r). \quad (2.20)$$

Here $\lambda_1,\dots,\lambda_N$ *are eigenvalues of* $\tilde{\mathcal{L}}(\lambda)$ *located in the strip* $a_1+1-l-2m < \operatorname{Im}\lambda < a+1-l-2m$;

$$u_n^{(k,j)}(\omega,r) = r^{i\lambda_n}\sum_{q=0}^{k}\frac{1}{q!}(i\ln r)^q\varphi_n^{(k-q,j)}(\omega) \quad (2.21)$$

are power solutions (*of order* k) *for homogeneous problem* (2.1), (2.2);

$$\{\varphi_n^{(0,j)},\dots,\varphi_n^{(\varkappa_{j,n}-1,j)}: j=1,\dots,J_n\}$$

is a canonical system of Jordan chains of the operator $\tilde{\mathcal{L}}(\lambda)$ *corresponding to the eigenvalue* λ_n, $n=1,\dots,N$; $c_n^{(k,j)}$ *are some constants;* u_1 *is a solution for problem* (2.1), (2.2) *from the space* $H^{l+2m}_{a_1}(K)$.

Remark 2.2. One can show that the formula (2.20) is valid even if the line $\operatorname{Im}\lambda = a+1-l-2m$ contains eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. We demand that the line $\operatorname{Im}\lambda = a+1-l-2m$ have no eigenvalues, since this condition will also be used for studying asymptotics of solutions for the adjoint problem (Theorem 4.2).

Remark 2.3. If the conditions of Theorem 2.2 are fulfilled and the strip $a_1+1-l-2m \leqslant \operatorname{Im}\lambda < a+1-l-2m$ contains no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$, then the solution u from Theorem 2.2 belongs to the space $H^{l+2m}_{a_1}(K)$.

3 ADJOINT NONLOCAL PROBLEMS IN ANGLES

1. In order to calculate the coefficients $c_\nu^{(k,j)}$ in asymptotic formula (2.20), we shall need operators that are adjoint to the operators of nonlocal problems.

Denote $W^l[b_1,b_2]^* = W^l(b_1,b_2)^*\times\mathbb{C}^{2m}$. Consider the operator $\tilde{\mathcal{L}}^*(\lambda)\colon W^l[b_1,b_2]^*\to W^{l+2m}(b_1,b_2)^*$, which is adjoint to the operator $\tilde{\mathcal{L}}(\bar\lambda)$ with regard to the extension of the inner product in $L_2(b_1,b_2)\times\mathbb{C}^{2m}$. The operator $\tilde{\mathcal{L}}^*(\lambda)$ takes $\{\psi,\chi_{\sigma\mu}\}\in W^l[b_1,b_2]^*$ to $\tilde{\mathcal{L}}^*(\lambda)\{\psi,\chi_{\sigma\mu}\}$ by the rule

$$\langle\varphi,\tilde{\mathcal{L}}^*(\lambda)\{\psi,\chi_{\sigma\mu}\}\rangle = \langle\tilde{\mathcal{P}}(\bar\lambda)\varphi,\psi\rangle + \sum_{\sigma=1,2}\sum_{\mu=1}^m \tilde{\mathcal{B}}_{\sigma\mu}(\bar\lambda)\varphi\cdot\overline{\chi_{\sigma\mu}}\quad\text{for all}\quad \varphi\in W^{l+2m}(b_1,b_2).$$

Here and further $\langle\cdot,\cdot\rangle$ is a sesquilinear form on the corresponding couple of adjoint spaces.

First of all we give a remark analogous to Remark 2.1.

Remark 3.1. An eigenvector and associated vectors $\{\psi^{(0)},\chi_{\sigma\mu}^{(0)}\},\dots,\{\psi^{(\varkappa-1)},\chi_{\sigma\mu}^{(\varkappa-1)}\}$ of the operator $\tilde{\mathcal{L}}^*(\lambda)$ corresponding to an eigenvalue $\bar\lambda_0$ satisfy the equalities

$$\sum_{q=0}^{\nu}\frac{1}{q!}\partial_\lambda^q\tilde{\mathcal{L}}^*(\bar\lambda_0)\{\psi^{(\nu-q)},\chi_{\sigma\mu}^{(\nu-q)}\}=0,\quad \nu=0,\dots,\varkappa-1. \tag{3.1}$$

From equalities (3.1) and Lemma A.2, it follows that the components $\psi^{(0)},\dots,\psi^{(\varkappa-1)}$ of an eigenvector and associated vectors of the operator $\tilde{\mathcal{L}}^*(\lambda)$ are infinitely differentiable functions in the intervals $[b_1,b]$ and $[b,b_2]$.

Denote $H_a^l(K,\gamma)^* = H_a^l(K)^*\times\prod\limits_{\sigma=1,2}\prod\limits_{\mu=1}^m H_a^{l+2m-m_{\sigma\mu}-1/2}(\gamma_\sigma)^*$. Let $\mathcal{L}^*\colon H_a^l(K,\gamma)^*\to H_a^{l+2m}(K)^*$ be the operator adjoint to the operator $\mathcal{L}$ with regard to the extension of the inner product in $L_2(K)\times\prod\limits_{\sigma=1,2}\prod\limits_{\mu=1}^m L_2(\gamma_\sigma)$. The operator $\mathcal{L}^*$ takes $\{v,w_{\sigma\mu}\}\in H_a^l(K,\gamma)^*$ to $\tilde{\mathcal{L}}^*\{v,w_{\sigma\mu}\}$ by the rule

$$\langle u,\mathcal{L}^*\{v,w_{\sigma\mu}\}\rangle = \langle\mathcal{P}(D_y)u,v\rangle + \sum_{\sigma=1,2}\sum_{\mu=1}^m\langle\mathcal{B}_{\sigma\mu}(D_y)u,w_{\sigma\mu}\rangle\quad\text{for all}\quad u\in H_a^{l+2m}(K). \tag{3.2}$$

Consider the homogeneous equation

$$\mathcal{L}^*\{v,w_{\sigma\mu}\}=0. \tag{3.3}$$

Lemma 3.1. *The function*

$$\{v,w_{\sigma\mu}\} = \Big\{r^{i\bar\lambda_0+2m-2}\sum_{q=0}^p\frac{1}{q!}(i\ln r)^q\psi^{(p-q)},\quad r^{i\bar\lambda_0+m_{\sigma\mu}-1}\sum_{q=0}^p\frac{1}{q!}(i\ln r)^q\chi_{\sigma\mu}^{(p-q)}\Big\}, \tag{3.4}$$

where $\{\psi^{(s)},\chi^{(s)}\}\in W^l[b_1,b_2]^*$, $s=0,\dots,\varkappa-1$, is a solution for homogeneous equation (3.3) *if and only if $\bar\lambda_0$ is an eigenvalue of the operator $\tilde{\mathcal{L}}^*(\lambda)$ and $\{\psi^{(0)},\chi_{\sigma\mu}^{(0)}\},\dots,\{\psi^{(\varkappa-1)},\chi_{\sigma\mu}^{(\varkappa-1)}\}$ is a Jordan chain corresponding to the eigenvalue $\bar\lambda_0$; $p\leqslant\varkappa-1$.*

Proof. By Remark 3.1 the functions $\psi^{(s)}$, $s=0,\dots,\varkappa-1$, belong to $L_2(b_1,b_2)$. Therefore, for any $u\in C_0^\infty(\bar K\setminus\{0\})$, the following identity holds:

$$\langle u,\mathcal{L}^*\{v,w_{\sigma\mu}\}\rangle = \int\limits_{b_1}^{b_2}\int\limits_0^\infty r^{-1}\tilde{\mathcal{P}}(rD_r)u\cdot\overline{r^{i\bar\lambda_0}\sum_{q=0}^p\frac{1}{q!}(i\ln r)^q\psi^{(p-q)}}\,dr\,d\omega$$
$$+\int\limits_0^\infty\sum_{\sigma=1,2}\sum_{\mu=1}^m r^{-1}\tilde{B}_{\sigma\mu}(rD_r)u\big|_{\omega=b_\sigma}\cdot\overline{r^{i\bar\lambda_0}\sum_{q=0}^p\frac{1}{q!}(i\ln r)^q\chi_{\sigma\mu}^{(p-q)}}\,dr$$

$$+\int_0^\infty \sum_{\sigma=1,2}\sum_{\mu=1}^{m} r^{-1}\tilde B^{\mathcal G}_{\sigma\mu}(rD_r)u\big|_{\omega=b}\overline{r^{i\bar\lambda_0}\beta_\sigma^{-m_{\sigma\mu}-i\bar\lambda_0}\sum_{q=0}^{p}\frac{1}{q!}(i\ln\beta_\sigma^{-1}+i\ln r)^q\chi_{\sigma\mu}^{(p-q)}}\,dr \tag{3.5}$$

(if we put $r' = r\beta_\sigma^{-1}$ in the last integral, then we obtain exactly formula (3.2)).

Denote by $\delta_{b'} = \delta_{b'}(\omega)$ the delta-function with support at the point b' ($b_1 \leqslant b' \leqslant b_2$). Let $\tilde{\mathcal P}^*(\lambda)$, $\tilde B^*_{\sigma\mu}(\lambda)$, and $(\tilde B^{\mathcal G}_{\sigma\mu})^*(\lambda)$ be the operators formally adjoint to $\tilde{\mathcal P}(\bar\lambda)$, $\tilde B_{\sigma\mu}(\bar\lambda)$, and $\tilde B^{\mathcal G}_{\sigma\mu}(\bar\lambda)$ respectively.

Note that identities of the form

$$\int_{b_1}^{b_2} D_\omega\varphi\cdot\overline{\psi^{(p-q)}}\,d\omega = \langle\varphi, D_\omega\psi^{(p-q)}\rangle,\quad D_\omega\varphi\big|_{\omega=b'}\cdot\overline{\chi^{(p-q)}} = \langle\varphi, D_\omega(\chi^{(p-q)}\otimes\delta_{b'})\rangle$$

(for $\varphi\in W^l(b_1,b_2)$) generate the distributions $D_\omega\psi^{(p-q)}$ and $D_\omega(\chi^{(p-q)}\otimes\delta_{b'})$ from the space $W^l(b_1,b_2)^*$. Therefore, integrating in (3.5) by parts (for fixed ω) and using the relations

$$\tilde{\mathcal P}^*(rD_r)\bigg(r^{i\bar\lambda_0}\sum_{q=0}^{p}\frac{1}{q!}(i\ln r)^q\psi^{(p-q)}\bigg) = r^{i\bar\lambda_0}\sum_{\nu=0}^{p}\frac{1}{\nu!}\partial_\lambda^\nu\tilde{\mathcal P}^*(\bar\lambda_0)\sum_{q=\nu}^{p}\frac{1}{(q-\nu)!}(i\ln r)^{q-\nu}\psi^{(p-q)},$$

$$\tilde B^*_{\sigma\mu}(rD_r)\bigg(r^{i\bar\lambda_0}\sum_{q=0}^{p}\frac{1}{q!}(i\ln r)^q\chi_{\sigma\mu}^{(p-q)}\otimes\delta_{b_\sigma}\bigg) = r^{i\bar\lambda_0}\sum_{\nu=0}^{p}\frac{1}{\nu!}\partial_\lambda^\nu\tilde B^*_{\sigma\mu}(\bar\lambda_0)\bigg(\sum_{q=\nu}^{p}\frac{1}{(q-\nu)!}(i\ln r)^{q-\nu}\chi_{\sigma\mu}^{(p-q)}\otimes\delta_{b_\sigma}\bigg),$$

$$(\tilde B^{\mathcal G}_{\sigma\mu})^*(rD_r)\bigg(r^{i\bar\lambda_0}\beta_\sigma^{-m_{\sigma\mu}-i\bar\lambda_0}\sum_{q=0}^{p}\frac{1}{q!}(i\ln\beta_\sigma^{-1}+i\ln r)^q\chi_{\sigma\mu}^{(p-q)}\otimes\delta_b\bigg)$$

$$= r^{i\bar\lambda_0}\sum_{\nu=0}^{p}\frac{1}{\nu!}\partial_\lambda^\nu(\beta_\sigma^{-m_{\sigma\mu}-i\lambda}(\tilde B^{\mathcal G}_{\sigma\mu})^*(\lambda))\big|_{\lambda=\bar\lambda_0}\bigg(\sum_{q=\nu}^{p}\frac{1}{(q-\nu)!}(i\ln r)^{q-\nu}\chi_{\sigma\mu}^{(p-q)}\otimes\delta_b\bigg)$$

(which are proved similarly to equalities (2.15), (2.16), and (2.18)), we conclude that the function $\{v, w_{\sigma\mu}\}$ satisfies homogeneous equation (3.3) if and only if

$$\sum_{h=0}^{k}\frac{1}{h!}\partial_\lambda^h\tilde{\mathcal L}^*(\bar\lambda_0)\{\psi^{(k-h)},\chi_{\sigma\mu}^{(k-h)}\} = 0,\quad k=0,\dots,p$$

(cf. the proof of Lemma 2.1). □

Any solution of the form (3.4) for homogeneous equation (3.3) is called a *power solution of order* p corresponding to the eigenvalue $\bar\lambda_0$.

2. Further we need a special choice of Jordan chains satisfying the conditions of biorthogonality and normalization. Such chains are described in the following lemma.

Lemma 3.2. *Suppose a canonical system of Jordan chains*

$$\{\varphi^{(0,j)},\dots,\varphi^{(\varkappa_j-1,j)}: j=1,\dots,J\}$$

corresponds to an eigenvalue λ_0 *of the operator* $\tilde{\mathcal L}(\lambda)$. *Then there exists a canonical system of Jordan chains*

$$\{\{\psi^{(0,j)},\chi_{\sigma\mu}^{(0,j)}\},\dots,\{\psi^{(\varkappa_j-1,j)},\chi_{\sigma\mu}^{(\varkappa_j-1,j)}\}: j=1,\dots,J\}$$

of the operator $\tilde{\mathcal L}^*(\lambda)$ *corresponding to the eigenvalue* $\bar\lambda_0$ *such that the following relations hold:*

$$\sum_{p=0}^{\nu}\sum_{q=0}^{k}\frac{1}{(\nu+k+1-p-q)!}\bigg\{(\partial_\lambda^{\nu+k+1-p-q}\tilde{\mathcal P}(\lambda_0)\varphi^{(q,\xi)},\psi^{(p,\zeta)})_{L_2(b_1,b_2)}$$
$$+\sum_{\sigma=1,2}\sum_{\mu=1}^{m}(\partial_\lambda^{\nu+k+1-p-q}\tilde{\mathcal B}_{\sigma\mu}(\lambda_0)\varphi^{(q,\xi)},\chi_{\sigma\mu}^{(p,\zeta)})_{\mathbb C}\bigg\} = \delta_{\xi,\zeta}\delta_{\varkappa_\xi-k-1,\nu}. \tag{3.6}$$

Here $\zeta,\xi=1,\dots,J$; $\nu=0,\dots,\varkappa_\zeta-1$; $k=0,\dots,\varkappa_\xi-1$; $\delta_{p,q}$ *is the Kronecker symbol.*

Proof. By Lemma 2.1 of [6], λ_0 is a normal eigenvalue of the operator $\tilde{\mathcal L}(\lambda)$, that is, $\dim\ker\tilde{\mathcal L}(\lambda_0)<\infty$, $\operatorname{codim}\mathcal R(\tilde{\mathcal L}(\lambda_0))<\infty$, and all points of the deleted neighborhood $0<|\lambda-\lambda_0|<\rho$ (for sufficiently small ρ) are regular ones for $\tilde{\mathcal L}(\lambda)$. Thus, the necessary result follows from Lemma 2.1 of [15]. □

4 CALCULATION OF THE COEFFICIENTS IN THE ASYMPTOTICS OF SOLUTIONS FOR NONLOCAL PROBLEMS IN ANGLES

1. In this section, we obtain explicit formulas for the coefficients $c_n^{(k,j)}$ in asymptotic formula (2.20). First we shall calculate the coefficients with the help of power solutions $\{v, w_{\sigma\mu}\}$ for homogeneous equation (3.3), and then we shall obtain a representation of the coefficients in terms of the Green formula.

Let $\bar{\lambda}_n$ be an eigenvalue of the operator $\tilde{\mathcal{L}}^*(\lambda)$, and let

$$\big\{\{\psi_n^{(0,j)}, \chi_{\sigma\mu,n}^{(0,j)}\}, \dots, \{\psi_n^{(\varkappa_{j,n}-1,j)}, \chi_{\sigma\mu,n}^{(\varkappa_{j,n}-1,j)}\} \colon j = 1, \dots, J_n\big\}$$

be Jordan chains of $\tilde{\mathcal{L}}^*(\lambda)$ corresponding to the eigenvalue $\bar{\lambda}_n$ and forming a canonical system. Consider the power solutions (of order ν) for Eq. (3.3)

$$\{v_n^{(\nu,j)}, w_{\sigma\mu,n}^{(\nu,j)}\} = \left\{ r^{i\bar{\lambda}_n+2m-2} \sum_{q=0}^{\nu} \frac{1}{q!}(i\ln r)^q \psi_n^{(\nu-q,j)}, \quad r^{i\bar{\lambda}_n+m_{\sigma\mu}-1} \sum_{q=0}^{\nu} \frac{1}{q!}(i\ln r)^q \chi_{\sigma\mu,n}^{(\nu-q,j)} \right\}, \tag{4.1}$$

where $\nu = 0, \dots, \varkappa_{j,n} - 1$.

Theorem 4.1. *Let the conditions of Theorem 2.2 hold; then the coefficients $c_n^{(k,j)}$ from (2.20) are calculated by the formulas*

$$c_n^{(k,j)} = (f, iv_n^{(\varkappa_{j,n}-k-1,j)})_{L_2(K)} + \sum_{\sigma=1,2} \sum_{\mu=1}^{m} (g_{\sigma\mu}, iw_{\sigma\mu,n}^{(\varkappa_{j,n}-k-1,j)})_{L_2(\gamma_\sigma)}, \tag{4.2}$$

where $\{v_n^{(\nu,j)}, w_{\sigma\mu,n}^{(\nu,j)}\}$ is the vector defined by equality (4.1), and the Jordan chains

$$\{\varphi_n^{(0,j)}, \dots, \varphi_n^{(\varkappa_{j,n}-1,j)} \colon j = 1, \dots, J_n\},$$
$$\big\{\{\psi_n^{(0,j)}, \chi_{\sigma\mu,n}^{(0,j)}\}, \dots, \{\psi_n^{(\varkappa_{j,n}-1,j)}, \chi_{\sigma\mu,n}^{(\varkappa_{j,n}-1,j)}\} \colon j = 1, \dots, J_n\big\},$$

appearing in (2.21) and (4.1), satisfy conditions (3.6) of biorthogonality and normalization.

Theorem 4.1 is proved similarly to Theorem 3.1 of [15].

Remark 4.1. By Remark 3.1, the functions $\psi_n^{(\nu,j)}$ belong to the space $L_2(b_1, b_2)$. From this and from equalities (4.1) and (4.2), it follows that

$$|c_n^{(k,j)}| \leqslant c(\|\{f, g_{\sigma\mu}\}\|_{H_a^l(K,\gamma)} + \|\{f, g_{\sigma\mu}\}\|_{H_{a_1}^l(K,\gamma)})$$

if $\{f, g_{\sigma\mu}\} \in H_a^l(K,\gamma) \cap H_{a_1}^l(K,\gamma)$ and $a_1 + 1 - l - 2m < \operatorname{Im}\lambda_n < a + 1 - l - 2m$.

From Theorems 2.2, 4.1, and the duality conception, one can obtain the following result concerning the asymptotics of solutions for the adjoint problem:

$$\mathcal{L}^*\{v, w_{\sigma\mu}\} = \Psi. \tag{4.3}$$

Theorem 4.2. *Suppose $\Psi \in H_a^{l+2m}(K)^* \cap H_{a_1}^{l+2m}(K)^*$, where $a > a_1$, and the lines $\operatorname{Im}\lambda = a_1 + 1 - l - 2m$ and $\operatorname{Im}\lambda = a + 1 - l - 2m$ contain no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. If $\{v, w_{\sigma\mu}\}$ is a solution for problem (4.3) from the space $H_{a_1}^l(K,\gamma)^*$, then*

$$\{v, w_{\sigma\mu}\} = \sum_{n=1}^{N} \sum_{j=1}^{J_n} \sum_{k=0}^{\varkappa_{j,n}-1} d_n^{(k,j)} \{v_n^{(k,j)}, w_{\sigma\mu,n}^{(k,j)}\} + \{V, W_{\sigma\mu}\}. \tag{4.4}$$

Here $\lambda_1, \dots, \lambda_N$ are eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ located in the strip $a_1 + 1 - l - 2m < \operatorname{Im}\lambda < a + 1 - l - 2m$; $\{v_n^{(k,j)}, w_{\sigma\mu,n}^{(k,j)}\}$ are the vectors defined by formula (4.1); $d_n^{(k,j)}$ are some constants; $\{V, W_{\sigma\mu}\}$ is a solution for problem (4.3) from the space $H_a^l(K,\gamma)^*$.

2. Consider the Green formula for nonlocal elliptic problems. For this, we introduce the set $\gamma=\{y\colon \varphi=b\}$, which is the support of nonlocal data in problem (2.1), (2.2). Denote $K_1=\{y\colon b_1<\varphi<b\}$ and $K_2=\{y\colon b<\varphi<b_2\}$. For functions $v(y)$ given in K we denote by $v_\sigma(y)$ their restrictions on K_σ, $\sigma=1,2$. We say that v belongs to $\mathcal{C}^\infty(\bar K\setminus\{0\})$ if v_σ belongs to $C^\infty(\bar K_\sigma\setminus\{0\})$, $\sigma=1,2$.

When considering the Green formula in the angle K, for short we shall omit the argument D_y in differential operators. Denote by $\mathcal{P}^*$ the operator formally adjoint to $\mathcal{P}$. By virtue of Theorem 4.1 of [12] (see also Theorem 1 of [13]), there exist (not unique) (1) a system $\{B'_{\sigma\mu}\}_{\mu=1}^m$ of normal on γ_σ operators of order $2m-1-m'_{\sigma\mu}$ with constant coefficients such that the system $\{B_{\sigma\mu},B'_{\sigma\mu}\}_{\mu=1}^m$ is a Dirichlet system on γ_σ[(1)] of order $2m$; (2) a Dirichlet system $\{B_\mu,B'_\mu\}_{\mu=1}^m$ on γ of order $2m$ such that the operators B_μ and B'_μ are of order $2m-\mu$ and $m-\mu$ respectively.

Whenever the choice has been made, there exist differential operators $C_{\sigma\mu}$, $C'_{\sigma\mu}$, T_ν, and $T^{\mathcal{G}}_{\sigma\nu}$ ($\sigma=1,2$; $\mu=1,\dots,m$; $\nu=1,\dots,2m$) with constant coefficients such that (I) the operators $C_{\sigma\mu}$, $C'_{\sigma\mu}$, T_ν, and $T^{\mathcal{G}}_{\sigma\nu}$ are of order $m'_{\sigma\mu}$, $2m-1-m_{\sigma\mu}$, $\nu-1$, and $\nu-1$ respectively; (II) the system $\{C_{\sigma\mu}\}_{\mu=1}^m$ covers the operator $\mathcal{P}^*$ on γ_σ and supplements $\{C'_{\sigma\mu}\}_{\mu=1}^m$ to a Dirichlet system on γ_σ of order $2m$; the system $\{T_\nu\}_{\nu=1}^{2m}$ is a Dirichlet system on γ of order $2m$; (III) for all $u\in C_0^\infty(\bar K\setminus\{0\})$, $v\in\mathcal{C}^\infty(\bar K_\sigma\setminus\{0\})$, the following Green formula is valid:

$$(\mathcal{P}u,v)_{L_2(K_\sigma)}+\sum_{\sigma=1,2}\sum_{\mu=1}^m\big(\mathcal{B}_{\sigma\mu}u,C'_{\sigma\mu}v_\sigma\big|_{\gamma_\sigma}\big)_{L_2(\gamma_\sigma)}+\sum_{\mu=1}^m\big(B_\mu u\big|_\gamma,\mathcal{T}_\mu v\big)_{L_2(\gamma)}$$
$$=\sum_{\sigma=1,2}(u,\mathcal{P}^*v_\sigma)_{K_\sigma}+\sum_{\sigma=1,2}\sum_{\mu=1}^m\big(B'_{\sigma\mu}u\big|_{\gamma_\sigma},C_{\sigma\mu}v_\sigma\big|_{\gamma_\sigma}\big)_{L_2(\gamma_\sigma)}+\sum_{\mu=1}^m\big(B'_\mu u\big|_\gamma,\mathcal{T}_{m+\mu}v\big)_{L_2(\gamma)}. \quad (4.5)$$

Here
$$\mathcal{T}_\nu v\equiv T_\nu v_1\big|_\gamma-T_\nu v_2\big|_\gamma+\sum_{k=1,2}(T^{\mathcal{G}}_{k\nu}v_k)(\mathcal{G}_k^{-1}y)\big|_\gamma,$$

where $\mathcal{G}_k^{-1}$ is the operator of rotation by the angle $-\omega_k$ and expansion by $1/\beta_k$ times in the plane $\{y\}$ ($k=1,2$; $\nu=1,\dots,2m$).

Formula (4.5) generates the problem formally adjoint to problem (2.1), (2.2):

$$\mathcal{P}^*(D_y)v_\sigma=f_\sigma(y)\quad (y\in K_\sigma;\ \sigma=1,2), \quad (4.6)$$
$$\mathcal{C}_{\sigma\mu}(D_y)v\equiv C_{\sigma\mu}(D_y)v_\sigma\big|_{\gamma_\sigma}=g_{\sigma\mu}(y)\quad (y\in\gamma_\sigma;\ \sigma=1,2;\ \mu=1,\dots,m), \quad (4.7)$$
$$\mathcal{T}_\nu(D_y)v\equiv T_\nu(D_y)v_1\big|_\gamma-T_\nu(D_y)v_2\big|_\gamma+\sum_{k=1,2}(T^{\mathcal{G}}_{k\nu}(D_y)v_k)(\mathcal{G}_k^{-1}y)\big|_\gamma=h_\nu(y)\quad (y\in\gamma;\ \nu=1,\dots,2m). \quad (4.8)$$

Problem (4.6)–(4.8) is called a *nonlocal transmission problem in the angle K* [12,13].

For functions $\tilde v(\omega)$ given in the interval (b_1,b_2), we denote by $\tilde v_1(\omega)$ and $\tilde v_2(\omega)$ their restrictions on the intervals (b_1,b) and (b,b_2) respectively. We say that $\tilde v$ belongs to $\mathcal{C}^\infty([b_1,b_2])$ if $\tilde v_1$ belongs to $C^\infty([b_1,b])$ and $\tilde v_2$ belongs to $C^\infty([b,b_2])$.

Write all the differential operators appearing in (4.5) in polar coordinates (omitting ω and D_ω): $\mathcal{P}(D_y)=r^{-2m}\tilde{\mathcal{P}}(rD_r)$, $B_{\sigma\mu}(D_y)=r^{-m_{\sigma\mu}}\tilde B_{\sigma\mu}(rD_r)$, etc. By Theorem 4.3 of [12], the following Green formula with the parameter λ is valid for any functions $\tilde u\in C^\infty([b_1,b_2])$ and $\tilde v\in\mathcal{C}^\infty([b_1,b_2])$:

$$(\tilde{\mathcal{P}}(\lambda)\tilde u,\tilde v)_{L_2(b_1,b_2)}+\sum_{\sigma=1,2}\sum_{\mu=1}^m\tilde{\mathcal{B}}_{\sigma\mu}(\lambda)\tilde u\cdot\overline{\tilde C'_{\sigma\mu}(\lambda')\tilde v_\sigma\big|_{\omega=b_\sigma}}+\sum_{\mu=1}^m\tilde B_\mu(\lambda)\tilde u\big|_{\omega=b}\cdot\overline{\tilde{\mathcal{T}}_\mu(\lambda')\tilde v}=(\tilde u,\tilde{\mathcal{P}}^*(\lambda')\tilde v_1)_{L_2(b_1,b)}$$
$$+(\tilde u,\tilde{\mathcal{P}}^*(\lambda')\tilde v_2)_{L_2(b,b_2)}+\sum_{\sigma=1,2}\sum_{\mu=1}^m\tilde B'_{\sigma\mu}(\lambda)\tilde u\big|_{\omega=b_\sigma}\cdot\overline{\tilde C_{\sigma\mu}(\lambda')\tilde v_\sigma\big|_{\omega=b_\sigma}}+\sum_{\mu=1}^m\tilde B'_\mu(\lambda)\tilde u\big|_{\omega=b}\cdot\overline{\tilde{\mathcal{T}}_{m+\mu}(\lambda')\tilde v}. \quad (4.9)$$

Here $\lambda'=\bar\lambda-2i(m-1)$;

$$\tilde{\mathcal{T}}_\nu(\lambda')\tilde v=\tilde T_\nu(\lambda')\tilde v_1(\omega)\big|_{\omega=b}-\tilde T_\nu(\lambda')\tilde v_2(\omega)\big|_{\omega=b}+\sum_{k=1,2}\beta_k^{-i\lambda'+(\nu-1)}\tilde T^{\mathcal{G}}_{k\nu}(\lambda')\tilde v_k(\omega-\omega_k)\big|_{\omega=b}.$$

(1) See [16, Chap. 2, Sect. 2.2] for the definition of a Dirichlet system.

Formula (4.9) generates a problem formally adjoint to problem (2.3), (2.4):

$$\tilde{\mathcal{P}}^*(\lambda)\tilde{v}_1(\omega) = 0 \quad (\omega \in (b_1, b)), \quad \tilde{\mathcal{P}}^*(\lambda)\tilde{v}_2(\omega) = 0 \quad (\omega \in (b, b_2)), \tag{4.10}$$

$$\tilde{\mathcal{C}}_{\sigma\mu}(\lambda)\tilde{v}(\omega) \equiv \tilde{C}_{\sigma\mu}(\lambda)\tilde{v}_\sigma(\omega)\big|_{\omega=\omega_\sigma} = 0 \quad (\sigma = 1, 2;\ \mu = 1, \dots, m), \tag{4.11}$$

$$\tilde{\mathcal{T}}_\nu(\lambda)\tilde{v}(\omega) \equiv \tilde{T}_\nu(\lambda)\tilde{v}_1(\omega)\big|_{\omega=b} - \tilde{T}_\nu(\lambda)\tilde{v}_2(\omega)\big|_{\omega=b} + \sum_{k=1,2} \beta_k^{-i\lambda+(\nu-1)}\tilde{T}^{\mathcal{G}}_{k\nu}(\lambda)\tilde{v}_k(\omega-\omega_k)\big|_{\omega=b} = 0 \quad (\nu = 1, \dots, 2m). \tag{4.12}$$

Problem (4.10)–(4.12) is called a *nonlocal transmission problem on the arc* (b_1, b_2) [12, 13].

Note that problem (4.10)–(4.12) can also be derived from problem (4.6)–(4.8) if in the latter we put $f_\sigma = 0$, $g_{\sigma\mu} = 0$, and $h_\nu = 0$ and formally perform the Mellin transformation.

The operator

$$\tilde{\mathcal{M}}(\lambda)\colon W^{l+2m}(b_1, b) \oplus W^{l+2m}(b, b_2) \to (W^l(b_1, b) \oplus W^l(b, b_2)) \times \mathbb{C}^{2m} \times \mathbb{C}^{2m},$$

acting by the formula

$$\tilde{\mathcal{M}}(\lambda)\tilde{v} = \{\tilde{z}, \tilde{\mathcal{C}}_{\sigma\mu}(\lambda)\tilde{v}, \tilde{\mathcal{T}}_\nu(\lambda)\tilde{v}\},$$

corresponds to problem (4.10)–(4.12). Here $\tilde{z}(\omega) = \tilde{\mathcal{P}}^*(\lambda)\tilde{v}_1(\omega)$ for $\omega \in (b_1, b)$ and $\tilde{z}(\omega) = \tilde{\mathcal{P}}^*(\lambda)\tilde{v}_2(\omega)$ for $\omega \in (b, b_2)$. Note that we cannot define $\tilde{z}$ by the formula $\tilde{z}(\omega) = \tilde{\mathcal{P}}^*(\lambda)\tilde{v}(\omega)$ for $\omega \in (b_1, b_2)$, since the function $\tilde{v} \in W^{l+2m}(b_1, b) \oplus W^{l+2m}(b, b_2)$ may be discontinuous at the point $\omega = b$.

3. Now we shall establish a connection between Jordan chains of the operators $\tilde{\mathcal{L}}^*(\lambda)$ and $\tilde{\mathcal{M}}(\lambda)$. Put

$$\tilde{\mathcal{C}}'_{\sigma\mu}(\lambda)\tilde{v} = \tilde{C}'_{\sigma\mu}(\lambda)\tilde{v}_\sigma(\omega)|_{\omega=b_\sigma}.$$

Repeating the proof of Proposition 2.5 [14, Chap. 1] and using Green formula (4.9) and Remark 3.1, we obtain the following result.

Lemma 4.1. *The vectors* $\{\psi^{(0)}, \chi^{(0)}_{\sigma\mu}\}, \dots, \{\psi^{(\varkappa-1)}, \chi^{(\varkappa-1)}_{\sigma\mu}\}$ *form a Jordan chain of the operator* $\tilde{\mathcal{L}}^*(\lambda)$ *corresponding to an eigenvalue* $\bar{\lambda}_0$ *if and only if the vectors* $\psi^{(0)}, \dots, \psi^{(\varkappa-1)}$ *form a Jordan chain of the operator* $\tilde{\mathcal{M}}(\lambda)$ *corresponding to the eigenvalue* $\bar{\lambda}_0 - 2i(m-1)$ *and the vectors* $\psi^{(k)}$ *and* $\chi^{(k)}_{\sigma\mu}$ *are connected by the relation*

$$\chi^{(k)}_{\sigma\mu} = \sum_{r=0}^{k} \frac{1}{r!}\partial_\lambda^r \tilde{\mathcal{C}}'_{\sigma\mu}(\bar{\lambda}_0 - 2i(m-1))\psi^{(k-r)}.$$

Combining Lemmas 3.2 and 4.1, we get the following condition of biorthogonality and normalization of Jordan chains in terms of the Green formula.

Lemma 4.2. *Suppose a canonical system*

$$\{\varphi^{(0,j)}, \dots, \varphi^{(\varkappa_j-1,j)}\colon j = 1, \dots, J\}$$

corresponds to an eigenvalue λ_0 *of the operator* $\tilde{\mathcal{L}}(\lambda)$. *Then there exist a canonical system of Jordan chains*

$$\{\psi^{(0,j)}, \dots, \psi^{(\varkappa_j-1,j)}\colon j = 1, \dots, J\}$$

of the operator $\tilde{\mathcal{M}}(\lambda)$ *corresponding to the eigenvalue* $\bar{\lambda}_0 - 2i(m-1)$ *such that the following relations are valid:*

$$\sum_{p=0}^{\nu}\sum_{q=0}^{k} \frac{1}{(\nu+k+1-p-q)!}\Big\{(\partial_\lambda^{\nu+k+1-p-q}\tilde{\mathcal{P}}(\lambda_0)\varphi^{(q,\xi)}, \psi^{(p,\zeta)})_{L_2(b_1,b_2)}$$
$$+ \sum_{\sigma=1,2}\sum_{\mu=1}^{m}\Big(\partial_\lambda^{\nu+k+1-p-q}\tilde{\mathcal{B}}_{\sigma\mu}(\lambda_0)\varphi^{(q,\xi)}, \sum_{r=0}^{p}\frac{1}{r!}\partial_\lambda^r\tilde{\mathcal{C}}'_{\sigma\mu}(\bar{\lambda}_0 - 2i(m-1))\psi^{(p-r,\zeta)}\Big)_{\mathbb{C}}\Big\} = \delta_{\xi,\zeta}\delta_{\varkappa_\xi-k-1,\nu}. \tag{4.13}$$

Put

$$\mathcal{C}'_{\sigma\mu}(D_y)v = C'_{\sigma\mu}(D_y)v_\sigma(y)|_{\gamma_\sigma}.$$

Let us formulate the main result on a representation of the coefficients $c_n^{(k,j)}$ from asymptotic formula (2.20) in terms of the Green formula.

Theorem 4.3. *Let conditions of Theorem 2.2 be fulfilled. Then the coefficients* $c_n^{(k,j)}$ *from* (2.20) *are calculated by the formulas*

$$c_n^{(k,j)} = (f, iv_n^{(\varkappa_{j,n}-k-1,j)})_{L_2(K)} + \sum_{\sigma=1,2}\sum_{\mu=1}^{m}(g_{\sigma\mu}, i\mathcal{C}'_{\sigma\mu}(D_y)v_n^{(\varkappa_{j,n}-k-1,j)})_{L_2(\gamma_\sigma)}. \tag{4.14}$$

Here $v_n^{(\nu,j)}$ *is a power solution for the homogeneous nonlocal transmission problem* (4.6)–(4.8) *given by*

$$v_n^{(\nu,j)} = r^{i\bar\lambda_n+2m-2}\sum_{q=0}^{\nu}\frac{1}{q!}(i\ln r)^q\psi_n^{(\nu-q,j)},$$

where $\{\psi_n^{(0,j)},\dots,\psi_n^{(\varkappa_{j,n}-1,j)}: j=1,\dots,J_n\}$ *is a canonical system of Jordan chains of the operator* $\tilde{\mathcal{M}}(\lambda)$ *corresponding to the eigenvalue* $\bar\lambda_n - 2i(m-1)$, *and the chains* $\{\varphi_n^{(0,j)},\dots,\varphi_n^{(\varkappa_{j,n}-1,j)}: j=1,\dots,J_n\}$ *(appearing in* (2.21)*) and* $\{\psi_n^{(0,j)},\dots,\psi_n^{(\varkappa_{j,n}-1,j)}: j=1,\dots,J_n\}$ *satisfy conditions* (4.13) *of biorthogonality and normalization.*

Proof. Similarly to the proof of Lemma 2.1, one can show that $v_n^{(\nu,j)}$ is a solution of the homogeneous problem (4.6)–(4.8) if and only if $\psi_n^{(0,j)},\dots,\psi_n^{(\varkappa_{j,n}-1,j)}$ is a Jordan chain of the operator $\tilde{\mathcal{M}}(\lambda)$ corresponding to the eigenvalue $\bar\lambda_n - 2i(m-1)$.

Further, we have

$$\begin{aligned}\mathcal{C}'_{\sigma\mu}(D_y)v_n^{(\nu,j)} &= r^{i\bar\lambda_n+m_{\sigma\mu}-1}\tilde{\mathcal{C}}'_{\sigma\mu}(\bar\lambda_n-2i(m-1)+rD_r)\sum_{q=0}^{\nu}\frac{1}{q!}(i\ln r)^q\psi_n^{(\nu-q,j)}\\ &= r^{i\bar\lambda_n+m_{\sigma\mu}-1}\sum_{s=0}^{\nu}\frac{1}{s!}\partial_\lambda^s\tilde{\mathcal{C}}'_{\sigma\mu}(\bar\lambda_n-2i(m-1))\sum_{q=s}^{\nu}\frac{1}{(q-s)!}(i\ln r)^{q-s}\psi_n^{(\nu-q,j)}.\end{aligned}$$

Changing the order of summation and applying Lemma 4.1, we get

$$\mathcal{C}'_{\sigma\mu}(D_y)v_n^{(\nu,j)} = r^{i\bar\lambda_n+m_{\sigma\mu}-1}\sum_{s=0}^{\nu}\frac{1}{s!}\partial_\lambda^s\tilde{\mathcal{C}}'_{\sigma\mu}(\bar\lambda_n-2i(m-1))\sum_{q=0}^{\nu-s}\frac{1}{q!}(i\ln r)^q\psi_n^{(\nu-q-s,j)}.$$

Now the necessary result follows from Theorem 4.1 and Lemma 4.2. □

3. Concluding this section, we consider the asymptotics of solutions for nonlocal problems in the angle with a special right-hand side. Put

$$F(\omega,r) = \sum_{q=0}^{M}\frac{1}{q!}(i\ln r)^q f^{(q)}(\omega),\quad G_{\sigma\mu}(r) = \sum_{q=0}^{M}\frac{1}{q!}(i\ln r)^q g_{\sigma\mu}^{(q)},\quad \{f^{(q)}, g_{\sigma\mu}^{(q)}\}\in W^l(b_1,b_2)\times\mathbb{C}^{2m}.$$

Let Λ be some complex number. If Λ is an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$, then denote by $\varkappa(\Lambda)$ the greatest of partial multiplicities of this eigenvalue; otherwise put $\varkappa(\Lambda)=0$.

Lemma 4.3. *For problem* (2.1), (2.2) *with right-hand side* $\{r^{i\Lambda-2m}F, r^{i\Lambda-m_{\sigma\mu}}G_{\sigma\mu}\}$, *there exists a solution*

$$u(\omega,r) = r^{i\Lambda}\sum_{q=0}^{M+\varkappa(\Lambda)}\frac{1}{q!}(i\ln r)^q u^{(q)}(\omega), \tag{4.15}$$

where $u^{(q)}\in W^{l+2m}(b_1,b_2)$. *A solution of such a form is unique if* $\varkappa(\Lambda)=0$ *(that is, if* Λ *is not an eigenvalue of* $\tilde{\mathcal{L}}(\lambda)$*). If* $\varkappa(\Lambda)>0$, *solution* (4.15) *is defined accurate to an arbitrary linear combination of power solutions* (2.19) *corresponding to the eigenvalue* Λ.

The proof is analogous to the proof of Lemma 3.1 [14, Chap. 3].

Remark 4.2. The results of Sects. 2–4 are generalized for the case of a system of equations as well as for the case of an arbitrary number of nonlocal terms with supports on different rays.

5 ASYMPTOTICS OF SOLUTIONS FOR LOCAL PROBLEMS IN $\mathbb{R}^2 \setminus \{0\}$

1. In investigating nonlocal elliptic problems in plane domains, one should consider solutions not in a whole domain G but in $G \setminus \mathcal{K}$, where $\mathcal{K}$ is a finite set of points (see [5,7]). And solutions may have power singularities near the set $\mathcal{K}$ which corresponds to some conditions of coherence. To study asymptotics of solutions for such problems, we need the results of Sects. 2–4 and of this section as well.

Let $\mathcal{P}(D_y)$ be a homogeneous properly elliptic differential operator of order $2m$ with constant coefficients.

Introduce the bounded operator $\mathcal{P} = \mathcal{P}(D_y)\colon H_a^{l+2m}(\mathbb{R}^2) \to H_a^l(\mathbb{R}^2)$. We shall study the asymptotics of solutions $u \in H_a^{l+2m}(\mathbb{R}^2)$ for the equation

$$\mathcal{P}u = f, \tag{5.1}$$

supposing that $f \in H_a^l(\mathbb{R}^2) \cap H_{a_1}^l(\mathbb{R}^2)$.

Write the operator $\mathcal{P}(D_y)$ in polar coordinates: $\mathcal{P}(D_y) = r^{-2m}\tilde{\mathcal{P}}(\omega, D_\omega, rD_r)$. The coefficients of the operator $\tilde{\mathcal{P}}(\omega, D_\omega, rD_r)$ as functions of ω belong to the set $C_{2\pi}^\infty[0, 2\pi]$ of 2π-periodic infinitely differentiable functions.

Introduce the bounded operator $\tilde{\mathcal{P}}(\lambda) = \tilde{\mathcal{P}}(\omega, D_\omega, \lambda) : W_{2\pi}^{l+2m}(0, 2\pi) \to W_{2\pi}^l(0, 2\pi)$, where $W_{2\pi}^l(0, 2\pi)$ is a completion of the set $C_{2\pi}^\infty[0, 2\pi]$ in $W^l(0, 2\pi)$.

From [5, Sect. 1], it follows that there exists a finite-meromorphic operator-valued function $\tilde{\mathcal{P}}^{-1}(\lambda)$ such that its poles coinciding with eigenvalues of $\tilde{\mathcal{P}}(\lambda)$ are located (except, perhaps, for a finite number) inside a double angle less than π containing the imaginary axis. If λ is not a pole, then $\tilde{\mathcal{P}}^{-1}(\lambda)$ is a bounded inverse operator for $\tilde{\mathcal{P}}(\lambda)$. If the line $\operatorname{Im}\lambda = a + 1 - l - 2m$ contains no poles of the operator $\tilde{\mathcal{P}}^{-1}(\lambda)$ (or no eigenvalues of the operator $\tilde{\mathcal{P}}(\lambda)$, which is the same), then by [5, Sect. 1] the operator $\mathcal{P}$ is an isomorphism.

Using the formulated results and repeating the considerations of [14, Chap. 3], we shall obtain most of the statements of this section.

Theorem 5.1. *Suppose $f \in H_a^l(\mathbb{R}^2) \cap H_{a_1}^l(\mathbb{R}^2)$, where $a > a_1$, and the lines $\operatorname{Im}\lambda = a_1 + 1 - l - 2m$ and $\operatorname{Im}\lambda = a + 1 - l - 2m$ contain no eigenvalues of the operator $\tilde{\mathcal{P}}(\lambda)$. If u is a solution of problem* (5.1) *from the space $H_a^{l+2m}(\mathbb{R}^2)$, then*

$$u(\omega, r) = \sum_{n=1}^{N} \sum_{j=1}^{J_n} \sum_{k=0}^{\varkappa_{j,n}-1} c_n^{(k,j)} u_n^{(k,j)}(\omega, r) + u_1(\omega, r). \tag{5.2}$$

Here $\lambda_1, \dots, \lambda_N$ are eigenvalues of $\tilde{\mathcal{P}}(\lambda)$ located in the strip $a_1 + 1 - l - 2m < \operatorname{Im}\lambda < a + 1 - l - 2m$;

$$u_n^{(k,j)}(\omega, r) = r^{i\lambda_n} \sum_{q=0}^{k} \frac{1}{q!}(i \ln r)^q \varphi_n^{(k-q,j)}(\omega) \tag{5.3}$$

are power (of order k) solutions for homogeneous problem (5.1)*;*

$$\{\varphi_n^{(0,j)}, \dots, \varphi_n^{(\varkappa_{j,n}-1,j)} : j = 1, \dots, J_n\}$$

is a canonical system of Jordan chains of the operator $\tilde{\mathcal{P}}(\lambda)$ corresponding to the eigenvalue λ_n, $n = 1, \dots, N$; $c_n^{(k,j)}$ are some constants; u_1 is a solution of problem (5.1) *from the space $H_{a_1}^{l+2m}(\mathbb{R}^2)$.*

Remark 5.1. Similarly to the case of plane angles, one can show that formula (5.2) is valid even if the line $\operatorname{Im}\lambda = a+1-l-2m$ contains eigenvalues of the operator $\tilde{\mathcal{P}}(\lambda)$. We demand that the line $\operatorname{Im}\lambda = a+1-l-2m$ have no eigenvalues, since this condition will also be used for studying asymptotics of solutions for the adjoint problem (Theorem 5.3).

2. Further we shall obtain explicit formulas for the coefficients $c_n^{(k,j)}$ in asymptotic formula (5.2). First we shall calculate the coefficients with the help of power solutions for a homogeneous adjoint equation and then we shall obtain a representation of the coefficients in terms of the Green formula.

Consider the operator $\mathcal{P}^*\colon H_a^l(\mathbb{R}^2)^* \to H_a^{l+2m}(\mathbb{R}^2)^*$ adjoint to $\mathcal{P}$ with respect to the extension of the inner product in $L_2(\mathbb{R}^2)$ and the operator $\tilde{\mathcal{P}}^*(\lambda)\colon W_{2\pi}^l(0, 2\pi)^* \to W_{2\pi}^{l+2m}(0, 2\pi)^*$ adjoint to $\tilde{\mathcal{P}}(\bar{\lambda})$ with respect to the extension of the inner product in $L_2(0, 2\pi)$.

Let $\bar{\lambda}_n$ be an eigenvalue of the operator $\tilde{\mathcal{P}}^*(\lambda)$. Let

$$\{\psi_n^{(0,j)}, \dots, \psi_n^{(\varkappa_{j,n}-1,j)} : j = 1, \dots, J_n\}$$

be Jordan chains of $\tilde{\mathcal{P}}^*(\lambda)$ corresponding to the eigenvalue $\bar{\lambda}_n$ and forming a canonical system. Using the ellipticity of the operator $\tilde{\mathcal{P}}^*(\omega, D_\omega, \lambda)$, the method of "frozen" coefficients, expansion of the functions $\psi_n^{(\nu,j)}$ in the Fourier series by the functions $e^{ik\omega}/\sqrt{2\pi}$,(2) and equalities of the type (3.1), one can show that $\psi_n^{(\nu,j)}$ are 2π-periodic infinitely differentiable functions in the interval $[0, 2\pi]$.

Consider the power solution (of order ν)

$$v_n^{(\nu,j)} = r^{i\bar{\lambda}_n+2m-2} \sum_{q=0}^{\nu} \frac{1}{q!} (i \ln r)^q \psi_n^{(\nu-q,j)}, \quad \nu = 0, \dots, \varkappa_{j,n} - 1, \tag{5.4}$$

for the equation $\mathcal{P}^* v = 0$ corresponding to the eigenvalue $\bar{\lambda}_n$ of the operator $\tilde{\mathcal{P}}^*(\lambda)$.

Theorem 5.2. *Let the conditions of Theorem* 5.1 *be fulfilled. Then the coefficients* $c_n^{(k,j)}$ *from* (5.2) *are calculated by the formulas*

$$c_n^{(k,j)} = (f, i v_n^{(\varkappa_{j,n}-k-1,j)})_{L_2(\mathbb{R}^2)}, \tag{5.5}$$

where $v_n^{(\nu,j)}$ *are defined by equalities* (5.4); *the Jordan chains* $\{\varphi_n^{(0,j)}, \dots, \varphi_n^{(\varkappa_{j,n}-1,j)} : j = 1, \dots, J_n\}$ *and* $\{\psi_n^{(0,j)}, \dots, \psi_n^{(\varkappa_{j,n}-1,j)} : j = 1, \dots, J_n\}$ *appearing in* (5.3) *and* (5.4) *satisfy conditions of biorthogonality and normalization analogous to* (3.6).

Remark 5.2. Since the functions $\psi_n^{(\nu,j)}$ are infinitely differentiable, from Eqs. (5.4) and (5.5) it follows that

$$|c_n^{(k,j)}| \leqslant c(\|f\|_{H_a^l(\mathbb{R}^2)} + \|f\|_{H_{a_1}^l(\mathbb{R}^2)})$$

if $f \in H_a^l(\mathbb{R}^2) \cap H_{a_1}^l(\mathbb{R}^2)$ and $a_1 + 1 - l - 2m < \operatorname{Im} \lambda_n < a + 1 - l - 2m$.

From Theorems 5.1, 5.2, and the duality conception, one can get the following result concerning the asymptotics of solutions for the adjoint problem:

$$\mathcal{P}^* v = \Psi. \tag{5.6}$$

Theorem 5.3. *Suppose* $\Psi \in H_a^{l+2m}(\mathbb{R}^2)^* \cap H_{a_1}^{l+2m}(\mathbb{R}^2)^*$, *where* $a > a_1$, *and the lines* $\operatorname{Im} \lambda = a_1 + 1 - l - 2m$ *and* $\operatorname{Im} \lambda = a + 1 - l - 2m$ *contain no eigenvalues of the operator* $\tilde{\mathcal{P}}(\lambda)$. *If* v *is a solution of problem* (4.3) *from the space* $H_{a_1}^l(\mathbb{R}^2)^*$, *then*

$$v = \sum_{n=1}^{N} \sum_{j=1}^{J_n} \sum_{k=0}^{\varkappa_{j,n}-1} d_n^{(k,j)} v_n^{(k,j)} + V. \tag{5.7}$$

Here $\lambda_1, \dots, \lambda_N$ *are eigenvalues of* $\tilde{\mathcal{P}}(\lambda)$ *located in the strip* $a_1 + 1 - l - 2m < \operatorname{Im} \lambda < a + 1 - l - 2m$; $v_n^{(k,j)}$ *are the vectors given by* (5.4); $d_n^{(k,j)}$ *are some constants;* V *is a solution for problem* (5.6) *from the space* $H_a^l(\mathbb{R}^2)^*$.

3. Consider the Green formula for local elliptic problems in $\mathbb{R}^2 \setminus \{0\}$. It is easy to see that, for any functions $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, $v \in C^\infty(\mathbb{R}^2 \setminus \{0\})$, the following Green formula is valid:

$$(\mathcal{P}(D_y)u, v)_{L_2(\mathbb{R}^2)} = (u, \mathcal{P}^*(D_y)v)_{L_2(\mathbb{R}^2)}. \tag{5.8}$$

Formula (5.8) generates a problem formally adjoint to problem (5.1):

$$\mathcal{P}^*(D_y)v = f(y) \quad (y \in \mathbb{R}^2 \setminus \{0\}). \tag{5.9}$$

---

(2) The possibility of expansion of a distribution $\psi \in W_{2\pi}^l(0, 2\pi)^*$ in the Fourier series by the functions $e_k(\omega) = e^{ik\omega}/\sqrt{2\pi}$ is justified by the following equalities: $\langle u, \psi \rangle = \Big\langle \sum_k (u, e_k)_{L_2(0,2\pi)} e_k, \psi \Big\rangle = \Big(u, \sum_k \psi_k e_k\Big)_{L_2(0,2\pi)}$, where $u \in W_{2\pi}^l(0, 2\pi)$ and $v_k = \langle e_k, v \rangle$.

Further, it is not hard to prove that, for any functions $\tilde{u} \in C^\infty_{2\pi}[0, 2\pi]$ and $\tilde{v} \in C^\infty_{2\pi}[0, 2\pi]$, the following Green formula with the parameter λ is valid:

$$(\tilde{\mathcal{P}}(\omega, D_\omega, \lambda)\tilde{u}, \tilde{v})_{L_2(0,2\pi)} = (\tilde{u}, \tilde{\mathcal{P}}^*(\omega, D_\omega, \lambda')\tilde{v})_{L_2(0,2\pi)}, \tag{5.10}$$

where $\lambda' = \bar{\lambda} - 2i(m-1)$.

Formula (5.10) generates the operator

$$\tilde{\mathcal{Q}}(\lambda) = \tilde{\mathcal{P}}^*(\omega, D_\omega, \lambda)\colon W^{l+2m}_{2\pi}(0, 2\pi) \to W^l_{2\pi}(0, 2\pi).$$

Using Green formula (5.10) and relations of the type (3.1), one can establish a connection between Jordan chains of the operators $\tilde{\mathcal{P}}^*(\lambda)$ and $\tilde{\mathcal{Q}}(\lambda)$.

Lemma 5.1. *The vectors $\psi^{(0)}, \dots, \psi^{(\varkappa-1)}$ form a Jordan chain of the operator $\tilde{\mathcal{P}}^*(\lambda)$ corresponding to an eigenvalue $\bar{\lambda}_0$ if and only if they form a Jordan chain of the operator $\tilde{\mathcal{Q}}(\lambda)$ corresponding to the eigenvalue $\bar{\lambda}_0 - 2i(m-1)$.*

Finally, using Lemma 5.1, we shall formulate the main result concerning a representation of the coefficients $c_n^{(k,j)}$ from asymptotic formula (5.2) in terms of the Green formula.

Theorem 5.4. *Let the conditions of Theorem* 5.1 *be fulfilled. Then the coefficients $c_n^{(k,j)}$ from* (5.2) *are calculated by the formula*

$$c_n^{(k,j)} = (f, iv_n^{(\varkappa_{j,n}-k-1,j)})_{L_2(\mathbb{R}^2)}. \tag{5.11}$$

Here $v_n^{(\nu,j)}$ is a power solution for homogeneous problem (5.9) *given by formula* (5.4)*; $\{\psi_n^{(0,j)}, \dots, \psi_n^{(\varkappa_{j,n}-1,j)}\colon j = 1, \dots, J_n\}$ is a canonical system of Jordan chains of the operator $\tilde{\mathcal{Q}}(\lambda)$ corresponding to the eigenvalue $\bar{\lambda}_n - 2i(m-1)$; the chains $\{\varphi_n^{(0,j)}, \dots, \varphi_n^{(\varkappa_{j,n}-1,j)}\colon j = 1, \dots, J_n\}$ (appearing in* (5.3)*) and $\{\psi_n^{(0,j)}, \dots, \psi_n^{(\varkappa_{j,n}-1,j)}\colon j = 1, \dots, J_n\}$ satisfy the conditions of biorthogonality and normalization analogous to* (4.13).

4. In investigating asymptotics of solutions for nonlocal problems in bounded domains, we need a result on the asymptotics of solutions for adjoint local problems in $\mathbb{R}^2 \setminus \{0\}$ with a special right-hand side. We focus our attention on the distinct from the model problem in the angle, where we needed a result on the asymptotics of solutions for the original (but not adjoint) problem with a special right-hand side.

Let Λ be some complex number. If $\bar{\Lambda}$ is an eigenvalue of the operator $\tilde{\mathcal{P}}^*(\lambda)$, then we denote by $\varkappa(\bar{\Lambda})$ the greatest of partial multiplicities of this eigenvalue. Otherwise we put $\varkappa(\bar{\Lambda}) = 0$.

Lemma 5.2. *For problem* (5.9) *with right-hand side $\Psi = r^{i\bar{\Lambda}-2} \sum\limits_{q=0}^{M} \frac{1}{q!}(i \ln r)^q \Psi^{(q)}$, $\Psi^{(q)} \in W^{l+2m}_{2\pi}(0, 2\pi)^*$, there exists a solution*

$$v = r^{i\bar{\Lambda}+2m-2} \sum_{q=0}^{M+\varkappa(\bar{\Lambda})} \frac{1}{q!}(i \ln r)^q v^{(q)}, \tag{5.12}$$

where $v^{(q)} \in W^l_{2\pi}(0, 2\pi)^*$. A solution of such a form is unique if $\varkappa(\bar{\Lambda}) = 0$ (that is, if $\bar{\Lambda}$ is not an eigenvalue of $\tilde{\mathcal{P}}^*(\lambda)$). If $\varkappa(\bar{\Lambda}) > 0$, then solution (5.12) *is defined accurate to an arbitrary linear combination of power solutions* (5.4) *corresponding to the eigenvalue $\bar{\Lambda}$.*

Proof. The idea of the proof is analogous to that of the proof of Lemma 3.1 of [14, Chap. 3]. To complete the picture we shall give a plan of the proof. One should substitute formula (5.12) of the solution into the equation

$$\mathcal{P}^* v = r^{i\bar{\Lambda}-2} \sum_{q=0}^{M} \frac{1}{q!}(i \ln r)^q \Psi^{(q)},$$

reduce the factor $r^{i\bar{\Lambda}-2}$, and gather the coefficients of the same powers of $i \ln r$. As a result, one obtains a system of $M + \varkappa(\bar{\Lambda})$ equations, from which one finds unknown $v^{(q)}$. The statement that a solution of the form (5.12) is unique (for $\varkappa(\bar{\Lambda}) = 0$) or defined accurate to an arbitrary linear combination of power solutions (5.4) corresponding to the eigenvalue $\bar{\Lambda}$ (for $\varkappa(\bar{\Lambda}) > 0$) follows from the result analogous to Lemma 1.3 [14, Chap. 3], which restricts the freedom in choosing power solutions for the equation $\mathcal{P}^* v = 0$. □

A SMOOTHNESS OF SOLUTIONS TO NONLOCAL PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

In this Appendix, we establish two auxiliary lemmas concerning the smoothness of the above-mentioned problems. These lemmas are necessary to prove the smoothness of eigenvectors and associated vectors of nonlocal elliptic problems.

Let $\tilde{\mathcal{P}}(\lambda)$, $\tilde{B}_{\sigma\mu}(\lambda)$, $\tilde{B}^{\mathcal{G}}_{\sigma\mu}(\lambda)$, and $\tilde{\mathcal{B}}_{\sigma\mu}(\lambda)$ be the differential operators defined in Sect. 2.

Consider the operator

$$\tilde{\mathcal{L}}_{(l)}(\lambda) = \{\tilde{\mathcal{P}}(\lambda), \tilde{\mathcal{B}}_{\sigma\mu}(\lambda)\}\colon W^{l+2m}(b_1, b_2) \to W^l[b_1, b_2] = W^l(b_1, b_2) \times \mathbb{C}^{2m}.$$

We study the smoothness of solutions for the nonlocal problem

$$\tilde{\mathcal{L}}_{(l)}(\lambda)u = \{f, g_{\sigma\mu}\}. \tag{A.1}$$

Lemma A.1. *Let $u \in W^{l+2m}(b_1, b_2)$ be a solution for problem* (A.1) *with right-hand side $\{f, g_{\sigma\mu}\} \in W^{l+k}(b_1, b_2)$. Then $u \in W^{l+2m+k}(b_1, b_2)$.*

Proof. The function $u(\omega)$ is a solution of the problem

$$\tilde{\mathcal{P}}(\lambda)u(\omega) = f(\omega) \quad (\omega \in (b_1, b_2)),$$
$$\tilde{B}_{\sigma\mu}(\lambda)u(\omega)\big|_{\omega=b_\sigma} = g_{\sigma\mu} - \beta_\sigma^{-m_{\sigma\mu}+i\lambda}\tilde{B}^{\mathcal{G}}_{\sigma\mu}(\lambda)u(\omega+\omega_\sigma, \lambda)\big|_{\omega=b_\sigma}, \quad \sigma = 1, 2; \quad \mu = 1, \dots, m,$$

Therefore, applying Theorem 5.1 [16, Chap. 2], we obtain $u \in W^{l+2m+k}(b_1, b_2)$. □

Consider the operator $\tilde{\mathcal{L}}^*_{(l)}(\lambda)\colon W^l[b_1, b_2]^* \to W^{l+2m}(b_1, b_2)^*$, adjoint to the operator $\tilde{\mathcal{L}}_{(l)}(\bar{\lambda})$ with regard to extension of the inner product in $L_2(b_1, b_2) \times \mathbb{C}^{2m}$ (see Sect. 3).

We shall investigate the smoothness of solutions for the adjoint nonlocal problem

$$\tilde{\mathcal{L}}^*_{(l)}(\lambda)\{v, w_{\sigma\mu}\} = \Psi. \tag{A.2}$$

Lemma A.2. *Let $\{v, w_{\sigma\mu}\} \in W^l[b_1, b_2]^*$ be a solution of problem* (A.2) *with right-hand side*

$$\Psi \in \begin{cases} W^{2m-k}(b_1, b_2)^*, & \text{if } 0 < k < 2m, \\ W^{-2m+k}(b_1, b) \oplus W^{-2m+k}(b, b_2), & \text{if } k \geqslant 2m. \end{cases}$$

Then $v \in W^k(b_1, b) \oplus W^k(b, b_2)$.

Proof.

(1) First, let us have $l = 0$. Denote $\tilde{\mathcal{L}}(\lambda) = \tilde{\mathcal{L}}_{(0)}(\lambda)$ and $\tilde{\mathcal{L}}^*(\lambda) = \tilde{\mathcal{L}}^*_{(0)}(\lambda)$.

Introduce the auxiliary operator $\tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)\colon L_2(b_1, b_2) \times \mathbb{C}^{2m} \times \mathbb{C}^{2m} \to W^{2m}(b_1, b_2)^*$, taking $\{v, w_{\sigma\mu}, w'_{\sigma\mu}\}$ to $\tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)\{v, w_{\sigma\mu}, w'_{\sigma\mu}\}$ by the rule

$$\langle u, \tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)\{v, w_{\sigma\mu}, w'_{\sigma\mu}\}\rangle = (\tilde{\mathcal{P}}(\lambda)u, v)_{L_2(b_1, b_2)} + \sum_{\sigma=1,2}\sum_{\mu=1}^{m} \tilde{B}_{\sigma\mu}(\lambda)u\big|_{\omega=b_\sigma} \cdot \overline{w_{\sigma\mu}}$$
$$+ \sum_{\sigma=1,2}\sum_{\mu=1}^{m} \beta_\sigma^{-m_{\sigma\mu}+i\lambda}\tilde{B}^{\mathcal{G}}_{\sigma\mu}(\lambda)u\big|_{\omega=b} \cdot \overline{w'_{\sigma\mu}} \quad \text{for all} \quad u \in W^{2m}(b_1, b_2).$$

Clearly,

$$\tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)\{v, w_{\sigma\mu}, w_{\sigma\mu}\} = \tilde{\mathcal{L}}^*(\lambda)\{v, w_{\sigma\mu}\}.$$

Introduce infinitely differentiable functions $\zeta_\sigma(\omega)$ $(\sigma = 1, 2)$, $\zeta(\omega)$,

$$\zeta_\sigma(\omega) = 1 \text{ for } |b_\sigma - \omega| < \frac{|b_\sigma - b|}{4}, \quad \zeta_\sigma(\omega) = 0 \text{ for } |b_\sigma - \omega| > \frac{|b_\sigma - b|}{2}; \quad \zeta(\omega) = 1 - \zeta_1(\omega) - \zeta_2(\omega).$$

(2) Consider the expression $\tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)(\zeta_1\{v, w_{\sigma\mu}, w_{\sigma\mu}\})$. Then we have

$$\langle u, \tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)(\zeta_1\{v, w_{\sigma\mu}, w_{\sigma\mu}\})\rangle = (\tilde{\mathcal{P}}(\lambda)u, \zeta_1 v)_{L_2(b_1,b_2)} + \sum_{\mu=1}^{m} \tilde{B}_{1\mu}(\lambda)u\big|_{\omega=b_1} \cdot \overline{w_{1\mu}} \quad \text{for all} \quad u \in W^{2m}(b_1, b_2).$$

Moreover, from Leibniz's formula, it follows that $\tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)(\zeta_1\{v, w_{\sigma\mu}, w_{\sigma\mu}\}) \in W^{2m-1}(b_1, b_2)^*$, since

$$\zeta_1\tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)\{v, w_{\sigma\mu}, w_{\sigma\mu}\} = \zeta_1\tilde{\mathcal{L}}^*(\lambda)\{v, w_{\sigma\mu}\} \in W^{2m-1}(b_1, b_2)^*$$

and $v \in L_2(b_1, b_2)$. Therefore we can use Theorem 5.1 [16, Chap. 2], which yields $\zeta_1 v \in W^1(b_1, b_2)$.

Similarly, we get $\zeta_2 v \in W^1(b_1, b_2)$.

(3) Consider the expression $\tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)(\zeta\{v, w_{\sigma\mu}, w_{\sigma\mu}\})$. Then we have

$$\langle u, \tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)(\zeta\{v, w_{\sigma\mu}, w_{\sigma\mu}\})\rangle = (\tilde{\mathcal{P}}(\lambda)u, \zeta v)_{L_2(-\infty,b)} \quad \text{for all} \quad u \in C_0^\infty(-\infty, b),$$

where $v(\omega)$ is extended by zero for $\omega \leqslant b_1$. Analogously to the above, we have $\tilde{\mathcal{L}}^*_{\mathcal{G}}(\lambda)(\zeta\{v, w_{\sigma\mu}, w_{\sigma\mu}\}) \in W^{-2m+1}(-\infty, b)$.(3) From this, the ellipticity of the operator $\tilde{\mathcal{P}}(\lambda)$, and the relation $v \in L_2(-\infty, b)$, it follows that the generalized derivative $\frac{d^{2m}(\zeta v)}{d\omega^{2m}}$ belongs to the space $W^{-2m+1}(-\infty, b)$. Therefore, by Lemma 12.3 [16, Chap. 1], we have $\zeta v \in W^1(-\infty, b)$. Similarly, one can prove that $\zeta v \in W^1(b, +\infty)$. Combining this with item (2) of the proof, we obtain $v \in W^1(b_1, b) \oplus W^1(b, b_2)$.

Repeating the described procedure, after a finite number of steps we shall get $v \in W^k(b_1, b) \oplus W^k(b, b_2)$.

(4) Finally, consider the case of an arbitrary $l \geqslant 0$. From Lemma A.1, it follows that

$$\mathcal{R}(\tilde{\mathcal{L}}_{(l)}(\lambda)) = \mathcal{R}(\tilde{\mathcal{L}}_{(0)}(\lambda)) \cap W^l[b_1, b_2]. \tag{A.3}$$

Moreover, by Lemma 2.1 of [6], $\mathcal{R}(\tilde{\mathcal{L}}_{(l)}(\lambda))$ is closed and $\operatorname{codim} \mathcal{R}(\tilde{\mathcal{L}}_{(l)}(\lambda))$ is finite. From this and from (A.3), it follows that the embedding $W^l[b_1, b_2]$ into $W^0[b_1, b_2]$ induces the isomorphism between the coset spaces $W^l[b_1, b_2]/\mathcal{R}(\tilde{\mathcal{L}}_{(l)}(\lambda))$ and $W^0[b_1, b_2]/\mathcal{R}(\tilde{\mathcal{L}}_{(0)}(\lambda))$.

Thus, we have $\operatorname{codim} \mathcal{R}(\tilde{\mathcal{L}}_{(l)}(\lambda)) = \operatorname{codim} \mathcal{R}(\tilde{\mathcal{L}}_{(0)}(\lambda))$, and hence $\dim \ker(\tilde{\mathcal{L}}^*_{(l)}(\lambda)) = \dim \ker(\tilde{\mathcal{L}}^*_{(0)}(\lambda))$. From this and from the obvious embedding $\ker(\tilde{\mathcal{L}}^*_{(0)}(\lambda)) \subset \ker(\tilde{\mathcal{L}}^*_{(l)}(\lambda))$, we obtain $\ker(\tilde{\mathcal{L}}^*_{(l)}(\lambda)) = \ker(\tilde{\mathcal{L}}^*_{(0)}(\lambda))$.

Further, since $\Psi \in \mathcal{R}(\tilde{\mathcal{L}}^*_{(l)}(\lambda))$, we have

$$\langle u, \Psi\rangle = 0 \quad \text{for all} \quad u \in \ker(\tilde{\mathcal{L}}_{(l)}(\lambda)).$$

But from Lemma A.1, it follows that $\ker(\tilde{\mathcal{L}}_{(l)}(\lambda)) = \ker(\tilde{\mathcal{L}}_{(0)}(\lambda))$. Therefore,

$$\langle u, \Psi\rangle = 0 \quad \text{for all} \quad u \in \ker(\tilde{\mathcal{L}}_{(0)}(\lambda)).$$

Hence, we have $\Psi \in \mathcal{R}(\tilde{\mathcal{L}}^*_{(0)}(\lambda))$ since $\Psi \in W^{2m}(b_1, b_2)^*$ by assumption. Let $\{f, g_{\sigma\mu}\} \in W^0[b_1, b_2]^* = W^0[b_1, b_2]$ be some solution of the problem $\tilde{\mathcal{L}}^*_{(0)}(\lambda)\{f, g_{\sigma\mu}\} = \Psi$. By what has been proved, we have $f \in W^k(b_1, b) \oplus W^k(b, b_2)$.

Clearly, $\{f, g_{\sigma\mu}\}$ is also a solution of the problem $\tilde{\mathcal{L}}^*_{(l)}(\lambda)\{f, g_{\sigma\mu}\} = \Psi$; therefore,

$$\{v, w_{\sigma\mu}\} - \{f,\ g_{\sigma\mu}\} \in \ker(\tilde{\mathcal{L}}^*_{(l)}(\lambda)) = \ker(\tilde{\mathcal{L}}^*_{(0)}(\lambda)).$$

Hence, v also belongs to $W^k(b_1, b) \oplus W^k(b, b_2)$. □

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(3) $W^{-s}(-\infty, b)$, $s \geqslant 0$, is the space adjoint to $\mathring{W}^s(-\infty, b)$, where $\mathring{W}^s(-\infty, b)$ is a completion of the set $C_0^\infty(-\infty, b)$ in the norm $\|u\| = \Big(\sum\limits_{j=0}^{s} \int\limits_{-\infty}^{b} \big|\frac{d^j u}{d\omega^j}\big|^2\, d\omega\Big)^{1/2}$.

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