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Solvability of Nonlocal Elliptic Problems in Sobolev Spaces

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Abstract. We study elliptic equations of order $2m$ with nonlocal boundary-value conditions in plane angles and in bounded domains, dealing with the case in which the support of nonlocal terms intersects the boundary. We establish necessary and sufficient conditions under which nonlocal problems are Fredholm in Sobolev spaces and in weighted spaces with small weight exponents, respectively. We also obtain an asymptotics of solutions of nonlocal problems near the conjugation points on the boundary, where solutions can have power singularities.

CONTENTS

INTRODUCTION

Nonlocal problems have been studied since the beginning of the 20th century, but only during the last two decades have these problems been investigated thoroughly. On one hand, this can be explained by significant theoretical achievements in that direction and, on the other hand, by various applications arising in diverse areas such as biophysics, theory of multidimensional diffusion processes [1], plasma theory [2], theory of sandwich shells and plates [3], and so on.

In the one-dimensional case, nonlocal problems were studied by Sommerfeld [4], Tamarkin [5], Picone [6], etc. In the two-dimensional case, one of the first works was due to Carleman [7] and treated the problem of finding a harmonic function, in a plane bounded domain, satisfying the following nonlocal condition on the boundary Υ:

$$u(y) + bu\big(\Omega(y)\big) = g(y), \qquad y \in \Upsilon,$$

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where $\Omega\colon \Upsilon \to \Upsilon$ stands for a transformation on the boundary such that $\Omega\big(\Omega(y)\big) \equiv y,\ y \in \Upsilon$. This setting of the problem originated further investigation of nonlocal problems with transformations mapping a boundary onto itself.

In 1969, Bitsadze and Samarskii [8] considered a nonlocal problem (of essentially different kind) arising in plasma theory: to find a function $u(y_1, y_2)$ harmonic on the rectangle

$$G = \{y \in \mathbb{R}^2 : -1 < y_1 < 1,\ 0 < y_2 < 1\},$$

continuous on $\bar{G}$, and satisfying the relations

$$u(y_1, 0) = f_1(y_1), \quad u(y_1, 1) = f_2(y_1), \quad -1 < y_1 < 1,$$
$$u(-1, y_2) = f_3(y_2), \quad u(1, y_2) = u(0, y_2), \quad 0 < y_2 < 1,$$

where f_1, f_2, f_3 are given continuous functions. This problem was solved in [8] by reducing it to a Fredholm integral equation and using the maximum principle. For arbitrary domains and general nonlocal conditions, such a problem was formulated as an unsolved one. Different generalizations of nonlocal problems with transformations mapping the boundary inside the closure of a domain were studied by Eidelman and Zhitarashu [9], Roitberg and Sheftel' [10], Kishkis [11], Gushchin and Mikhailov [12], etc.

The most complete theory for elliptic equations of order $2m$ with general nonlocal conditions in multidimensional domains was developed by Skubachevskii and his disciples, see [13–20]: a classification with respect to types of nonlocal conditions was suggested, the Fredholm solvability in the corresponding spaces was investigated, index properties were studied, and the asymptotics of solutions near special conjugation points was obtained. It turns out that the most difficult situation occurs if the support of nonlocal terms intersects the boundary. In that case, the generalized solutions of nonlocal problems can have power singularities near some points even if the boundary and the right-hand sides are infinitely smooth [14,19]. For this reason, to investigate such problems, weighted spaces (introduced by Kondrat'ev for boundary-value problems in nonsmooth domains [21]) are naturally applied.

In the present paper, we study nonlocal elliptic problems in plane domains in Sobolev spaces $W^l(G) = W_2^l(G)$ (with no weight), dealing with the case in which the support of nonlocal terms can intersect the boundary. Let us consider the following example. Denote by $G \subset \mathbb{R}^2$ a bounded domain with the boundary $\partial G = \Upsilon_1 \cup \Upsilon_2 \cup \{g_1, g_2\}$, where Υ_i are open sets (in the topology of ∂G) given by C^∞-curves and g_1 and g_2 are the endpoints of the curves $\bar{\Upsilon}_1$ and $\bar{\Upsilon}_2$. Let the domain G coincide with plane angles in some neighborhoods of g_1 and g_2. We consider the following nonlocal problem in G:

$$\Delta u = f_0(y) \quad (y \in G), \tag{0.1}$$
$$u|_{\Upsilon_i} - b_i u\big(\Omega_i(y)\big)\big|_{\Upsilon_i} = f_i(y) \quad (y \in \Upsilon_i;\ i = 1, 2). \tag{0.2}$$

Here $b_1, b_2 \in \mathbb{R}$; Ω_i is an infinitely differentiable nondegenerate transformation mapping some neighborhood $\mathcal{O}_i$ of the curve Υ_i onto $\Omega(\mathcal{O}_i)$ in such a way that $\Omega_i(\Upsilon_i) \subset G$ and $\overline{\omega_i(\Upsilon_i)} \cap \partial G \neq \varnothing$ (see Fig. 0.1). We seek a solution $u \in W^{l+2}(G)$ under the assumption that $f_0 \in W^l(G)$, $f_i \in W^{l+3/2}(\Upsilon_i)$.

In this work, we obtain necessary and sufficient conditions under which a problem of type (0.1), (0.2) is Fredholm. It is shown that the solvability of such a problem is influenced by (I) spectral properties of model nonlocal problems with parameter and (II) the validity of some algebraic relations between the differential operator and nonlocal boundary-value operators at the points of conjugation of nonlocal conditions (points g_1 and g_2 at Fig. 0.1). We consider nonlocal problems for both nonhomogeneous and homogeneous boundary-value conditions, which turn out to be not equivalent with respect to Fredholm solvability. Near the conjugation points, the asymptotics of solutions is obtained.

We note that nonlocal problems in Sobolev spaces for the case in which the support of nonlocal terms does not intersect the boundary was thoroughly investigated by Skubachevskii [13, 17].

However, elliptic equations of order $2m$ with general nonlocal conditions for the case in which the support of nonlocal terms intersects the boundary is studied in Sobolev spaces for the first time.

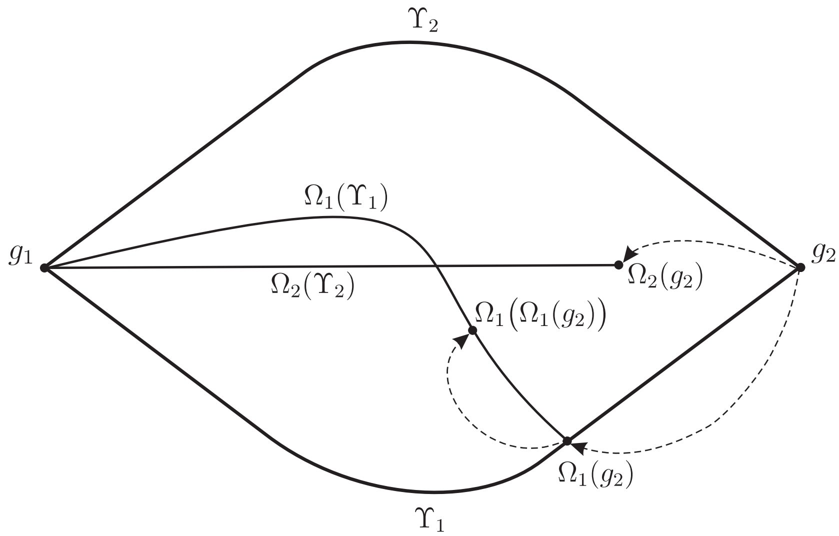

Fig. 0.1: Domain G with the boundary $\partial G = \bar{\Upsilon}_1 \cup \bar{\Upsilon}_2$.

The paper is organized as follows. The setting of the problem is presented in Sec. 1. In the same section, we define model problems in plane angles and problems with a parameter that correspond to the points of conjugation of nonlocal conditions. Properties of the original problem crucially depend on whether or not some line of the form

$$\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda = \Lambda\} \tag{0.3}$$

(where $\Lambda \in \mathbb{R}$ is defined by the order of differential equation and the order of the corresponding Sobolev spaces) contains eigenvalues of model problems with parameter. In Sec. 2 we study nonlocal problems in plane angles for the case in which the line (0.3) contains no eigenvalues, and in Sec. 3 we deal with the case in which this line contains a single proper eigenvalue (see Definition 3.1). We use the results of Sec. 2 in Sec. 4 to investigate the Fredholm solvability of the original problem in a bounded domain, and in Sec. 5 to obtain an asymptotics of solutions of nonlocal problems near the conjugation points.

In [14, 16, 18], the authors consider nonlocal problems in weighted spaces $H_a^l(G)$ with the norm

$$\|u\|_{H_a^k(G)} = \left(\sum_{|\alpha| \leqslant k} \int_G \rho^{2(a-k+|\alpha|)} |D^\alpha u|^2 \right)^{1/2}.$$

Here $k \geqslant 0$ is an integer, $a \in \mathbb{R}$, and $\rho = \rho(y)$ is the distance between the point y and the set of conjugation points. For problem (0.1), (0.2), we have $\rho(y) = \operatorname{dist}(y, \{g_1, g_2\})$. In [16, 18] it is proved that, if

$$f_0 \in H_a^l(G), \qquad f_i \in H_a^{l+3/2}(\Upsilon_i), \qquad a > l+1,$$

and the function $\{f_0, f_i\}$ satisfies finitely many orthogonality conditions, then problem (0.1), (0.2) admits a solution $u \in H_a^{l+2}(G)$. If $a \leqslant l+1$, then the following difficulty arises: generally, the relation $u \in H_a^{l+2}(G)$ does not imply that $u\big(\Omega_i(y)\big)\big|_{\Upsilon_i} \in H_a^{l+3/2}(\Upsilon_i)$. To avoid this difficulty, one can introduce the spaces (for problem (0.1), (0.2)) with the weight function

$$\hat{\rho}(y) = \operatorname{dist}\big(y, \{g_1, g_2, \Omega_1(g_2), \Omega_1\big(\Omega_1(g_2)\big), \Omega_2(g_2)\}\big)$$

and an arbitrary $a \in \mathbb{R}$ and prove the Fredholm solvability of nonlocal problems in these spaces (see [14]). However, the presence of the weight function $\hat\rho(y)$ means that we impose a restriction both on the right-hand side and on the solution not only near the conjugation points g_1 and g_2 but also near the point $\Omega_1(g_2)$ lying on a smooth part of the boundary and near the points $\Omega_1\big(\Omega_1(g_2)\big)$ and $\Omega_2(g_2)$ lying inside the domain (see Fig. 0.1).

In Sec. 6, we prove that the following assertion holds despite the fact that, for $a \leqslant l+1$, the relation $u \in H_a^{l+2}(G)$ does not imply the relation $u\big(\Omega_i(y)\big)\big|_{\Upsilon_i} \in H_a^{l+3/2}(\Upsilon_i)$. If $a>0$, $f_0 \in H_a^l(G)$, $f_i \in H_a^{l+3/2}(\Upsilon_i)$, and $\{f_0, f_i\}$ satisfies finitely many orthogonality conditions, then problem (0.1), (0.2) still admits a solution $u \in H_a^{l+2}(G)$. In this case, as above, the line (0.3) (with Λ depending now on the exponent a as well) must contain no eigenvalues of model problems with parameter.

In Sec. 7, using the results of Sec. 3, we study nonlocal problems in bounded domains for the special case in which the line (0.3) contains only a proper eigenvalue of model problems with parameter. In this case, to ensure the existence of solutions, we impose additional consistency conditions on the right-hand side at the conjugation points. The first part of the paper contains Secs. 1–3.

Let us also describe the contents of the second part of the paper, which will be published in the next issue of the Journal. The most complicated considerations in Sections 4, 6, and 7 are related to constructing right regularizers for nonlocal problems in bounded domains. In all these sections, to construct a regularizer, we use one and the same scheme described in detail in Sec. 4. This allows us to dwell only on the most important points in out treatments in Sections 6 and 7.

Finally, in Sec. 8, by using the results of Sections 4 and 7, we obtain a criteria for the Fredholm solvability of elliptic problems with homogeneous nonlocal conditions. Here algebraic relations between the differential operator and nonlocal boundary-value operators play an essential role. Two examples illustrating the results of this paper are given in Sec. 9.

1. SETTING OF NONLOCAL PROBLEMS IN BOUNDED DOMAINS

1.1. *Setting of Nonlocal Problem*

Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary ∂G. We introduce a set $\mathcal{K} \subset \partial G$ consisting of finitely many points and assume that

$$\partial G \setminus \mathcal{K} = \bigcup_{i=1}^{N_0} \Upsilon_i,$$

where Υ_i are open C^∞-curves (in the topology of ∂G). We assume that the domain G coincides with some plane angle in some neighborhood of each of the points $g \in \mathcal{K}$.

Denote by $\mathbf{P}(y, D_y)$ and $B_{i\mu s}(y, D_y)$ differential operators of orders $2m$ and $m_{i\mu}$, respectively, with complex-valued C^∞-coefficients ($i = 1, \dots, N_0$; $\mu = 1, \dots, m$; $s = 0, \dots, S_i$). Throughout the paper, we assume that the operator $\mathbf{P}(y, D_y)$ is properly elliptic for all $y \in \bar G$ and the system of operators $\{B_{i\mu 0}(y, D_y)\}_{\mu=1}^m$ covers $\mathbf{P}(y, D_y)$ for all $i = 1, \dots, N_0$ and $y \in \bar\Upsilon_i$ (see, e.g., [22, Ch. 2, § 1]).

For an integer $k \geqslant 0$, denote by $W^k(G) = W_2^k(G)$ the Sobolev space with the norm

$$\|u\|_{W^k(G)} = \left(\sum_{|\alpha| \leqslant k} \int_G |D^\alpha u|^2 \, dy \right)^{1/2}$$

(we set $W^0(G) = L_2(G)$ for $k=0$). For an integer $k \geqslant 1$, we introduce the space $W^{k-1/2}(\Upsilon)$ of traces on a smooth curve $\Upsilon \subset \bar G$ with the norm

$$\|\psi\|_{W^{k-1/2}(\Upsilon)} = \inf \|u\|_{W^k(G)} \quad (u \in W^k(G) : u|_\Upsilon = \psi). \tag{1.1}$$

Consider the operators

$$\mathbf{P} : W^{l+2m}(G) \to W^l(G), \qquad \mathbf{B}_{i\mu}^0 : W^{l+2m}(G) \to W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$$

given by $\mathbf{P}u = \mathbf{P}(y, D_y)u$ and $\mathbf{B}^0_{i\mu}u = B_{i\mu 0}(y, D_y)u(y)|_{\Upsilon_i}$. From now on, we always assume that $l + 2m - m_{i\mu} \geqslant 1$. The operators $\mathbf{P}$ and $\mathbf{B}^0_{i\mu}$ will *correspond to a "local" boundary-value problem.*

Now we proceed by defining the operators corresponding to nonlocal conditions near the set $\mathcal{K}$. Let Ω_{is} ($i = 1, \dots, N_0$; $s = 1, \dots, S_i$) be an infinitely differentiable nondegenerate transformation mapping some neighborhood $\mathcal{O}_i$ of the curve $\overline{\Upsilon_i \cap \mathcal{O}_{2\varepsilon_0}(\mathcal{K})}$ onto the set $\Omega_{is}(\mathcal{O}_i)$ in such a way that $\Omega_{is}(\Upsilon_i) \subset G$ and

$$\Omega_{is}(g) \in \mathcal{K} \quad \text{for } \in \bar{\Upsilon}_i \cap \mathcal{K}, \tag{1.2}$$

Where $\varepsilon_0 > 0$ and $\mathcal{O}_{2\varepsilon_0}(\mathcal{K}) = \{y \in \mathbb{R}^2 : \operatorname{dist}(y, \mathcal{K}) < 2\varepsilon_0\}$ is the $2\varepsilon_0$-neighborhood of the set $\mathcal{K}$. Thus, under the transformations Ω_{is}, the curves Υ_i are mapped strictly inside the domain G, whereas the set of endpoints of Υ_i is mapped to itself.

Let ε_0 be taken so small (see Remark 1.2 below) that, in the $2\varepsilon_0$-neighborhood $\mathcal{O}_{2\varepsilon_0}(g)$ of each point $g \in \mathcal{K}$, the domain G coincides with a plane angle. Let us specify the structure of the transformation Ω_{is} near the set $\mathcal{K}$.

Denote by the symbol Ω^{+1}_{is} the transformation $\Omega_{is}\colon \mathcal{O}_i \to \Omega_{is}(\mathcal{O}_i)$ and by Ω^{-1}_{is} the transformation $\Omega^{-1}_{is}\colon \Omega_{is}(\mathcal{O}_i) \to \mathcal{O}_i$ inverse to Ω_{is}. The set of all points

$$\Omega^{\pm 1}_{i_q s_q}(\dots \Omega^{\pm 1}_{i_1 s_1}(g)) \in \mathcal{K} \qquad (1 \leqslant s_j \leqslant S_{i_j}, \quad j = 1, \dots, q),$$

i.e., points which can be obtained by consecutively applying the transformations $\Omega^{+1}_{i_j s_j}$ or $\Omega^{-1}_{i_j s_j}$ (taking the points of $\mathcal{K}$ to $\mathcal{K}$) to the point g) is called an *orbit* of $g \in \mathcal{K}$ and is denoted by $\operatorname{Orb}(g)$.

Clearly, for any $g, g' \in \mathcal{K}$, either $\operatorname{Orb}(g) = \operatorname{Orb}(g')$ or $\operatorname{Orb}(g) \cap \operatorname{Orb}(g') = \varnothing$. Thus, we have

$$\mathcal{K} = \bigcup_{p=1}^{N_1} \operatorname{Orb}_p, \quad \text{where} \quad \operatorname{Orb}_{p_1} \cap \operatorname{Orb}_{p_2} = \varnothing \quad (p_1 \neq p_2),$$

and, for each $p = 1, \dots, N_1$, the set Orb_p coincides with an orbit of some point $g \in \mathcal{K}$. Let each orbit Orb_p consist of the points g^p_j, $j = 1, \dots, N_{1p}$.

For every point $g \in \mathcal{K}$, consider neighborhoods

$$\hat{\mathcal{V}}(g) \supset \mathcal{V}(g) \supset \mathcal{O}_{2\varepsilon_0}(g) \tag{1.3}$$

such that

(1) the boundary ∂G coincides with a plane angle in the neighborhood $\hat{\mathcal{V}}(g)$;
(2) $\overline{\hat{\mathcal{V}}(g)} \cap \overline{\hat{\mathcal{V}}(g')} = \varnothing$ for any $g, g' \in \mathcal{K}$, $g \neq g'$;
(3) if $g^p_j \in \bar{\Upsilon}_i \cap \operatorname{Orb}_p$ and $\Omega_{is}(g^p_j) = g^p_k$, then $\mathcal{V}(g^p_j) \subset \mathcal{O}_i$ and $\Omega_{is}\big(\mathcal{V}(g^p_j)\big) \subset \hat{\mathcal{V}}(g^p_k)$.

For each $g^p_j \in \bar{\Upsilon}_i \cap \operatorname{Orb}_p$, we fix the transformation $y \mapsto y'(g^p_j)$ of the argument; this transformation is the composition of the shift by the vector $-\overrightarrow{Og^p_j}$ and a rotation by some angle such that the set $\mathcal{V}(g^p_j)$ ($\hat{\mathcal{V}}(g^p_j)$) maps onto a neighborhood $\mathcal{V}^p_j(0)$ ($\hat{\mathcal{V}}^p_j(0)$) of the origin, whereas the sets

$$G \cap \mathcal{V}(g^p_j) \ (G \cap \hat{\mathcal{V}}(g^p_j)) \qquad \text{and} \qquad \Upsilon_i \cap \mathcal{V}(g^p_j) \ (\Upsilon_i \cap \hat{\mathcal{V}}(g^p_j))$$

are taken to the intersection of the plane angle

$$K^p_j = \{y \in \mathbb{R}^2 : \ r > 0, \ |\omega| < b^p_j < \pi\}$$

with $\mathcal{V}^p_j(0)$ ($\hat{\mathcal{V}}^p_j(0)$) and to the intersection of a side of the angle K^p_j with $\mathcal{V}^p_j(0)$ ($\hat{\mathcal{V}}^p_j(0)$), respectively.

Condition 1.1. *The above change of variable $y \mapsto y'(g)$ for $y \in \mathcal{V}(g)$, $g \in \mathcal{K} \cap \bar{\Upsilon}_i$, reduces the transformation $\Omega_{is}(y)$ $(i = 1, \dots N_0,\ s = 1, \dots, S_i)$ to the composition of a rotation and a homothety in the new variables y'.*

Remark 1.1. In particular, Condition 1.1 combined with the assumption $\Omega_{is}(\Upsilon_i) \subset G$ means that *if $g \in \Omega_{is}(\bar{\Upsilon}_i \setminus \Upsilon_i) \cap \bar{\Upsilon}_j \cap \mathcal{K} \neq \varnothing$, then the curves $\Omega_{is}(\bar{\Upsilon}_i)$ and $\bar{\Upsilon}_j$ are not tangent to each other at the point g.*

We introduce the bounded operators $\mathbf{B}^1_{i\mu}\colon W^{l+2m}(G) \to W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$ by the formula

$$\mathbf{B}^1_{i\mu}u = \sum_{s=1}^{S_i} \bigl(B_{i\mu s}(y, D_y)(\zeta u)\bigr)\bigl(\Omega_{is}(y)\bigr)\big|_{\Upsilon_i},$$

where $\bigl(B_{i\mu s}(y, D_y)v\bigr)\bigl(\Omega_{is}(y)\bigr) = B_{i\mu s}(y', D_{y'})v(y')|_{y'=\Omega_{is}(y)}$ and the function $\zeta \in C^\infty(\mathbb{R}^2)$ satisfies

$$\zeta(y) = 1\ (y \in \mathcal{O}_{\varepsilon_0/2}(\mathcal{K})), \quad \zeta(y) = 0 \qquad (y \notin \mathcal{O}_{\varepsilon_0}(\mathcal{K})). \tag{1.4}$$

Since $\mathbf{B}^1_{i\mu}u = 0$ whenever $\operatorname{supp} u \subset \bar{G} \setminus \overline{\mathcal{O}_{\varepsilon_0}(\mathcal{K})}$, we say that the operator $\mathbf{B}^1_{i\mu}$ *corresponds to nonlocal terms with support near the set $\mathcal{K}$.*

We also introduce a bounded operator $\mathbf{B}^2_{i\mu}\colon W^{l+2m}(G) \to W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$ satisfying the following condition.

Condition 1.2. *There exist numbers $\varkappa_1 > \varkappa_2 > 0$ and $\rho > 0$ such that the inequalities*

$$\|\mathbf{B}^2_{i\mu}u\|_{W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)} \leqslant c_1\|u\|_{W^{l+2m}(G\setminus\overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})}, \tag{1.5}$$

$$\|\mathbf{B}^2_{i\mu}u\|_{W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i\setminus\overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})})} \leqslant c_2\|u\|_{W^{l+2m}(G_\rho)}, \tag{1.6}$$

hold for any

$$u \in W^{l+2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})}) \cup W^{l+2m}(G_\rho),$$

where $i = 1, \dots, N_0$, $\mu = 1, \dots, m$, $c_1, c_2 > 0$, and $G_\rho = \{y \in G : \operatorname{dist}(y, \partial G) > \rho\}$.

It follows from (1.5) that $\mathbf{B}^2_{i\mu}u = 0$ whenever $\operatorname{supp} u \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$. For this reason, we say that the operator $\mathbf{B}^2_{i\mu}$ *corresponds to nonlocal terms supported outside the set $\mathcal{K}$.*

We will suppose throughout that Conditions 1.1 and 1.2 are satisfied.

Note that we *a priori* assume no connection between the numbers $\varkappa_1, \varkappa_2, \rho$ in Condition 1.2 and the number ε_0 in Condition 1.1.

We study the following nonlocal elliptic problem:

$$\mathbf{P}u = f_0(y) \quad (y \in G), \tag{1.7}$$

$$\mathbf{B}^0_{i\mu}u + \mathbf{B}^1_{i\mu}u + \mathbf{B}^2_{i\mu}u = f_{i\mu}(y) \quad (y \in \Upsilon_i;\ i = 1, \dots, N_0;\ \mu = 1, \dots, m). \tag{1.8}$$

Let us introduce the following operator corresponding to problem (1.7), (1.8):

$$\mathbf{L} = \{\mathbf{P},\ \mathbf{B}^0_{i\mu} + \mathbf{B}^1_{i\mu} + \mathbf{B}^2_{i\mu}\} : W^{l+2m}(G) \to \mathcal{W}^l(G, \Upsilon),$$

where

$$\mathcal{W}^l(G, \Upsilon) = W^l(G) \times \prod_{i=1}^{N_0}\prod_{\mu=1}^{m} W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i).$$

Remark 1.2. In what follows, we need the assumption that ε_0 is sufficiently small (whereas $\varkappa_1, \varkappa_2, \rho$ can be arbitrary). Let us show that this leads to no loss of generality.

Let us take a number $\hat{\varepsilon}_0$ such that $0 < \hat{\varepsilon}_0 < \varepsilon_0$. We consider a function $\hat{\zeta} \in C^\infty(\mathbb{R}^2)$ for which

$$\hat{\zeta}(y) = 1 \ (y \in \mathcal{O}_{\hat{\varepsilon}_0/2}(\mathcal{K})), \quad \hat{\zeta}(y) = 0 \ (y \notin \mathcal{O}_{\hat{\varepsilon}_0}(\mathcal{K}))$$

and introduce an operator $\hat{\mathbf{B}}^1_{i\mu} \colon W^{l+2m}(G) \to W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$ by the formula

$$\hat{\mathbf{B}}^1_{i\mu} u = \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)(\hat{\zeta} u)\big)\big(\Omega_{is}(y)\big)\big|_{\Upsilon_i}.$$

Clearly,

$$\mathbf{B}^0_{i\mu} + \mathbf{B}^1_{i\mu} + \mathbf{B}^2_{i\mu} = \mathbf{B}^0_{i\mu} + \hat{\mathbf{B}}^1_{i\mu} + \hat{\mathbf{B}}^2_{i\mu}, \quad \text{where} \quad \hat{\mathbf{B}}^2_{i\mu} = \mathbf{B}^1_{i\mu} - \hat{\mathbf{B}}^1_{i\mu} + \mathbf{B}^2_{i\mu}.$$

It follows from Example 1.1 (see Sec. 1.2) that the operator $\mathbf{B}^1_{i\mu} - \hat{\mathbf{B}}^1_{i\mu}$ satisfies Condition 1.2 for some $\varkappa_1, \varkappa_2, \rho$. Therefore, we can always choose ε_0 to be as small as necessary (possibly at the expense of a modification of the operator $\mathbf{B}^2_{i\mu}$ and the values of $\varkappa_1, \varkappa_2, \rho$).

1.2. *Example of Nonlocal Problem*

In the following example we give a concrete realization for the abstract nonlocal operators $\mathbf{B}^2_{i\mu}$.

Example 1.1. Let the operators $\mathbf{P}(y, D_y)$ and $B_{i\mu s}(y, D_y)$ be as above. Let Ω_{is} $(i = 1, \dots, N_0,$ $s = 1, \dots, S_i)$ be an infinitely differentiable nondegenerate transformation mapping some neighborhood $\mathcal{O}_i$ of the curve Υ_i onto $\Omega_{is}(\mathcal{O}_i)$ in such a way that $\Omega_{is}(\Upsilon_i) \subset G$. Note that it is not assumed in this example that *condition* (1.2) *holds for any* Ω_{is}.

Consider the following nonlocal problem:

$$\mathbf{P}(y, D_y)u = f_0(y) \quad (y \in G), \tag{1.9}$$

$$B_{i\mu 0}(y, D_y)u(y)|_{\Upsilon_i} + \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)u\big)\big(\Omega_{is}(y)\big)\big|_{\Upsilon_i} = f_{i\mu}(y) \quad (y \in \Upsilon_i;\ i = 1, \dots, N_0;\ \mu = 1, \dots, m). \tag{1.10}$$

We take a small number $\varepsilon_0 > 0$ in such a way that, for any point $g \in \mathcal{K}$, the set $\overline{\mathcal{O}_{\varepsilon_0}(g)}$ intersects the curve $\overline{\Omega_{is}(\Upsilon_i)}$ only for $g \in \mathcal{K} \cap \overline{\Omega_{is}(\Upsilon_i)}$.

Let a point $g \in \mathcal{K} \cap \bar{\Upsilon}_i$ be such that $\Omega_{is}(g) \in \mathcal{K}$. Then we define the orbit $\operatorname{Orb}(g)$ of the point g as above and assume that Condition 1.1 holds for each point of the orbit $\operatorname{Orb}(g)$.

Remark 1.3. According to Remark 1.1, Condition 1.1 is a restriction upon the geometric structure of the support of nonlocal terms near the set $\mathcal{K}$. However, if $\Omega_{is}(\bar{\Upsilon}_i \setminus \Upsilon_i) \subset \partial G \setminus \mathcal{K}$, then we impose no restrictions upon the geometric structure of the curve $\Omega_{is}(\bar{\Upsilon}_i)$ near ∂G (cf. [14, 16]).

We set

$$\mathbf{P}u = \mathbf{P}(y, D_y)u, \qquad \mathbf{B}^0_{i\mu} u = B_{i\mu 0}(y, D_y)u(y)|_{\Upsilon_i},$$

$$\mathbf{B}^1_{i\mu} u = \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)(\zeta u)\big)\big(\Omega_{is}(y)\big)\big|_{\Upsilon_i}, \qquad \mathbf{B}^2_{i\mu} u = \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)((1-\zeta) u)\big)\big(\Omega_{is}(y)\big)\big|_{\Upsilon_i},$$

where ζ is defined by (1.4) (see Figures 1.1 and 1.2). Then problem (1.9), (1.10) acquires the form (1.7), (1.8).

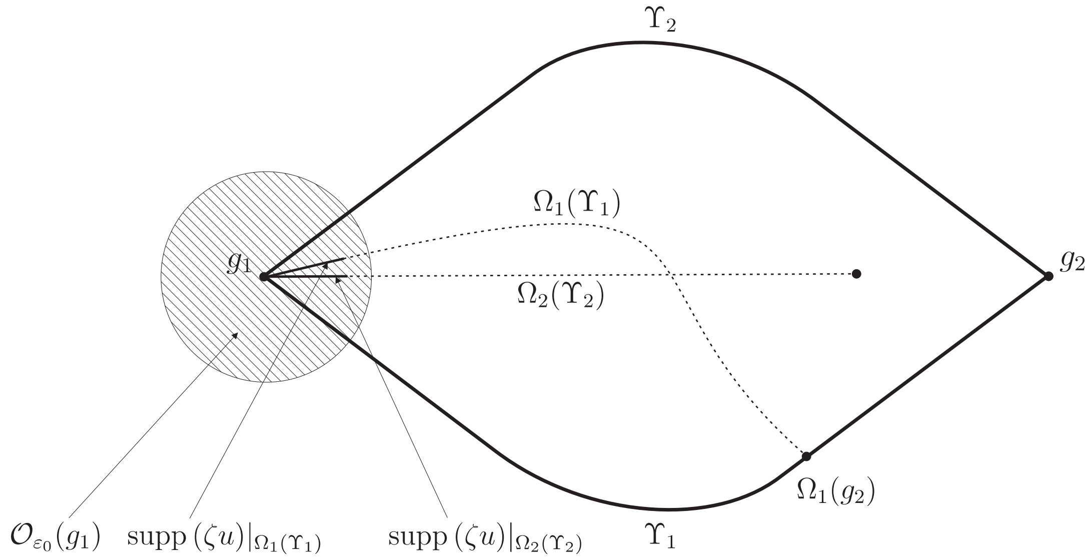

Fig. 1.1: Dotted lines denote the support of nonlocal terms corresponding to the operator $\mathbf{B}^2_{i\mu}$.

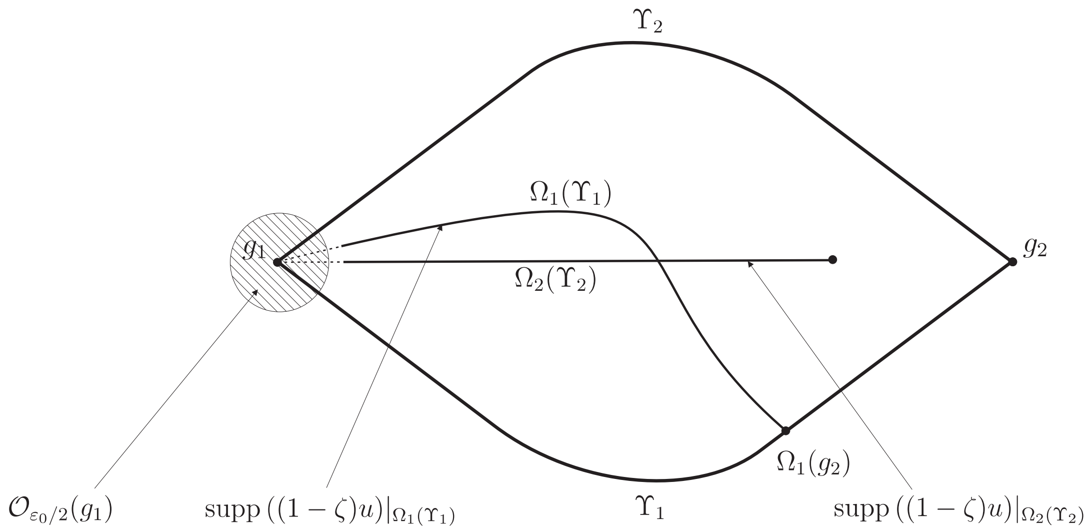

Fig. 1.1: Dotted lines denote the support of nonlocal terms corresponding to the operator $\mathbf{B}^2_{i\mu}$.

As in the proof of Lemma 2.5 [16] (where the weighted spaces must be replaced by the corresponding Sobolev spaces), one can show that the operator $\mathbf{B}^2_{i\mu}$ satisfies Condition 1.2. For example, let us prove inequality (1.5). Clearly, it suffices to consider an arbitrary term of the form $\psi = \big(B_{i\mu s}(y, D_y)((1-\zeta)u)\big)\big(\Omega_{is}(y)\big)\big|_{\Upsilon_i}$. We introduce a function $v \in C_0^\infty\big(\Omega_{is}(\mathcal{O}_i)\big)$ such that

$$v|_{\Omega_{is}(\Upsilon_i)} = \big(B_{i\mu s}(y, D_y)((1-\zeta)u)\big)\big|_{\Omega_{is}(\Upsilon_i)}, \tag{1.11}$$

$$\|v\|_{W^{l+2m-m_{i\mu}}(\Omega_{is}(\mathcal{O}_i))} \leqslant 2\big\|\big(B_{i\mu s}(y, D_y)((1-\zeta)u)\big)\big|_{\Omega_{is}(\Upsilon_i)}\big\|_{W^{l+2m-m_{i\mu}-1/2}(\Omega_{is}(\Upsilon_i))}, \tag{1.12}$$

It follows from (1.11) that

$$v\big(\Omega_{is}(y)\big)\big|_{\Upsilon_i} = \psi.$$

Combining this condition with the boundedness of the trace operator in Sobolev spaces and with inequality (1.12), we obtain

$$\begin{aligned}\|\psi\|_{W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)} &= \big\|v\big(\Omega_{is}(y)\big)\big|_{\Upsilon_i}\big\|_{W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)} \leqslant \big\|v\big(\Omega_{is}(y)\big)\big\|_{W^{l+2m-m_{i\mu}}(\mathcal{O}_i)} \\ &\leqslant k_1\|v\|_{W^{l+2m-m_{i\mu}}(\Omega_{is}(\mathcal{O}_i))} \leqslant 2k_1\|B_{i\mu s}(y, D_y)((1-\zeta)u)|_{\Upsilon_i}\|_{W^{l+2m-m_{i\mu}-1/2}(\Omega_{is}(\Upsilon_i))} \\ &\leqslant k_2\|(1-\zeta)u\|_{W^{l+2m}(G)}. \end{aligned} \tag{1.13}$$

Thus, by setting $\varkappa_1 = \varepsilon_0/2$, we see that relation (1.13) implies the estimate (1.5). Note that, in this case, the numbers $\varkappa_1$ and ε_0 turn out to be related to each other.

Similar considerations enable one to obtain the estimate (1.6). The proof is based on the boundedness of the trace operator, on the smoothness of the transformations Ω_{is}, and on the relation

$$\Omega_{is}(\Upsilon_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})}) \subset G_\rho$$

(which holds for any $\varkappa_2 < \varkappa_1$ and for a sufficiently small $\rho = \rho(\varkappa_2)$). The last relation follows from the embedding $\Omega_{is}(\Upsilon_i) \subset G$ and from the continuity of Ω_{is}.

1.3. *Nonlocal Problems Near the Set $\mathcal{K}$*

When studying problem (1.7), (1.8), one must pay special attention to the behavior of solutions in a neighborhood of the set $\mathcal{K}$ which consists of conjugation points. Let us consider the corresponding model problems in plane angles. To this end, we formally assume that

$$\mathbf{B}^2_{i\mu} = 0, \quad i = 1, \dots, N_0, \ \mu = 1, \dots, m. \tag{1.14}$$

Let us fix some orbit $\mathrm{Orb}_p \subset \mathcal{K}$ $(p = 1, \dots, N_1)$ and suppose that

$$\operatorname{supp} u \subset \Big(\bigcup_{j=1}^{N_{1p}} \mathcal{V}(g_j^p) \Big) \cap \bar{G}.$$

We denote by $u_j(y)$ the function $u(y)$ for $y \in \hat{\mathcal{V}}(g_j^p) \cap G$. If

$$g_j^p \in \bar{\Upsilon}_i, \qquad y \in \mathcal{V}(g_j^p), \qquad \text{and} \qquad \Omega_{is}(y) \in \hat{\mathcal{V}}(g_k^p),$$

denote $u(\Omega_{is}(y))$ by $u_k(\Omega_{is}(y))$. Then, by virtue of assumption (1.14), the nonlocal problem (1.7), (1.8) becomes

$$\mathbf{P}(y, D_y)u_j = f_0(y) \quad (y \in \mathcal{V}(g_j^p) \cap G),$$

$$B_{i\mu 0}(y, D_y)u_j(y)|_{\mathcal{V}(g_j^p)\cap\Upsilon_i} + \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)(\zeta u_k)\big)\big(\Omega_{is}(y)\big)\big|_{\mathcal{V}(g_j^p)\cap\Upsilon_i} = f_{i\mu}(y)$$

$$\big(y \in \mathcal{V}(g_j^p) \cap \Upsilon_i;\ i \in \{1 \leqslant i \leqslant N_0 : g_j^p \in \bar{\Upsilon}_i\};\ j = 1, \dots, N_{1p};\ \mu = 1, \dots, m\big).$$

Let $y \mapsto y'(g_j^p)$ be the above change of variable. Introduce the function $U_j(y') = u_j(y(y'))$ and denote y' by y again. For a chosen p, we set $N = N_{1p}$, $b_j = b_j^p$, $K_j = K_j^p$ (see Sec. 1.1), and

$$\gamma_{j\sigma} = \{y \in \mathbb{R}^2 : r > 0,\ \omega = (-1)^\sigma b_j\} \qquad (\sigma = 1, 2),$$

where (ω, r) are polar coordinates with the pole at the origin. Now, using Condition 1.1, we can represent problem (1.7), (1.8) as follows:

$$\mathbf{P}_j(y, D_y)U_j = f_j(y) \quad (y \in K_j), \tag{1.15}$$

$$\mathbf{B}_{j\sigma\mu}(y, D_y)U|_{\gamma_{j\sigma}} \equiv \sum_{k,s} (B_{j\sigma\mu ks}(y, D_y)U_k)(\mathcal{G}_{j\sigma ks}y)|_{\gamma_{j\sigma}} = f_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma}). \tag{1.16}$$

Here (and below unless otherwise stated)

$$j, k = 1, \dots, N = N_{1p}; \qquad \sigma = 1, 2; \qquad \mu = 1, \dots, m; \qquad s = 0, \dots, S_{j\sigma k};$$

$\mathbf{P}_j(y, D_y)$ and $B_{j\sigma\mu ks}(y, D_y)$ are operators of orders $2m$ and $m_{j\sigma\mu}$, respectively, with variable C^∞-coefficients; $\mathcal{G}_{j\sigma ks}$ is the operator of rotation by an angle of $\omega_{j\sigma ks}$ and of the homothety with the coefficient $\chi_{j\sigma ks}$ $(\chi_{j\sigma ks} > 0)$ in the y-plane. Moreover,

$$|(-1)^\sigma b_j + \omega_{j\sigma ks}| < b_k \qquad \text{for} \qquad (j, 0) \neq (k, s), \qquad \omega_{j\sigma j0} = 0, \qquad \text{and} \qquad \chi_{j\sigma j0} = 1.$$

Since $\mathcal{V}(0) \supset \mathcal{O}_{\varepsilon_0}(0)$ (see. (1.3)), it follows that

$$B_{j\sigma\mu ks}(y, D_y)v(y) = 0 \quad \text{for} \quad |y| \geqslant \varepsilon_0, \quad (k, s) \neq (j, 0), \tag{1.17}$$

for any function v (which need not be compactly supported). Moreover, since we consider problem (1.15), (1.16) for functions U with compact support, we can assume that the coefficients of the operators $\mathbf{P}_j(y, D_y)$ and $B_{j\sigma\mu j0}(y, D_y)$ vanish outside a disk of sufficiently large radius.

Let us introduce the following spaces of vector functions:

$$W^{l+2m,N}(K) = \prod_{j=1}^{N} W^{l+2m}(K_j),\ \mathcal{W}^{l,N}(K,\gamma) = \prod_{j=1}^{N} \mathcal{W}^{l}(K_j,\gamma_j),$$

$$\mathcal{W}^{l}(K_j,\gamma_j) = W^{l}(K_j) \times \prod_{\sigma=1,2}\prod_{\mu=1}^{m} W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}).$$

We consider the operator $\mathbf{L}_p\colon W^{l+2m,N}(K) \to \mathcal{W}^{l,N}(K,\gamma)$ corresponding to problem (1.15), (1.16) and given by

$$\mathbf{L}_p U = \{\mathbf{P}_j(y, D_y)U_j,\ \mathbf{B}_{j\sigma\mu}(y, D_y)U|_{\gamma_{j\sigma}}\}.$$

The subscript p means that the operator $\mathbf{L}_p$ is related to the orbit Orb_p.

Denote by $\mathcal{P}_j(D_y)$ and $B_{j\sigma\mu ks}(D_y)$ the principal homogeneous parts of the operators $\mathbf{P}_j(0, D_y)$ and $B_{j\sigma\mu ks}(0, D_y)$, respectively. Along with problem (1.15), (1.16), we study the model nonlocal problem

$$\mathcal{P}_j(D_y)U_j = f_j(y) \quad (y \in K_j), \tag{1.18}$$

$$\mathcal{B}_{j\sigma\mu}(D_y)U|_{\gamma_{j\sigma}} \equiv \sum_{k,s}(B_{j\sigma\mu ks}(D_y)U_k)(\mathcal{G}_{j\sigma ks}y)|_{\gamma_{j\sigma}} = f_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma}). \tag{1.19}$$

Introduce the operator $\mathcal{L}_p\colon W^{l+2m,N}(K) \to \mathcal{W}^{l,N}(K,\gamma)$ corresponding to problem (1.18), (1.19) and given by

$$\mathcal{L}_p U = \{\mathcal{P}_j(D_y)U_j,\ \mathcal{B}_{j\sigma\mu}(D_y)U|_{\gamma_{j\sigma}}\}.$$

Let us represent the operators $\mathcal{P}_j(D_y)$ and $B_{j\sigma\mu ks}(D_y)$ in polar coordinates:

$$\mathcal{P}_j(D_y) = r^{-2m}\tilde{\mathcal{P}}_j(\omega, D_\omega, rD_r), \qquad B_{j\sigma\mu ks}(D_y) = r^{-m_{j\sigma\mu}}\tilde{B}_{j\sigma\mu ks}(\omega, D_\omega, rD_r).$$

Introduce the following spaces of vector functions:

$$W^{l+2m,N}(-b,b) = \prod_{j=1}^{N} W^{l+2m}(-b_j,b_j), \qquad \mathcal{W}^{l,N}[-b,b] = \prod_{j=1}^{N} \mathcal{W}^{l}[-b_j,b_j],$$

$$\mathcal{W}^{l}[-b_j,b_j] = W^{l}(-b_j,b_j) \times \mathbb{C}^{2m}$$

and consider the analytic operator-valued function $\tilde{\mathcal{L}}_p(\lambda)\colon W^{l+2m,N}(-b,b) \to \mathcal{W}^{l,N}[-b,b]$ given by

$$\tilde{\mathcal{L}}_p(\lambda)\varphi = \{\tilde{\mathcal{P}}_j(\omega, D_\omega, \lambda)\varphi_j,\ \sum_{k,s}(\chi_{j\sigma ks})^{i\lambda-m_{j\sigma\mu}}\tilde{B}_{j\sigma\mu ks}(\omega, D_\omega, \lambda)\varphi_k(\omega+\omega_{j\sigma ks})|_{\omega=(-1)^\sigma b_j}\}.$$

Main definitions and facts concerning eigenvalues, eigenvectors, and associate vectors of analytic operator-valued functions can be found in [23]. In what follows, it is fundamental that the spectrum of the operator $\tilde{\mathcal{L}}_p(\lambda)$ is discrete (see Lemma 2.1 [15]).

Below we show that the Fredholm solvability of problem (1.7), (1.8) in Sobolev spaces depends on the location of eigenvalues of the model operators $\tilde{\mathcal{L}}_p(\lambda)$ corresponding to the points of $\mathcal{K}$. Note that the solvability of the same problem in weighted spaces depends on the location of eigenvalues of the model operators corresponding not only to the points of $\mathcal{K}$ but also on the validity of the conditions $\Omega_{is}(\mathcal{K}) \subset \bar{G}$ and $\Omega_{i's'}(\Omega_{is}(\mathcal{K}) \cap \Upsilon_{i'}) \subset G$ (see [14, 16]). This can be explained as follows: the points of the above sets are related by means of the transformations Ω_{is}. For this reason, the singularities of solutions occurring near the set $\mathcal{K}$ can be "transferred" to other points both on the boundary and strictly inside the domain. However, in the case under consideration, we shall prove below that, if the right-hand side of problem (1.7), (1.8) is subjected to finitely many orthogonality conditions in the Sobolev space $\mathcal{W}^l(G,\Upsilon)$, then the solutions belong to the Sobolev space $W^{l+2m}(G)$. Therefore, these solutions have no singularities.

2. NONLOCAL PROBLEMS IN PLANE ANGLES FOR THE CASE IN WHICH LINE $\operatorname{Im}\lambda = 1 - L - 2M$ CONTAINS NO EIGENVALUES OF $\tilde{\mathcal{L}}_P(\lambda)$

In this section we construct an operator acting on Sobolev spaces defined for compactly supported functions; this operator is the right inverse for the operator $\mathbf{L}_p$ up to a sum of small and compact perturbations. (Recall that $\mathbf{L}_p$ corresponds to the model problem (1.15), (1.16).)

2.1. *Weighted Spaces* $H_a^k(Q)$

Throughout the section, we suppose that the orbit Orb_p is fixed; therefore, for brevity, we denote the operators $\mathbf{L}_p$, $\mathcal{L}_p$, and $\tilde{\mathcal{L}}_p(\lambda)$ by $\mathfrak{L}$, $\mathcal{L}$, and $\tilde{\mathcal{L}}(\lambda)$, respectively.

In the investigation of the solvability of problem (1.15), (1.16) in Sobolev spaces we use the results on the solvability of problem (1.18), (1.19) in weighted spaces. Let us introduce these spaces and list their properties.

For any set $X \in \mathbb{R}^n$ $(n \geqslant 1)$, denote by $C_0^\infty(X)$ the set of infinitely differentiable functions on $\bar{X}$ which are compactly supported in X. Let

$$Q = K_j, \qquad Q = K_j \cap \{y \in \mathbb{R}^2 : \ |y| < d\} \qquad (d > 0), \qquad \text{or} \qquad Q = \mathbb{R}^2.$$

Denote by $H_a^k(Q)$ the completion of the set $C_0^\infty(\bar{Q} \setminus \{0\})$ with respect to the norm

$$\|w\|_{H_a^k(Q)} = \left(\sum_{|\alpha| \leqslant k} \int_Q r^{2(a-k+|\alpha|)} |D_y^\alpha w|^2 dy \right)^{1/2},$$

where $a \in \mathbb{R}$ and $k \geqslant 0$ is an integer. For $k \geqslant 1$, denote by $H_a^{k-1/2}(\gamma)$ the space of traces on a smooth curve $\gamma \subset \bar{Q}$ with the norm

$$\|\psi\|_{H_a^{k-1/2}(\gamma)} = \inf \|w\|_{H_a^k(Q)} \quad (w \in H_a^l(Q) : w|_\gamma = \psi).$$

We introduce the following spaces of vector functions:

$$H_a^{l+2m,N}(K) = \prod_{j=1}^N H_a^{l+2m}(K_j), \ \mathcal{H}_a^{l,N}(K,\gamma) = \prod_{j=1}^N \mathcal{H}_a^l(K_j,\gamma_j),$$
$$\mathcal{H}_a^l(K_j,\gamma_j) = H_a^l(K_j) \times \prod_{\sigma=1,2} \prod_{\mu=1}^m H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}).$$

The bounded operator $\mathcal{L}_a : H_a^{l+2m,N}(K) \to \mathcal{H}_a^{l,N}(K,\gamma)$ given by

$$\mathcal{L}_a U = \{\mathcal{P}_j(D_y)U_j, \ \mathcal{B}_{j\sigma\mu}(D_y)U|_{\gamma_{j\sigma}}\} \tag{2.1}$$

corresponds to problem (1.18), (1.19) in the weighted spaces. It follows from Theorem 2.1 [15] that the operator $\mathcal{L}_a$ has bounded inverse if and only if the line $\operatorname{Im}\lambda = 1-l-2m$ contains no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. In this section and in the next one, we study the solvability of problems (1.18), (1.19) and (1.15), (1.16) in Sobolev spaces by using the invertibility of $\mathcal{L}_a$. To this end, we need some auxiliary results (Lemmas 2.1 and 2.2) concerning the relation between the spaces $H_a^k(\cdot)$ and $W^k(\cdot)$.

Lemma 2.1. *Let $u \in W^k(Q)$ $(k \geqslant 2)$, $u(y) = 0$ for $|y| \geqslant 1$, and $D^\alpha u|_{y=0} = 0$ $(|\alpha| \leqslant k-2)$. Then we have*

$$\|u\|_{H_a^k(Q)} \leqslant c_a \|u\|_{W^k(Q)}, \qquad a > 0. \tag{2.2}$$

If we additionally assume that[1] *$D^{k-1}u \in H_0^1(Q)$, then*

$$\|u\|_{H_0^k(Q)} \leqslant c \sum_{|\alpha|=k-1} \|D^\alpha u\|_{H_0^1(Q)}. \tag{2.3}$$

Here Q is the same domain as above and $c_a > 0$ does not depend on u.

[1]If some assertion is stated for a function $D^l u$, then we mean that this assertion holds for all functions of the form $D^\alpha u$, $|\alpha| = l$.

Proof. It follows from Lemma 4.9 [21] that

$$\|D^{k-1}u\|_{H_a^1(Q)} \leqslant c\|D^{k-1}u\|_{W^1(Q)} \leqslant c\|u\|_{W^k(Q)}$$

for each $a > 0$. Combining this estimate (or the relation $D^{k-1}u \in H_0^1(Q)$) with Lemma 4.12 in [21][2] yields inequality (2.2) for $0 < a < 1$ (or inequality (2.3), respectively). Since the support of u is compact, it follows that inequality (2.2) holds for any $a > 0$.

Lemma 2.2. *Let $u \in W^1(\mathbb{R}^2)$, and let $u(y) = 0$ for $|y| \geqslant 1$. Then*

$$\|u(y) - u(\mathcal{G}_0 y)\|_{H_0^1(\mathbb{R}^2)} \leqslant c\|u\|_{W^1(\mathbb{R}^2)},$$

where $\mathcal{G}_0$ is the composition of a rotation by an angle of ω_0 ($-\pi < \omega_0 \leqslant \pi$) and an expansion with some coefficient χ_0 ($\chi_0 > 0$).

Proof. Writing a function u in the polar coordinates (ω, r) yields

$$u(y) - u(\mathcal{G}_0 y) = u(\omega, r) - u(\omega + \omega_0, \chi_0 r) = v_1 + v_2,$$

where $v_1(\omega, r) = u(\omega, r) - u(\omega + \omega_0, r)$, $v_2(\omega, r) = u(\omega + \omega_0, r) - u(\omega + \omega_0, \chi_0 r)$.

Let us consider the function v_1. By Lemma 4.15 [21], we obtain

$$\int_0^\infty r^{-1}|v_1(0, r)|^2 dr \leqslant k_1\|u\|_{W^1(\mathbb{R}^2)}.$$

It follows from this inequality and from Lemma 4.8 in [21] that $v_1 \in H_0^1(\mathbb{R}^2)$ and

$$\|v_1\|_{H_0^1(\mathbb{R}^2)} \leqslant k_2\|u\|_{W^1(\mathbb{R}^2)}. \tag{2.4}$$

To prove the lemma, it remains to show that

$$\int_{\mathbb{R}^2} r^{-2}|v_2|^2 dy \leqslant k_3\|u\|_{W^1(\mathbb{R}^2)}. \tag{2.5}$$

If $\chi_0 > 1$ (the case in which $0 < \chi_0 < 1$ can be treated similarly), then

$$\int_{\mathbb{R}^2} r^{-2}|v_2|^2 dy = \int_{-\pi}^{\pi} d\omega \int_0^\infty r^{-1}|v_2(\omega, r)|^2 dr = \int_{-\pi+\omega_0}^{\pi+\omega_0} d\omega \int_0^\infty r^{-1} dr \left|\int_r^{\chi_0 r} \frac{\partial u(\omega, t)}{\partial t} dt\right|^2.$$

Using first the Cauchy–Schwarz inequality and then changing the limits of integration, we obtain an estimate of the form (2.5), namely,

$$\int_{\mathbb{R}^2} r^{-2}|v_2|^2 dy \leqslant (\chi_0 - 1)\int_{-\pi+\omega_0}^{\pi+\omega_0} d\omega \int_0^\infty dr \int_r^{\chi_0 r} \left|\frac{\partial u(\omega, t)}{\partial t}\right|^2 dt$$
$$= \frac{(\chi_0 - 1)^2}{\chi_0}\int_{-\pi+\omega_0}^{\pi+\omega_0} d\omega \int_0^\infty \left|\frac{\partial u(\omega, t)}{\partial t}\right|^2 t\, dt \leqslant \frac{(\chi_0 - 1)^2}{\chi_0}\|u\|^2_{W^1(\mathbb{R}^2)}.$$

Let us prove another auxiliary result.

Lemma 2.3. *Let H, H_1, and H_2 be Hilbert spaces, $\mathfrak{A} : H \to H_1$ a linear bounded operator, and $\mathcal{T} : H \to H_2$ a compact operator. Suppose that, for some $\varepsilon > 0$, $c > 0$, and $f \in H$, the following inequality holds:*

$$\|\mathfrak{A}f\|_{H_1} \leqslant \varepsilon\|f\|_H + c\|\mathcal{T}f\|_{H_2}. \tag{2.6}$$

Then there exist operators $\mathcal{M}, \mathcal{F} : H \to H_1$ such that

$$\mathfrak{A} = \mathcal{M} + \mathcal{F},$$

$\|\mathcal{M}\| \leqslant 2\varepsilon$, and the operator $\mathcal{F}$ is finite-dimensional.

[2] Lemma 4.12 [21] was proved by Kondrat'ev for $a = 0$; however, his proof remains valid for any $a < 1$ with minor modifications.

Proof. As is well known (see, e.g., [24, Ch. 5, § 85]), any compact operator is the limit of a norm convergent sequence of finite-dimensional operators. Therefore, there exist bounded operators $\mathcal{M}_0, \mathcal{F}_0 : H \to H_2$ such that $\mathcal{T} = \mathcal{M}_0 + \mathcal{F}_0$, $\|\mathcal{M}_0\| \leqslant c^{-1}\varepsilon$, and the operator $\mathcal{F}_0$ is finite-dimensional. This, together with (2.6), implies that

$$\|\mathfrak{A}f\|_{H_1} \leqslant 2\varepsilon\|f\|_H + c\|\mathcal{F}_0 f\|_{H_2} \quad \text{for any} \quad f \in H. \tag{2.7}$$

Denote by $\ker(\mathcal{F}_0)^\perp$ the orthogonal complement in H to the kernel of the operator $\mathcal{F}_0$. Since the finite-dimensional operator $\mathcal{F}_0$ maps $\ker(\mathcal{F}_0)^\perp$ onto its image bijectively, it follows that the subspace $\ker(\mathcal{F}_0)^\perp$ is finite-dimensional. Let $\mathcal{I}$ denote the identity operator in H, and let $\mathcal{P}_0$ be the orthogonal projection to $\ker(\mathcal{F}_0)^\perp$. Clearly, $\mathfrak{A}\mathcal{P}_0 \colon H \to H_1$ is a finite-dimensional operator. Moreover, since $\mathcal{I} - \mathcal{P}_0$ is the orthogonal projection onto $\ker(\mathcal{F}_0)$, it follows that $\mathcal{F}_0(\mathcal{I} - \mathcal{P}_0) = 0$. Therefore, substituting the function $(\mathcal{I} - \mathcal{P}_0)f$ for f in (2.7), we obtain

$$\|\mathfrak{A}(\mathcal{I} - \mathcal{P}_0)f\|_{H_1} \leqslant 2\varepsilon\|(\mathcal{I} - \mathcal{P}_0)f\|_H \leqslant 2\varepsilon\|f\|_H \quad \text{for any} \quad f \in H.$$

Write $\mathcal{M} = \mathfrak{A}(\mathcal{I} - \mathcal{P}_0)$ and $\mathcal{F} = \mathfrak{A}\mathcal{P}_0$. This completes the proof.

2.2. *Construction of the Operator* $\mathfrak{R}$

In this subsection, we construct an operator $\mathfrak{R}$ acting on a subspace $\mathcal{S}^{l,N}(K,\gamma)$ of $W^{l,N}(K,\gamma)$ defined for compactly supported functions. This operator is a right inverse of the operator $\mathfrak{L}$ up to a sum of small and compact perturbations (see Theorem 2.1). To construct the operator $\mathfrak{R}$, we assume that the following condition holds.

Condition 2.1. *The line* $\operatorname{Im}\lambda = 1 - l - 2m$ *contains no eigenvalues of the operator* $\tilde{\mathcal{L}}(\lambda)$.

Denote by $\mathcal{S}^{l,N}(K,\gamma)$ the subspace of $\mathcal{W}^{l,N}(K,\gamma)$ consisting of the functions $\{f_j, f_{j\sigma\mu}\}$ such that

$$D^\alpha f_j|_{y=0} = 0, \qquad |\alpha| \leqslant l-2, \tag{2.8}$$

$$\left.\frac{\partial^\beta f_{j\sigma\mu}}{\partial\tau_{j\sigma}^\beta}\right|_{y=0} = 0, \qquad \beta \leqslant l + 2m - m_{j\sigma\mu} - 2, \tag{2.9}$$

where $\tau_{j\sigma}$ is the unit vector directed along the ray $\gamma_{j\sigma}$. If $l-2<0$ or $l+2m-m_{j\sigma\mu}-2<0$, then the corresponding conditions are absent. It follows from Sobolev's embedding theorem and from Riesz' theorem on the general form of a linear continuous functional on a Hilbert space that the set $\mathcal{S}^{l,N}(K,\gamma)$ is closed and of finite codimension in $\mathcal{W}^{l,N}(K,\gamma)$.

Let us consider the operators

$$\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\mathcal{B}_{j\sigma\mu}(D_y)U \equiv \frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\Big(\sum_{k,s}(B_{j\sigma\mu ks}(D_y)U_k)(\mathcal{G}_{j\sigma ks}y)\Big).$$

Using the chain rule, we can write

$$\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\mathcal{B}_{j\sigma\mu}(D_y)U \equiv \sum_{k,s}(\hat{B}_{j\sigma\mu ks}(D_y)U_k)(\mathcal{G}_{j\sigma ks}y), \tag{2.10}$$

where $\hat{B}_{j\sigma\mu ks}(D_y)$ are some homogenous differential operators of order $l+2m-1$ with constant coefficients. In particular, we have

$$\hat{B}_{j\sigma\mu j0}(D_y) = \frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}B_{j\sigma\mu j0}(D_y)$$

because $\mathcal{G}_{j\sigma j0}y \equiv y$. Formally replacing the nonlocal operators in (2.10) by the corresponding local ones, introduce the operators

$$\hat{\mathcal{B}}_{j\sigma\mu}(D_y)U \equiv \sum_{k,s}\hat{B}_{j\sigma\mu ks}(D_y)U_k(y). \tag{2.11}$$

Along with system (2.11), consider the operators (for $l \geqslant 1$)

$$D^\xi \mathcal{P}_j(D_y)U_j(y), \quad |\xi| = l-1. \tag{2.12}$$

The system of operators (2.11) and (2.12) plays an essential role in the proof of the following lemma, which is used below in the construction of the operator $\mathfrak{R}$.

Lemma 2.4. *Let Condition 2.1 hold. Then, for any ε, $0<\varepsilon<1$, there exists a bounded operator*

$$\mathcal{A} : \{f \in \mathcal{S}^{l,N}(K,\gamma) : \operatorname{supp} f \subset \mathcal{O}_\varepsilon(0)\} \to W^{l+2m,N}(K)$$

such that, for any $f = \{f_j, f_{j\sigma\mu}\} \in \operatorname{Dom}(\mathcal{A})$, the function $V = \mathcal{A}f$ satisfies the following conditions:

$$V = 0 \quad \text{for} \quad |y| \geqslant 1,$$
$$\|\mathcal{L}V - f\|_{\mathcal{H}_0^{l,N}(K)} \leqslant c\|f\|_{\mathcal{W}^{l,N}(K,\gamma)}, \tag{2.13}$$
$$\|V\|_{H_a^{l+2m,N}(K)} \leqslant c_a\|f\|_{\mathcal{W}^{l,N}(K,\gamma)} \quad \text{for any } a>0. \tag{2.14}$$

Proof. 1. Introduce the operator

$$f_{j\sigma\mu} \mapsto \Phi_{j\sigma\mu} \tag{2.15}$$

taking each function $f_{j\sigma\mu} \in W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})$ to its extension $\Phi_{j\sigma\mu} \in W^{l+2m-m_{j\sigma\mu}}(\mathbb{R}^2)$ to $\mathbb{R}^2$ such that $\Phi_{j\sigma\mu} = 0$ for $|y| \geqslant 2$. We also consider an extension of the function f_j from K_j to $\mathbb{R}^2$ such that the extended function (which we also denote by f_j) is equal to zero for $|y| \geqslant 2$. The corresponding extension operators can be chosen to be linear and bounded (see [25, Ch. 6, § 3]).

Let us consider the following linear algebraic system for the partial derivatives $D^\alpha W_j$, $|\alpha| = l+2m-1$, $j=1,\dots,N$:

$$\hat{\mathcal{B}}_{j\sigma\mu}(D_y)W = \frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\Phi_{j\sigma\mu}, \tag{2.16}$$
$$D^\xi \mathcal{P}_j(D_y)W_j = D^\xi f_j \tag{2.17}$$

($j=1,\dots,N$; $\sigma=1,2$; $\mu=1,\dots,m$; $|\xi|=l-1$). Recall that each of the operators $\hat{\mathcal{B}}_{j\sigma\mu}(D_y)$ given by (2.11) is the sum of "local" operators, which enables us to regard system (2.16), (2.17) as an algebraic system. Let us assume that system (2.16), (2.17) admits a unique solution for any right-hand side. Denote by $W_{j\alpha}$ the solution of system (2.16), (2.17). It is obvious that $W_{j\alpha} \in W^1(\mathbb{R}^2)$ and $W_{j\alpha} = 0$ for $|y| \geqslant 2$. By virtue of Lemma 4.17 [21], there exists a bounded linear operator

$$\{W_{j\alpha}\}_{|\alpha|=l+2m-1} \mapsto V_j \tag{2.18}$$

taking the system

$$\{W_{j\alpha}\}_{|\alpha|=l+2m-1} \in \prod_{|\alpha|=l+2m-1} W^1(\mathbb{R}^2)$$

to a function $V_j \in W^{l+2m}(\mathbb{R}^2)$ such that

$$V_j = 0 \quad \text{for} \quad |y| \geqslant 1,$$
$$D^\alpha V_j|_{y=0} = 0, \quad |\alpha| \leqslant l+2m-2, \tag{2.19}$$
$$D^\alpha V_j - W_{j\alpha} \in H_0^1(\mathbb{R}^2), \quad |\alpha| = l+2m-1. \tag{2.20}$$

2. Let us show that $V = (V_1,\dots,V_N)$ is the desired function. Inequality (2.14) follows from relations (2.19) and from Lemma 2.1 because the operator (2.18) is bounded.

Let us prove (2.13). Since the functions $W_{j\alpha}$ are solutions of the algebraic system (2.16), (2.17) and the functions V_j satisfy (2.20), it follows that

$$\hat{\mathcal{B}}_{j\sigma\mu}(D_y)V - \frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\Phi_{j\sigma\mu} \in H_0^1(\mathbb{R}^2), \tag{2.21}$$

$$D^{l-1}(\mathcal{P}_j(D_y)V_j - f_j) \in H_0^1(\mathbb{R}^2). \tag{2.22}$$

Moreover, by (2.19) and (2.8) we have

$$D^\alpha(\mathcal{P}_j(D_y)V_j - f_j)|_{y=0} = 0, \quad |\alpha| \leqslant l-2.$$

Combining this with relations (2.22) and Lemma 2.1, we see that $\mathcal{P}_j(D_y)V_j - f_j \in H_0^l(K_j)$.

Now let us show that

$$\mathcal{B}_{j\sigma\mu}(D_y)V|_{\gamma_{j\sigma}} - f_{j\sigma\mu} \in H_0^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}). \tag{2.23}$$

To do this, we return in (2.21) from the "local" operators $\hat{\mathcal{B}}_{j\sigma\mu}(D_y)$ to the nonlocal ones, i.e., to $\dfrac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\mathcal{B}_{j\sigma\mu}(D_y)$. Then, using Lemma 2.2, by (2.21) we see that

$$\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}(\mathcal{B}_{j\sigma\mu}(D_y)V - \Phi_{j\sigma\mu}) \in H_0^1(\mathbb{R}^2). \tag{2.24}$$

The inclusions (2.24) and Lemma 4.18 [21] imply

$$\int_0^\infty r^{-1}\left|\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}(\mathcal{B}_{j\sigma\mu}(D_y)V|_{\gamma_{j\sigma}} - f_{j\sigma\mu})\right|^2 dr \leqslant k_1\left\|\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}(\mathcal{B}_{j\sigma\mu}(D_y)V - \Phi_{j\sigma\mu})\right\|^2_{H_0^1(K_j)}. \tag{2.25}$$

It follows from inequality (2.25), from relations (2.8) and (2.19), and from Lemma 4.7 in [21] that

$$\int_0^\infty r^{1-2(l+2m-m_{j\sigma\mu})}|\mathcal{B}_{j\sigma\mu}(D_y)V|_{\gamma_{j\sigma}} - f_{j\sigma\mu}|^2 dr \leqslant k_2\left\|\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}(\mathcal{B}_{j\sigma\mu}(D_y)V - \Phi_{j\sigma\mu})\right\|^2_{H_0^1(K_j)}. \tag{2.26}$$

Combining this inequality with the relation $\mathcal{B}_{j\sigma\mu}(D_y)V|_{\gamma_{j\sigma}} - f_{j\sigma\mu} \in W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})$ and using (2.26) and Lemma 4.16 [21], we obtain (2.23). Using the boundedness of the operators (2.15) and (2.18), one can easily prove the estimate (2.13) as well.

3. Now it remains to show that system (2.16), (2.17) admits a unique solution for any right-hand side. Obviously, this system consists of $(l+2m)N$ equations for $(l+2m)N$ unknowns. Therefore, it suffices to show that the corresponding homogeneous system has a trivial solution only. Assume the contrary. Let there exist a nontrivial numerical vector $\{q_{j\alpha}\}$ ($j = 1, \dots, N$, $|\alpha| = l+2m-1$) such that the right-hand side of system (2.16), (2.17) vanishes after substituting the numbers $q_{j\alpha}$ for $D^\alpha W_j$ into the left-hand side of the system. Let us consider the homogeneous polynomial $Q_j(y)$ of order $l+2m-1$ such that $D^\alpha Q_j(y) \equiv q_{j\alpha}$. Then we have $\mathcal{P}_j(D_y)Q_j(y) \equiv 0$ (since $D^\xi\mathcal{P}_j(D_y)Q_j(y) \equiv 0$ for all $|\xi| = l-1$), and

$$\hat{\mathcal{B}}_{j\sigma\mu}(D_y)Q(y) \equiv \sum_{k,s}\hat{B}_{j\sigma\mu ks}(D_y)Q_k(y) \equiv 0 \quad \big(Q = (Q_1, \dots, Q_N)\big). \tag{2.27}$$

Note that $\hat{B}_{j\sigma\mu ks}(D_y)Q_k(y) \equiv \text{const}$, whereas every operator $\mathcal{G}_{j\sigma ks}$ of a rotation or an expansion takes a constant to itself. Therefore, along with (2.27), the following identity holds:

$$\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\Big(\mathcal{B}_{j\sigma\mu}(D_y)Q(y)\Big) \equiv \sum_{k,s}(\hat{B}_{j\sigma\mu ks}(D_y)Q_k)(\mathcal{G}_{j\sigma ks}y) \equiv 0. \tag{2.28}$$

Since $\mathcal{B}_{j\sigma\mu}(D_y)Q$ is a homogeneous polynomial of order $l+2m-m_{j\sigma\mu}-1$, it follows from (2.28) that $\mathcal{B}_{j\sigma\mu}(D_y)Q|_{\gamma_{j\sigma}} \equiv 0$. Thus, we see that the vector-valued function $Q = (Q_1, \dots, Q_N)$ is a solution of the homogeneous problem (1.18), (1.19). Therefore,

$$\begin{gathered}\tilde{\mathcal{P}}_j(\omega, D_\omega, rD_r)\big(r^{l+2m-1}\tilde{Q}_j(\omega)\big) \equiv 0,\\ \sum_{k,s}(\chi_{j\sigma ks})^{(l+2m-1)-m_{j\sigma\mu}}\tilde{B}_{j\sigma\mu ks}(\omega, D_\omega, rD_r)\big(r^{l+2m-1}\tilde{Q}_k(\omega+\omega_{j\sigma ks})\big)|_{\omega=(-1)^\sigma b_j} \equiv 0,\end{gathered} \tag{2.29}$$

where $Q_j(y) \equiv r^{l+2m-1}\tilde{Q}_j(\omega)$. However, identities (2.29) mean that $\tilde{\mathcal{L}}(-i(l+2m-1))\tilde{Q}(\omega) \equiv 0$, where $\tilde{Q} = (\tilde{Q}_1, \dots, \tilde{Q}_N)$. This contradicts the assumption that the line $\operatorname{Im}\lambda = 1-l-2m$ contains no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$.

Corollary 2.1. *The function V constructed in Lemma* 2.4 *satisfies the following inequality*:

$$\|\mathfrak{L}V - f\|_{\mathcal{H}_0^{l,N}(K)} \leqslant c\|f\|_{\mathcal{W}^{l,N}(K,\gamma)}. \tag{2.30}$$

Proof. By virtue of inequality (2.13), it suffices to estimate the differences

$$(\mathbf{P}_j(y, D_y) - \mathcal{P}_j(D_y))V_j, \qquad (\mathbf{B}_{j\sigma\mu}(y, D_y) - \mathcal{B}_{j\sigma\mu}(D_y))V|_{\gamma_{j\sigma}}.$$

The former contains terms of the form

$$\big(a_\alpha(y) - a_\alpha(0)\big)D^\alpha V_j \ (|\alpha| = 2m), \qquad a_\beta(y)D^\beta V_j \ (|\beta| \leqslant 2m-1),$$

where a_α and a_β are infinitely differentiable functions. Choosing some a, $0 < a < 1$, taking into account that $V = 0$ for $|y| \geqslant 1$, and using Lemma 3.3′ in [21] and inequality (2.14), we obtain

$$\begin{aligned}\|\big(a_\alpha(y) - a_\alpha(0)\big)D^\alpha V_j\|_{H_0^l(K_j)} &\leqslant k_1\|\big(a_\alpha(y) - a_\alpha(0)\big)D^\alpha V_j\|_{H_{a-1}^l(K_j)}\\ &\leqslant k_2\|D^\alpha V_j\|_{H_a^l(K_j)} \leqslant k_3\|f\|_{\mathcal{W}^{l,N}(K,\gamma)}.\end{aligned}$$

Similarly, it follows from the definition of weighted spaces and from inequality (2.14) that

$$\|a_\beta(y)D^\beta V_j\|_{H_0^l(K_j)} \leqslant k_4\|a_\beta(y)D^\beta V_j\|_{H_a^{l+1}(K_j)} \leqslant k_5\|V_j\|_{H_a^{l+2m}(K_j)} \leqslant k_6\|f\|_{\mathcal{W}^{l,N}(K,\gamma)}.$$

The expressions $(\mathbf{B}_{j\sigma\mu}(y, D_y) - \mathcal{B}_{j\sigma\mu}(D_y))V|_{\gamma_{j\sigma}}$ can be estimated in the same way.

Using Lemma 2.4, we can construct an operator $\mathfrak{R}$ with the desired properties.

Theorem 2.1. *Let Condition* 2.1 *hold. Then, for any ε, $0 < \varepsilon < 1$, there exist bounded operators*

$$\begin{gathered}\mathfrak{R} : \{f \in \mathcal{S}^{l,N}(K,\gamma) : \operatorname{supp} f \subset \mathcal{O}_\varepsilon(0)\} \to \{U \in W^{l+2m,N}(K) : \operatorname{supp} U \subset \mathcal{O}_{\varepsilon_1}(0)\},\\ \mathfrak{M}, \mathfrak{T} : \{f \in \mathcal{S}^{l,N}(K,\gamma) : \operatorname{supp} f \subset \mathcal{O}_\varepsilon(0)\} \to \{f \in \mathcal{S}^{l,N}(K,\gamma) : \operatorname{supp} f \subset \mathcal{O}_{2\varepsilon_1}(0)\}\end{gathered}$$

with[3] $\varepsilon_1 = \max\{\varepsilon, \varepsilon_0/\min\{\chi_{j\sigma ks}, 1\}\}$ *such that* $\|\mathfrak{M}f\|_{\mathcal{W}^{l,N}(K,\gamma)} \leqslant c\varepsilon_1 \|f\|_{\mathcal{W}^{l,N}(K,\gamma)}$, *where* $c > 0$ *depends only on the coefficients of the operators* $\mathcal{P}_j(D_y)$ *and* $B_{j\sigma\mu ks}(D_y)$. *Moreover, the operator* $\mathfrak{T}$ *is compact, and*

$$\mathfrak{L}\mathfrak{R}f = f + \mathfrak{M}f + \mathfrak{T}f. \tag{2.31}$$

Proof. By Lemma 2.4 we have $f - \mathcal{L}\mathcal{A}f \in \mathcal{H}_0^{l,N}(K,\gamma)$. Therefore,

$$\mathcal{L}_0^{-1}(f - \mathcal{L}\mathcal{A}f) \in H_0^{l+2m,N}(K),$$

where $\mathcal{L}_0 : H_0^{l+2m,N}(K) \to \mathcal{H}_0^{l,N}(K,\gamma)$ is the operator given by (2.1) for $a = 0$. Set

$$\mathfrak{R}f = \psi U, \quad U = \mathcal{L}_0^{-1}(f - \mathcal{L}\mathcal{A}f) + \mathcal{A}f.$$

Here $\psi \in C_0^\infty(\mathbb{R}^2)$ satisfies

$$\psi(y) = 1 \qquad \text{for} \qquad |y| \leqslant \varepsilon_1 = \max\{\varepsilon, \varepsilon_0/\min\{\chi_{j\sigma ks}, 1\}\}, \qquad \operatorname{supp}\psi \subset \mathcal{O}_{2\varepsilon_1}(0),$$

and ψ does not depend on the polar angle ω. Let us show that the operator $\mathfrak{R}$ has the desired properties. Since the embedding $H_0^{l+2m,N}(K) \subset W^{l+2m,N}(K)$ is continuous for the compactly supported functions and the operators $\mathcal{A}$ are bounded, we can use inequality (2.13) and obtain

$$\|\mathfrak{R}f\|_{W^{l+2m,N}(K)} \leqslant c\|f\|_{W^{l+2m,N}(K)}.$$

Let us prove relation (2.31). Since $\mathcal{P}_j(D_y)U_j = f_j$ and $\psi f_j = f_j$, it follows that

$$\mathbf{P}_j(y, D_y)(\psi U_j) - f_j = [\mathbf{P}_j(y, D_y), \psi]U_j + \psi(y)\big(\mathbf{P}_j(y, D_y) - \mathcal{P}_j(D_y)\big)U_j, \tag{2.32}$$

where $[\cdot,\cdot]$ stands for the commutator.

Let $b(y)$ be an arbitrary coefficient of the operator $B_{j\sigma\mu ks}(y, D_y)$ with $(k,s) \neq (j,0)$. By virtue of (1.17) and by the choice of the function ψ, we have

$$b(\mathcal{G}_{j\sigma ks}y) = 0 \quad \text{for} \quad |y| \geqslant \varepsilon_0/\chi_{j\sigma ks},$$
$$(D_y^\alpha\psi)(\mathcal{G}_{j\sigma ks}y) = D_y^\alpha\psi(y) \quad \text{for} \quad |y| \leqslant \varepsilon_0/\chi_{j\sigma ks}$$

(the last expression, for $|y| \leqslant \varepsilon_0/\chi_{j\sigma ks}$, is equal to 1 for $|\alpha| = 0$ and to 0 for $|\alpha| \geqslant 1$). Thus,

$$(bvD_y^\alpha\psi)(\mathcal{G}_{j\sigma ks}y) \equiv D_y^\alpha\psi(y)(bv)(\mathcal{G}_{j\sigma ks}y) \quad \text{for any} \quad v. \tag{2.33}$$

Obviously, if $(k,s) = (j,0)$, then identity (2.33) is also true. Therefore, taking into account the relations $\mathcal{B}_{j\sigma\mu}(D_y)U|_{\gamma_{j\sigma}} = f_{j\sigma\mu}$ and $\psi f_{j\sigma\mu} = f_{j\sigma\mu}$, we see that

$$\mathbf{B}_{j\sigma\mu}(y, D_y)(\psi U)|_{\gamma_{j\sigma}} - f_{j\sigma\mu} = [\mathbf{B}_{j\sigma\mu}(y, D_y), \psi]U|_{\gamma_{j\sigma}} + \psi(y)\big(\mathbf{B}_{j\sigma\mu}(y, D_y) - \mathcal{B}_{j\sigma\mu}(D_y)\big)U|_{\gamma_{j\sigma}}. \tag{2.34}$$

It follows from (2.32)–(2.34) and from Leibniz' formula that $\operatorname{supp}(\mathfrak{L}\mathfrak{R}f - f) \subset \mathcal{O}_{2\varepsilon_1}(0)$ and

$$\|\mathfrak{L}\mathfrak{R}f - f\|_{\mathcal{W}^{l,N}(K,\gamma)} \leqslant k_1\varepsilon_1\|f\|_{\mathcal{W}^{l,N}(K,\gamma)} + k_2(\varepsilon_1)\|\psi_1 U\|_{W^{l+2m-1,N}(K)}, \tag{2.35}$$

[3]Recall that the number ε_0 defines the diameter of the support for the function ζ occurring in the definition of the nonlocal operator $\mathbf{B}^1_{i\mu}$ (see Sec. 1). In other words, the number ε_0 defines the diameter for the support of the coefficients of the model operators $B_{j\sigma\mu ks}(y, D_y)$, $(k,s) \neq (j,0)$ (see (1.17)).

where $\psi_1 \in C_0^\infty(\mathbb{R}^2)$ is equal to 1 on the support of ψ. Note that the function $\mathcal{L}_0^{-1}(f - \mathcal{L}\mathcal{A}f)$ belongs to $H_0^{l+2m,N}(K)$, and therefore vanishes at $y = 0$ together with all its derivatives of order $\leqslant l + 2m - 2$. By virtue of Lemma 2.4 (in particular, see (2.19)), the function $\mathcal{A}f$ possesses the same property. Hence, $\mathfrak{L}\mathfrak{R}f - f \in \mathcal{S}^{l,N}(K,\gamma)$.

Moreover, by virtue of Lemma 2.4 and by the compactness of the embedding

$$\{\psi_1 U : U \in W^{l+2m,N}(K)\} \subset W^{l+2m-1,N}(K),$$

the operator

$$f \mapsto \psi_1 U$$

(see the second norm on the right-hand side of (2.35)) compactly maps

$$\{f \in \mathcal{S}^{l,N}(K,\gamma) : \operatorname{supp} f \subset \mathcal{O}_\varepsilon(0)\}$$

into $W^{l+2m-1,N}(K)$. Combining this with inequality (2.35) and Lemma 2.3, we complete the proof.

The operator $\mathfrak{R}$ has the "demerit" that the diameter of the support of $\mathfrak{R}f$ depends on ε_0 and cannot be reduced by reducing the diameter of the support of f. However, to construct a right regularizer for problem (1.7), (1.8) in the entire domain G, we need a modification $\mathfrak{R}'$ of $\mathfrak{R}$ which is free of this demerit. In the following theorem we construct such a modification $\mathfrak{R}'$ defined for the functions $f' = \{f_{j\sigma\mu}\}$.

Theorem 2.2. *Let Condition* 2.1 *hold. Then, for any* ε, $0 < \varepsilon < 1$, *there exist bounded operators*

$$\mathfrak{R}' : \{f' : \{0, f'\} \in \mathcal{S}^{l,N}(K,\gamma),\ \operatorname{supp} f' \subset \mathcal{O}_\varepsilon(0)\} \to \{U \in W^{l+2m,N}(K) : \operatorname{supp} U \subset \mathcal{O}_{2\varepsilon}(0)\},$$
$$\mathfrak{M}', \mathfrak{T}' : \{f' : \{0, f'\} \in \mathcal{S}^{l,N}(K,\gamma),\ \operatorname{supp} f' \subset \mathcal{O}_\varepsilon(0)\} \to \{f \in \mathcal{S}^{l,N}(K,\gamma) : \operatorname{supp} f \subset \mathcal{O}_{2\varepsilon_2}(0)\},$$

$\varepsilon_2 = \varepsilon / \min\{\chi_{j\sigma ks}, 1\}$, *such that* $\|\mathfrak{M}'f'\|_{\mathcal{W}^{l,N}(K,\gamma)} \leqslant c\varepsilon \|\{0, f'\}\|_{\mathcal{W}^{l,N}(K,\gamma)}$, *where the constant* $c > 0$ *depends only on the coefficients of the operators* $\mathcal{P}_j(D_y)$ *and* $B_{j\sigma\mu ks}(D_y)$, *the operator* $\mathfrak{T}'$ *is compact, and*

$$\mathfrak{L}\mathfrak{R}'f' = \{0, f'\} + \mathfrak{M}'f' + \mathfrak{T}'f'.$$

Proof. Write

$$\mathfrak{R}'f' = \psi U, \quad U = \mathcal{L}_0^{-1}\big(\{0, f'\} - \mathcal{L}\mathcal{A}\{0, f'\}\big) + \mathcal{A}\{0, f'\},$$

where $\psi \in C_0^\infty(\mathbb{R}^2)$ is such that $\psi(y) = 1$ for $|y| \leqslant \varepsilon$, $\operatorname{supp}\psi \subset \mathcal{O}_{2\varepsilon}(0)$, and ψ does not depend on the polar angle ω.

The rest of the proof coincides with that of Theorem 2.1 except for one item. Namely, identity (2.33) can fail for the case in question, and therefore, instead of (2.34), we will have

$$\begin{aligned}\mathbf{B}_{j\sigma\mu}(y, D_y)(\psi U)|_{\gamma_{j\sigma}} - f_{j\sigma\mu} &= [\mathbf{B}_{j\sigma\mu}(y, D_y), \psi]U|_{\gamma_{j\sigma}} + \psi(y)\big(\mathbf{B}_{j\sigma\mu}(y, D_y) - \mathcal{B}_{j\sigma\mu}(D_y)\big)U|_{\gamma_{j\sigma}} \\ &+ \sum_{(k,s)\neq(j,0)} \big(\psi(\mathcal{G}_{j\sigma ks}y) - \psi(y)\big)\big(B_{j\sigma\mu ks}(y, D_y)U_k\big)\big(\mathcal{G}_{j\sigma ks}y\big)\big|_{\gamma_{j\sigma}}. \end{aligned} \quad (2.36)$$

Thus, to prove the theorem, it suffices to show that each of the operators

$$U_k \mapsto J_{j\sigma\mu ks} = \big(\psi(\mathcal{G}_{j\sigma ks}y) - \psi(y)\big)\big(B_{j\sigma\mu ks}(y, D_y)U_k\big)\big(\mathcal{G}_{j\sigma ks}y\big)\big|_{\gamma_{j\sigma}} \quad (2.37)$$

compactly maps $W^{l+2m}(K_k)$ into $W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})$.

Note that, if $(k, s) \neq (j, 0)$, the operator $\mathcal{G}_{j\sigma ks}$ maps the ray $\gamma_{j\sigma}$ onto the ray

$$\{y \in \mathbb{R}^2 : r > 0,\ \omega = (-1)^\sigma b_j + \omega_{j\sigma ks}\},$$

which is strictly inside the angle K_k. Therefore, there exists a function $\xi \in C_0^\infty\big((-b_k, b_k)\big)$ which equals 1 at the point $\omega = (-1)^\sigma b_j + \omega_{j\sigma ks}$.

Moreover, note that the difference $\psi(y) - \psi(\mathcal{G}_{j\sigma ks}^{-1} y)$ is compactly supported and vanishes near the origin. Therefore, there exists a function $\psi_1 \in C_0^\infty(K_k)$ vanishing near the origin and equal to 1 on the support of the function $\xi(\omega)\big(\psi(y) - \psi(\mathcal{G}_{j\sigma ks}^{-1} y)\big)$.

Thus, we have

$$\begin{aligned} \|J_{j\sigma\mu ks}\|_{W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} &\leqslant k_1 \|\xi(\omega)\big(\psi(y) - \psi(\mathcal{G}_{j\sigma ks}^{-1} y)\big) B_{j\sigma\mu ks}(y, D_y) U_k\|_{W^{l+2m-m_{j\sigma\mu}}(K_k)} \\ &\leqslant k_2 \|\psi_1 U_k\|_{W^{l+2m}(K_k)}. \end{aligned} \tag{2.38}$$

Let us estimate the norm on the right-hand side of the last inequality by using Theorem 5.1 [22, Ch. 2] and taking into account that (I) the function ψ_1 is compactly supported and vanishes both near the origin and near the sides of the angle K_k and (II) $\mathcal{P}_k(D_y)U_k = 0$. As a result, using Leibniz' formula, we obtain

$$\|J_{j\sigma\mu ks}\|_{W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} \leqslant k_3 \|\psi_2 U_k\|_{W^{l+2m-1}(K_k)}, \tag{2.39}$$

where $\psi_2 \in C_0^\infty(K_k)$ is equal to 1 on the support of ψ_1. It follows from the estimate (2.39) and the Rellich theorem that the operator (2.37) is compact.

Remark 2.1. It follows from the proofs of Theorems 2.1 and 2.2 that

$$D^\alpha \mathfrak{R} f|_{y=0} = 0, \quad D^\alpha \mathfrak{R}' f'|_{y=0} = 0, \quad |\alpha| \leqslant l + 2m - 2.$$

In Sec. 6 (in the second part of the paper), we study nonlocal problems in weighted spaces with small values of the weight exponent a. The role of model operators in weighted spaces is played by the bounded operator $\mathfrak{L}_a : H_a^{l+2m,N}(K) \to \mathcal{H}_a^{l,N}(K,\gamma)$ given by

$$\mathfrak{L}_a U = \{\mathbf{P}_j(y, D_y)U_j,\ \mathbf{B}_{j\sigma\mu}(y, D_y)U|_{\gamma_{j\sigma}}\}.$$

Let us formulate an analog of Theorem 2.2 for weighted spaces.

Theorem 2.3. *Let the line* $\operatorname{Im}\lambda = a + 1 - l - 2m$ *contain no eigenvalues of* $\tilde{\mathcal{L}}(\lambda)$. *Then, for any* ε, $0 < \varepsilon < 1$, *there exist bounded operators*

$$\begin{aligned} \mathfrak{R}'_a &: \{f' : \{0, f'\} \in \mathcal{H}_a^{l,N}(K,\gamma),\ \operatorname{supp} f' \subset \mathcal{O}_\varepsilon(0)\} \to \{U \in H_a^{l+2m,N}(K) : \operatorname{supp} U \subset \mathcal{O}_{2\varepsilon}(0)\}, \\ \mathfrak{M}'_a, \mathfrak{T}'_a &: \{f' : \{0, f'\} \in \mathcal{H}_a^{l,N}(K,\gamma),\ \operatorname{supp} f' \subset \mathcal{O}_\varepsilon(0)\} \to \{f \in \mathcal{H}_a^{l,N}(K,\gamma) : \operatorname{supp} f \subset \mathcal{O}_{2\varepsilon_2}(0)\}, \end{aligned}$$

$\varepsilon_2 = \varepsilon/\min\{\chi_{j\sigma ks}, 1\}$. *such that* $\|\mathfrak{M}'_a f'\|_{\mathcal{H}_a^{l,N}(K,\gamma)} \leqslant c\varepsilon \|\{0, f'\}\|_{\mathcal{H}_a^{l,N}(K,\gamma)}$, *where the positive constant c depends only on the coefficients of the operators* $\mathcal{P}_j(D_y)$ *and* $B_{j\sigma\mu ks}(D_y)$, *the operator* $\mathfrak{T}'_a$ *is compact, and*

$$\mathfrak{L}_a \mathfrak{R}'_a f' = \{0, f'\} + \mathfrak{M}'_a f' + \mathfrak{T}'_a f'.$$

Proof. It follows from Theorem 2.1 [15] that the operator $\mathcal{L}_a$ has bounded inverse. Write

$$\mathfrak{R}'_a f' = \psi U, \qquad U = \mathcal{L}_a^{-1}\{0, f'\},$$

where ψ is the same function as in the proof of Theorem 2.2. The remaining part of the proof is similar to that of Theorem 2.2.

3. NONLOCAL PROBLEMS IN PLANE ANGLES FOR THE CASE IN WHICH LINE $\operatorname{Im}\lambda = 1 - L - 2M$ CONTAINS A PROPER EIGENVALUE OF $\tilde{\mathcal{L}}_P(\lambda)$

3.1. *Spaces* $\hat{\mathcal{S}}^{l,N}(K,\gamma)$

In this section, we still denote the operators $\mathbf{L}_p$, $\mathcal{L}_p$, and $\tilde{\mathcal{L}}_p(\lambda)$ by $\mathfrak{L}$, $\mathcal{L}$, and $\tilde{\mathcal{L}}(\lambda)$, respectively. Let us consider the situation in which the line $\operatorname{Im}\lambda = 1 - l - 2m$ contains eigenvalues of $\tilde{\mathcal{L}}(\lambda)$. Let $\lambda = \lambda_0$ be one of these eigenvalues.

Definition 3.1. We say that $\lambda = \lambda_0$ is a *proper eigenvalue* if (I) none of the corresponding eigenvectors $\varphi(\omega) = (\varphi_j(\omega), \dots, \varphi_N(\omega))$ has associate vectors and (II) the functions $r^{i\lambda_0}\varphi_j(\omega)$, $j = 1, \dots, N$, are polynomials in y_1, y_2.

Definition 3.2. An eigenvalue $\lambda = \lambda_0$ which is not proper is said to be an *improper eigenvalue*.

Remark 3.1. The notion of proper eigenvalue was originally proposed by Kondrat'ev [21] for "local" elliptic boundary-value problems in angular or conical domains.

Clearly, if λ_0 is a proper eigenvalue, then $\operatorname{Re}\lambda_0 = 0$. Therefore, the line $\operatorname{Im}\lambda = 1 - l - 2m$ can contain at most one proper eigenvalue. In this section, we investigate the case in which the following condition holds.

Condition 3.1. *The line* $\operatorname{Im}\lambda = 1 - l - 2m$ *contains only the eigenvalue* $\lambda_0 = i(1 - l - 2m)$ *and it is proper.*

In this case, the conclusion of Lemma 2.4 fails because the algebraic system (2.16), (2.17) can have no solution for some right-hand side and the system of operators (2.11), (2.12) is not linearly independent. Indeed, let $\varphi(\omega) = (\varphi_1(\omega), \dots, \varphi_N(\omega))$ be an eigenvector corresponding to the proper eigenvalue $\lambda_0 = i(1 - l - 2m)$. Then, by the definition of proper eigenvalue, $Q_j(y) = r^{l+2m-1}\varphi_j(\omega)$ is a polynomial (obviously homogeneous) of degree $l + 2m - 1$ with respect to $y = (y_1, y_2)$. Repeating the arguments of assertion 3 in the proof of Lemma 2.4, we see that, after substituting $q_{j\alpha} = D^\alpha Q_j$ for $D^\alpha W_j$ in the left-hand side of system (2.16), (2.17), the right-hand side of this system vanishes. Therefore, system (2.11), (2.12) is linearly dependent. Nevertheless, provided that Condition 3.1 holds, it turns out to be possible to construct an operator $\hat{\mathfrak{R}}$ defined for compactly supported functions in a certain space $\hat{\mathcal{S}}^{l,N}(K,\gamma)$ so that this operator is a right inverse for $\mathfrak{L}$ (see Theorem 3.1). However, in contrast to $\mathcal{S}^{l,N}(K,\gamma)$, the set $\hat{\mathcal{S}}^{l,N}(K,\gamma)$ is not closed in the topology of the space $\mathcal{W}^{l,N}(K,\gamma)$.

In system (2.11) formed by homogeneous operators of order $l + 2m - 1$, we choose maximally many linearly independent operators and denote them by

$$\hat{\mathcal{B}}_{j'\sigma'\mu'}(D_y)U. \tag{3.1}$$

Any operator $\hat{\mathcal{B}}_{j\sigma\mu}(D_y)$ not included in system (3.1) can be represented in the form

$$\hat{\mathcal{B}}_{j\sigma\mu}(D_y)U = \sum_{j',\sigma',\mu'} p_{j\sigma\mu}^{j'\sigma'\mu'} \hat{\mathcal{B}}_{j'\sigma'\mu'}(D_y)U, \tag{3.2}$$

where $p_{j\sigma\mu}^{j'\sigma'\mu'}$ are some constants.

Let us consider the functions $f = \{f_j, f_{j\sigma\mu}\} \in \mathcal{W}^{l,N}(K,\gamma)$ satisfying the condition

$$\mathcal{T}_{j\sigma\mu} f \equiv \frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\Phi_{j\sigma\mu} - \sum_{j',\sigma',\mu'} p_{j\sigma\mu}^{j'\sigma'\mu'} \frac{\partial^{l+2m-m_{j'\sigma'\mu'}-1}}{\partial\tau_{j'\sigma'}^{l+2m-m_{j'\sigma'\mu'}-1}}\Phi_{j'\sigma'\mu'} \in H_0^1(\mathbb{R}^2). \tag{3.3}$$

Here the indices j', σ', μ' correspond to the operators (3.1), whereas the indices j, σ, μ correspond to the operators in system (2.11) which are not included in (3.1), the symbols $\Phi_{j\sigma\mu}$ stand for the fixed extensions of the functions $f_{j\sigma\mu}$ to $\mathbb{R}^2$ (these extensions are defined by the operator (2.15)), and $p_{j\sigma\mu}^{j'\sigma'\mu'}$ for the constants appearing in relation (3.2). If system 2.11 is linearly independent, then the set of conditions (3.3) is empty.

Note that the validity of conditions (3.3) does not depend on the choice of the extension of $f_{j\sigma\mu}$ to $\mathbb{R}^2$. Indeed, let $\hat{\Phi}_{j\sigma\mu}$ be an extension distinct from $\Phi_{j\sigma\mu}$. Then $(\Phi_{j\sigma\mu} - \hat{\Phi}_{j\sigma\mu})|_{\gamma_{j\sigma}} = 0$; therefore,

$$\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}(\Phi_{j\sigma\mu} - \hat{\Phi}_{j\sigma\mu}) \in H_0^1(\mathbb{R}^2)$$

by Theorem 4.8 [21].

Now let us complete system (3.1) with operators of order $l+2m-1$ in system (2.12) in such a way that the resulting system consists of linearly independent operators

$$\hat{\mathcal{B}}_{j'\sigma'\mu'}(D_y)U, \qquad D^{\xi'}\mathcal{P}_{j'}(D_y)U_{j'}, \tag{3.4}$$

and any operator $D^{\xi}\mathcal{P}_j(D_y)U_j$ not belonging to (3.4) can be represented in the following form:

$$D^{\xi}\mathcal{P}_j(D_y)U_j = \sum_{j',\sigma',\mu'} p_{j\xi}^{j'\sigma'\mu'}\hat{\mathcal{B}}_{j'\sigma'\mu'}(D_y)U + \sum_{j',\xi'} p_{j\xi}^{j'\xi'} D^{\xi'}\mathcal{P}_{j'}(D_y)U_{j'}, \tag{3.5}$$

where $p_{j\xi}^{j',\sigma',\mu'}$ and $p_{j\xi}^{j',\xi'}$ are some constants.

Let us extend the components $f_j \in W^l(K_j)$ of the vector f to $\mathbb{R}^2$. The extended functions are also denoted by $f_j \in W^l(\mathbb{R}^2)$. We consider the functions f satisfying

$$\mathcal{T}_{j\xi}f \equiv D^{\xi}f_j - \sum_{j',\sigma',\mu'} p_{j\xi}^{j'\sigma'\mu'}\frac{\partial^{l+2m-m_{j'\sigma'\mu'}-1}}{\partial\tau_{j'\sigma'}^{l+2m-m_{j'\sigma'\mu'}-1}}\Phi_{j'\sigma'\mu'} - \sum_{j',\xi'} p_{j\xi}^{j'\xi'} D^{\xi'} f_{j'} \in H_0^1(\mathbb{R}^2). \tag{3.6}$$

Here the indices j', σ', μ' and j', ξ' correspond to the operators (3.4), whereas the indices j, ξ correspond to the operators of system (2.12) that are not included in (3.4), and $p_{j\xi}^{j'\sigma'\mu'}$ and $p_{j\xi}^{j'\xi'}$ stand for constants entering relations (3.5). As above, one can show that the validity of conditions (3.6) does not depend on the choice of the extension of f_j and $f_{j\sigma\mu}$ to $\mathbb{R}^2$. Note that the set of conditions (3.6) is empty if either $l = 0$ or $l \geqslant 1$ and system (3.4) contains all operators in (2.12).

Let us introduce an analog of the set $\mathcal{S}^{l,N}(K,\gamma)$ used above for the case in which Condition 3.1 holds. Denote by $\hat{\mathcal{S}}^{l,N}(K,\gamma)$ the set of functions $f \in \mathcal{W}^{l,N}(K,\gamma)$ satisfying conditions (2.8), (2.9), (3.3), and (3.6). Supplying $\hat{\mathcal{S}}^{l,N}(K,\gamma)$ with the norm

$$\|f\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)} = \Big(\|f\|^2_{\mathcal{W}^{l,N}(K,\gamma)} + \sum_{j,\sigma,\mu}\|\mathcal{T}_{j\sigma\mu}f\|^2_{H_0^1(\mathbb{R}^2)} + \sum_{j,\xi}\|\mathcal{T}_{j\xi}f\|^2_{H_0^1(\mathbb{R}^2)}\Big)^{1/2} \tag{3.7}$$

makes $\hat{\mathcal{S}}^{l,N}(K,\gamma)$ a complete space. (In the definition of the norm (3.7), the indices j, σ, μ and j, ξ correspond to operators not occurring in system (3.4).)

Let us establish some important properties of the space $\hat{\mathcal{S}}^{l,N}(K,\gamma)$. The following lemma shows that, if we impose finitely many orthogonality conditions of the form

$$D^{\alpha}U|_{y=0} = 0, \quad |\alpha| \leqslant l+2m-2, \tag{3.8}$$

on a compactly supported function $U \in W^{l+2m,N}(K)$, then the right-hand side of the corresponding nonlocal problem belongs to $\hat{\mathcal{S}}^{l,N}(K,\gamma)$.

Lemma 3.1. *Let Condition* 3.1 *hold. Let* $U \in W^{l+2m,N}(K)$, $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon \min\{\chi_{j\sigma ks},1\}}(0)$, *and let relations* (3.8) *hold. Then*

$$\|\mathcal{L}U\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)} \leqslant c\|U\|_{W^{l+2m,N}(K)}, \quad \|\mathfrak{L}U\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)} \leqslant c\|U\|_{W^{l+2m,N}(K)}. \tag{3.9}$$

Proof. 1. Set $f = \{f_j, f_{j\sigma\mu}\} = \mathcal{L}U$. It follows from the assumptions of the lemma that $f \in \mathcal{W}^{l,N}(K,\gamma)$, $\operatorname{supp} f \subset \mathcal{O}_\varepsilon(0)$, and the functions f_j and $f_{j\sigma\mu}$ satisfy relations (2.8) and (2.9), respectively.

We denote by $\Phi_{j\sigma\mu} \in W^{l+2m-m_{j\sigma\mu}}(\mathbb{R}^2)$ the extension of $f_{j\sigma\mu}$ defined by the operator (2.15). Let us show that

$$\hat{\mathcal{B}}_{j\sigma\mu}(D_y)U - \frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\Phi_{j\sigma\mu} \in H_0^1(\mathbb{R}^2). \tag{3.10}$$

By Lemma 2.2,

$$\hat{\mathcal{B}}_{j\sigma\mu}(D_y)U - \frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\mathcal{B}_{j\sigma\mu}(D_y)U \in H_0^1(\mathbb{R}^2);$$

thus, to prove (3.10), it suffices to show that

$$\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}(\mathcal{B}_{j\sigma\mu}(D_y)U - \Phi_{j\sigma\mu}) \in H_0^1(\mathbb{R}^2). \tag{3.11}$$

However,

$$\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}(\mathcal{B}_{j\sigma\mu}(D_y)U - \Phi_{j\sigma\mu}) \in W^1(\mathbb{R}^2)$$

and

$$\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}(\mathcal{B}_{j\sigma\mu}(D_y)U - \Phi_{j\sigma\mu})\big|_{\gamma_{j\sigma}} = 0;$$

hence, relation (3.11) follows from Lemma 4.8 [21]. This also proves relation (3.10).

The operators $\hat{\mathcal{B}}_{j\sigma\mu}(D_y)U$ satisfy relations (3.2); therefore, by virtue of (3.10), the functions $\Phi_{j\sigma\mu}$ satisfy relations (3.3).

Similarly, it follows from (3.10), from equalities $\mathcal{P}_j(D_y)U_j - f_j = 0$, and from relations (3.5) that the function f satisfies relations (3.6). Therefore, $f \in \hat{\mathcal{S}}^{l,N}(K,\gamma)$, and one can readily see that the first inequality in (3.9) holds.

2. Now, to prove that $\mathfrak{L}U \in \hat{\mathcal{S}}^{l,N}(K,\gamma)$, it suffices to establish the relations

$$D^{l-1}\big(\mathbf{P}_j(y,D_y) - \mathcal{P}_j(D_y)\big)U_j \in H_0^1(\mathbb{R}^2), \quad \frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\big(\mathbf{B}_{j\sigma\mu}(y,D_y)U - \mathcal{B}_{j\sigma\mu}(D_y)U\big) \in H_0^1(\mathbb{R}^2),$$

where $U_j \in W^{l+2m}(\mathbb{R}^2)$ is an extension of $U_j \in W^{l+2m}(K_j)$ to $\mathbb{R}^2$ (which is also denoted by U_j). These expressions consist of the terms

$$\big(a_\alpha(y) - a_\alpha(0)\big)D^\alpha U_j \ (|\alpha| = l+2m-1), \qquad a_\beta(y)D^\beta U_j \ (|\beta| \leqslant l+2m-2),$$

where a_α and a_β are infinitely differentiable functions.

Since $U_j \in W^{l+2m}(\mathbb{R}^2)$, it follows that $D^\alpha U_j \in H_1^1(\mathbb{R}^2)$. This property, together with Lemma 3.3′ in [21], implies that

$$\big(a_\alpha(y) - a_\alpha(0)\big)D^\alpha U_j \in H_0^1(\mathbb{R}^2).$$

The function $a_\beta D^\beta U_j$ ($|\beta| \leqslant l+2m-2$) belongs to $W^2(\mathbb{R}^2)$. It follows from this fact, together with relations (3.8) and Lemma 2.1, that

$$a_\beta D^\beta U_j \in H_a^2(\mathbb{R}^2) \subset H_{a-1}^1(\mathbb{R}^2), \qquad a>0.$$

Let us choose some a such that $0<a<1$. Since the supports of U_j are compact, we obtain

$$a_\beta D^\beta U_j \in H_0^1(\mathbb{R}^2).$$

Moreover, one can readily show that the second inequality in (3.9) also holds.

The following lemma shows that the set $\hat{\mathcal{S}}^{l,N}(K,\gamma)$ is not closed in the topology of $\mathcal{W}^{l,N}(K,\gamma)$.

Lemma 3.2. *Let Condition* 3.1 *hold. Then there exists a family of functions* $f^\delta \in \hat{\mathcal{S}}^{l,N}(K,\gamma)$, $\delta>0$, *such that* $\operatorname{supp} f^\delta \subset \mathcal{O}_\varepsilon(0)$ *and* f^δ *converges in* $\mathcal{W}^{l,N}(K,\gamma)$ *to a function* $f^0 \notin \hat{\mathcal{S}}^{l,N}(K,\gamma)$ *as* $\delta\to 0$.

Proof. 1. As was shown above, if $\lambda_0 = i(1-l-2m)$ is a proper eigenvalue of $\tilde{\mathcal{L}}(\lambda)$, then system (2.11), (2.12) is linearly dependent. We consider the two possible cases: (a) system (2.11) is linearly dependent or (b) system (2.11) is linearly independent but system (2.11), (2.12) is linearly dependent.

2. Suppose first that system (2.11) is linearly dependent. Then the set of conditions (3.3) is not empty. In this case, for some j,σ,μ, the norm (3.7) contains the corresponding term of the form $\|\mathcal{T}_{j\sigma\mu} f\|_{H_0^1(\mathbb{R}^2)}$. Let us fix the related subscripts j,σ,μ. Without loss of generality, one can assume that $\gamma_{j\sigma}$ coincides with the axis Oy_1. We introduce some functions $f^\delta = \{0, f^\delta_{j_1\sigma_1\mu_1}\}$ ($0\leqslant\delta\leqslant 1$) such that $f^\delta_{j_1\sigma_1\mu_1}=0$ for $(j_1,\sigma_1,\mu_1)\neq(j,\sigma,\mu)$ and

$$f^\delta_{j\sigma\mu}(y_1) = \psi(y_1) y_1^{l+2m-m_{j\sigma\mu}-1+\delta},$$

where $\psi\in C_0^\infty\big([0,\infty)\big)$, $\psi(y_1)=1$ for $0\leqslant y_1\leqslant\varepsilon/2$, and $\psi(y_1)=0$ for $y_1\geqslant 2\varepsilon/3$. Clearly,

$$\hat{\Phi}^\delta_{j\sigma\mu}(y) = \psi(r) y_1^{l+2m-m_{j\sigma\mu}-1} r^\delta$$

is an extension of the function $f^\delta_{j\sigma\mu}$ to $\mathbb{R}^2$. Moreover, the extension operator defined for the functions $f^\delta_{j\sigma\mu}$ ($0\leqslant\delta\leqslant 1$) is bounded as an operator from $W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})$ to $W^{l+2m-m_{j\sigma\mu}}(\mathbb{R}^2)$ (this holds because

$$\|f^\delta_{j\sigma\mu}\|_{W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} \geqslant c_1 \qquad\text{and}\qquad \|\hat{\Phi}^\delta_{j\sigma\mu}\|_{W^{l+2m-m_{j\sigma\mu}}(\mathbb{R}^2)} \leqslant c_2,$$

where $c_1,c_2>0$ do not depend on $0\leqslant\delta\leqslant 1$).

Thus, for $0<\delta\leqslant 1$,

$$\begin{aligned}
\|f^\delta\|^2_{\mathcal{W}^{l,N}(K,\gamma)} &= \|f^\delta_{j\sigma\mu}\|^2_{W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})},\\
\|f^\delta\|^2_{\hat{\mathcal{S}}^{l,N}(K,\gamma)} &\approx \|f^\delta_{j\sigma\mu}\|^2_{W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} + \left\|\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial y_1^{l+2m-m_{j\sigma\mu}-1}}\hat{\Phi}^\delta_{j\sigma\mu}\right\|^2_{H_0^1(\mathbb{R}^2)}
\end{aligned} \tag{3.12}$$

(the fact that the norms (3.12) are finite for any $\delta>0$ can be verified by the straightforward calculations). Here the symbol "≈" means that the corresponding norms are equivalent. Moreover, one can directly see that

$$\hat{\Phi}^\delta_{j\sigma\mu}\to\hat{\Phi}^0_{j\sigma\mu} \quad\text{in}\quad W^{l+2m-m_{j\sigma\mu}}(\mathbb{R}^2) \quad\text{as}\quad \delta\to 0.$$

Therefore,

$$f^{\delta}_{j\sigma\mu} \to f^{0}_{j\sigma\mu} \quad \text{in} \quad W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}) \quad \text{as} \quad \delta \to 0.$$

However, the corresponding function $f^0 = \{0, f^0_{j\sigma\mu}\}$ does not belong to $\hat{\mathcal{S}}^{l,N}(K,\gamma)$. Indeed, assuming the contrary, by virtue of (3.12), we have

$$\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial y_1^{l+2m-m_{j\sigma\mu}-1}} \hat{\Phi}^0_{j\sigma\mu} \in H^1_0(\mathbb{R}^2),$$

which is wrong because, near the origin, the function

$$\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial y_1^{l+2m-m_{j\sigma\mu}-1}} \hat{\Phi}^0_{j\sigma\mu}$$

is equal to a nonzero constant.

3. Now let system (2.11) be linearly independent; then system (2.11), (2.12) is linearly dependent. In this case, conditions (3.3) are absent, but the set of conditions (3.6) is not empty. Therefore, for some j and ξ, the norm (3.7) contains the corresponding term $\|\mathcal{T}_{j\xi} f\|_{H^1_0(\mathbb{R}^2)}$. We fix these indices j, ξ and introduce functions

$$f^\delta = \{f^\delta_{j_1}, 0\} \quad (0 \leqslant \delta \leqslant 1) \quad \text{such that} \quad f^\delta_{j_1} = 0 \quad \text{for} \quad j_1 \neq j \quad \text{and} \quad f^\delta_j = \psi(r) y^\xi r^\delta.$$

One can directly see that

$$f^\delta_j \to f^0_j \quad \text{in} \quad W^l(\mathbb{R}^2) \quad \text{as} \quad \delta \to 0;$$

however, $f^0 = \{f^0_j, f^0_{j\sigma\mu}\} \notin \hat{\mathcal{S}}^{l,N}(K,\gamma)$ because $D^\xi f^0_j \notin H^1_0(\mathbb{R}^2)$.

3.2. *Construction of the Operator* $\hat{\mathfrak{R}}$

Let us prove an analog of Lemma 2.4 which will be used later on to construct the operator $\hat{\mathfrak{R}}$ acting on the space $\hat{\mathcal{S}}^{l,N}(K,\gamma)$.

Lemma 3.3. *Let Condition* 3.1 *hold. Then, for any* ε, $0 < \varepsilon < 1$, *there exists a bounded operator*

$$\hat{\mathcal{A}} : \{f \in \hat{\mathcal{S}}^{l,N}(K,\gamma) \colon \operatorname{supp} f \subset \mathcal{O}_\varepsilon(0)\} \to W^{l+2m,N}(K)$$

such that, for any $f = \{f_j, f_{j\sigma\mu}\} \in \operatorname{Dom}(\hat{\mathcal{A}})$, *the function* $V = \hat{\mathcal{A}} f$ *satisfies the following conditions*:

$$\begin{gathered} V = 0 \quad \textit{for} \quad |y| \geqslant 1, \\ \|\mathcal{L}V - f\|_{\mathcal{H}^{l,N}_0(K)} \leqslant c \|f\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)}, \end{gathered} \tag{3.13}$$

and inequality (2.14) *holds.*

Proof. 1. Similarly to the proof of Lemma 2.4, we consider the following algebraic system for all partial derivatives $D^\alpha W_j$, $|\alpha| = l + 2m - 1$, $j = 1, \dots, N$:

$$\begin{aligned} \hat{\mathcal{B}}_{j'\sigma'\mu'}(D_y) W &= \frac{\partial^{l+2m-m_{j'\sigma'\mu'}-1}}{\partial \tau_{j'\sigma'}^{l+2m-m_{j'\sigma'\mu'}-1}} \Phi_{j'\sigma'\mu'}, \\ D^{\xi'} \mathcal{P}_{j'}(D_y) W_{j'} &= D^{\xi'} f_{j'}, \end{aligned} \tag{3.14}$$

where $\Phi_{j'\sigma'\mu'}$ and $f_{j'}$ are the extensions of $f_{j'\sigma'\mu'}$ and $f_{j'}$ to $\mathbb{R}^2$ described in the proof of Lemma 2.4. Now the left-hand side of system (3.14) contains only operators occurring in system (3.4). The matrix of system (3.14) consists of $(l+2m)N$ columns and q, $q < (l+2m)N$, linearly independent rows.

Choosing q linearly independent columns and assuming that the unknowns $D^\alpha W_j$ corresponding to the remaining $(l+2m)N-q$ columns are equal to zero, we obtain a system of q equations for q unknowns, and this system admits a unique solution. Thus, we defined a bounded linear operator

$$\left\{\frac{\partial^{l+2m-m_{j'\sigma'\mu'}-1}}{\partial\tau_{j'\sigma'}^{l+2m-m_{j'\sigma'\mu'}-1}}\Phi_{j'\sigma'\mu'},\ D^{\xi'}f_{j'}\right\}\mapsto\{D^\alpha W_j\}\equiv\{W_{j\alpha}\} \tag{3.15}$$

from $W^{1,q}(\mathbb{R}^2)$ to $W^{1,(l+2m)N}(\mathbb{R}^2)$, and $W_{j\alpha}(y)=0$ for $|y|\geqslant 2$. Using the functions $D^\alpha W_j$ and the operator (2.18), we obtain functions V_j, $j=1,\dots,N$, satisfying relations (2.19) and (2.20). Let us show that $V=(V_1,\dots,V_N)$ is a desired function.

2. Similarly to the proof of Lemma 2.4, one can prove the estimate (2.14) for the function V. Let us prove inequality (3.13). Since $\{W_{j\alpha}\}$ is a solution of system (3.14) and the functions V_j satisfy conditions (2.20), it follows that

$$\hat{\mathcal{B}}_{j'\sigma'\mu'}(D_y)V-\frac{\partial^{l+2m-m_{j'\sigma'\mu'}-1}}{\partial\tau_{j'\sigma'}^{l+2m-m_{j'\sigma'\mu'}-1}}\Phi_{j'\sigma'\mu'}\in H_0^1(\mathbb{R}^2), \tag{3.16}$$

$$D^{\xi'}(\mathcal{P}_{j'}(D_y)V_{j'}-f_{j'})\in H_0^1(\mathbb{R}^2). \tag{3.17}$$

Let us consider an arbitrary operator $\hat{\mathcal{B}}_{j\sigma\mu}(D_y)$ not entering system (3.4). Using (3.2), we obtain

$$\begin{aligned}\hat{\mathcal{B}}_{j\sigma\mu}(D_y)V-\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\Phi_{j\sigma\mu}&=\sum_{j',\sigma',\mu'}p_{j\sigma\mu}^{j'\sigma'\mu'}\left(\hat{\mathcal{B}}_{j'\sigma'\mu'}(D_y)V-\frac{\partial^{l+2m-m_{j'\sigma'\mu'}-1}}{\partial\tau_{j'\sigma'}^{l+2m-m_{j'\sigma'\mu'}-1}}\Phi_{j'\sigma'\mu'}\right)\\&+\sum_{j',\sigma',\mu'}p_{j\sigma\mu}^{j'\sigma'\mu'}\frac{\partial^{l+2m-m_{j'\sigma'\mu'}-1}}{\partial\tau_{j'\sigma'}^{l+2m-m_{j'\sigma'\mu'}-1}}\Phi_{j'\sigma'\mu'}-\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\Phi_{j\sigma\mu}.\end{aligned} \tag{3.18}$$

However, $f\in\hat{\mathcal{S}}^{l,N}(K,\gamma)$; therefore, conditions (3.3) hold. These conditions, together with relations (3.16) and (3.18), imply the relations

$$\hat{\mathcal{B}}_{j\sigma\mu}(D_y)V-\frac{\partial^{l+2m-m_{j\sigma\mu}-1}}{\partial\tau_{j\sigma}^{l+2m-m_{j\sigma\mu}-1}}\Phi_{j\sigma\mu}\in H_0^1(\mathbb{R}^2) \tag{3.19}$$

for any j, σ, AND μ. Similarly, one can consider the operators $D^\xi P_j(D_y)$ that do not occur in system (3.4) and prove that

$$D^\xi(\mathcal{P}_j(D_y)V_j-f_j)\in H_0^1(\mathbb{R}^2) \tag{3.20}$$

for all j and ξ by using relations (3.2) and (3.3), (3.5), and (3.6), as well as (3.16) and (3.17).

The estimate (3.13) follows from (3.19) and (3.20) by repeating the arguments of the proof of Lemma 2.4.

The proof of the following corollary of Lemma 3.3 is just like that of Corollary 2.1.

Corollary 3.1. *The function V constructed in Lemma* 3.2 *satisfies the following inequality*:

$$\|\mathfrak{L}V-f\|_{\mathcal{H}_0^{l,N}(K)}\leqslant c\|f\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)}. \tag{3.21}$$

Let us now use Lemma 3.3 to construct a right inverse of the operator $\mathcal{L}$ defined for the compactly supported functions $f\in\hat{\mathcal{S}}^{l,N}(K,\gamma)$ and prove an analog of Theorem 2.1. However, we cannot formally repeat the arguments of the proof of Theorem 2.1 because they use the invertibility of the operator $\mathcal{L}_0$ given by (2.1) in weighted spaces. In the present case, by Theorem 2.1 [15], the operator $\mathcal{L}_0$ is not invertible because the line $\operatorname{Im}\lambda=1-l-2m$ contains the eigenvalue $\lambda_0=i(1-l-2m)$ of the operator $\tilde{\mathcal{L}}(\lambda)$. However, as was mentioned above, the spectrum of $\tilde{\mathcal{L}}(\lambda)$ is discrete; hence, there is an $a>0$ such that the line $\operatorname{Im}\lambda=a+1-l-2m$ contains no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$, which implies that the operator $\mathcal{L}_a$ is invertible. In order to pass from $a>0$ to $a=0$, we make use of the following result.

Lemma 3.4. *Let $W \in H_a^{l+2m,N}(K)$ for some $a > 0$, and let $f = \mathcal{L}_a W \in \mathcal{H}_0^{l,N}(K,\gamma)$. Suppose that the closed strip $1-l-2m \leqslant \operatorname{Im}\lambda \leqslant a+1-l-2m$ contains only the eigenvalue $\lambda_0 = i(1-l-2m)$ of $\tilde{\mathcal{L}}(\lambda)$, and let this eigenvalue be proper. Then*

$$\|D^{l+2m}W\|_{H_0^{0,N}(K)} \leqslant c\|f\|_{\mathcal{H}_0^{l,N}(K,\gamma)}. \tag{3.22}$$

Lemma 3.4 will be proved in Sec. 3.3. Let us now study the solvability of problems (1.18), (1.19) and (1.15), (1.16), respectively.

We write $K_j^d = K_j \cap \{y \in \mathbb{R}^2 :\ |y| < d\}$, $W^{k,N}(K^d) = \prod_{j=1}^N W^k(K_j^d)$, and

$$H_a^{k,N}(K^d) = \prod_{j=1}^N H_a^k(K_j^d).$$

Lemma 3.5. *Let Condition* 3.1 *hold. Then, for any $f \in \hat{\mathcal{S}}^{l,N}(K,\gamma)$ with $\operatorname{supp} f \subset \mathcal{O}_\varepsilon(0)$, there exists a solution U of problem* (1.18), (1.19) *such that $U \in W^{l+2m,N}(K^d)$ for any $d > 0$ and U satisfies relations* (3.8) *and the inequalities*

$$\|U\|_{W^{l+2m,N}(K^d)} \leqslant c_d\|f\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)}, \tag{3.23}$$

$$\|U\|_{H_0^{l+2m-1,N}(K^d)} \leqslant c_d\|f\|_{\mathcal{W}^{l,N}(K,\gamma)}. \tag{3.24}$$

Proof. 1. Choose an a, $0 < a < 1$, such that the strip $1-l-2m < \operatorname{Im}\lambda \leqslant a+1-l-2m$ contains no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$. (Such a value a exists because the spectrum of $\tilde{\mathcal{L}}(\lambda)$ is discrete.) It follows from the definition of the space $\hat{\mathcal{S}}^{l,N}(K,\gamma)$ that relations (2.8) and (2.9) hold for any function $f = \{f_j, f_{j\sigma\mu}\}$ satisfying the assumptions of the lemma. Combining this fact with Lemma 2.1, we obtain

$$\|f\|_{\mathcal{H}_a^{l,N}(K,\gamma)} \leqslant k_1\|f\|_{\mathcal{W}^{l,N}(K,\gamma)}. \tag{3.25}$$

Let us consider the function $f - \mathcal{L}V$, where $V = \hat{\mathcal{A}}f \in W^{l+2m,N}(K) \cap H_a^{l+2m,N}(K)$ is the function defined in Lemma 3.2. It follows from inequalities (2.14) and (3.25) that

$$\|f - \mathcal{L}V\|_{\mathcal{H}_a^{l,N}(K,\gamma)} \leqslant k_2\|f\|_{\mathcal{W}^{l,N}(K,\gamma)}. \tag{3.26}$$

Therefore, the function $f - \mathcal{L}V \in \mathcal{H}_a^{l,N}(K,\gamma)$ belongs to the domain of the operator $\mathcal{L}_a^{-1}$. Writing $W = \mathcal{L}_a^{-1}(f - \mathcal{L}V)$, we see that $U = V + W$ is a solution of problem (1.18), (1.19).

2. Let us prove (3.24). Since the operator $\mathcal{L}_a^{-1}$ is bounded, it follows from inequality (3.26) that

$$\|W\|_{H_a^{l+2m}(K)} \leqslant k_3\|f\|_{\mathcal{W}^{l,N}(K,\gamma)}. \tag{3.27}$$

Now the estimate (3.24) follows from inequalities (3.27) and (2.14) and from the fact that the embedding $H_a^{l+2m,N}(K) \subset H_0^{l+2m-1,N}(K^d)$ is bounded.

3. Let us prove (3.23). Since the operator

$$\hat{\mathcal{A}}\colon \hat{\mathcal{S}}^{l,N}(K,\gamma) \to W^{l+2m,N}(K)$$

is bounded and inequality (3.27) holds, it suffices to estimate the functions $D^{l+2m}W$. It follows from Lemma 3.2 that $f - \mathcal{L}V \in \mathcal{H}_0^{l,N}(K,\gamma)$ and the estimate (3.13) is valid. Therefore, applying Lemma 3.4 to the function $W = \mathcal{L}_a^{-1}(f - \mathcal{L}V)$ and using the estimate (3.13), we obtain

$$\|D^{l+2m}W\|_{H_0^{0,N}(K)} \leqslant k_4\|f - \mathcal{L}V\|_{\mathcal{H}_0^{l,N}(K,\gamma)} \leqslant k_5\|f\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)}.$$

We now note that $H_0^0(K_j) = L_2(K_j)$ and thus complete the proof of (3.23).

4. The validity of relations (3.8) follows from the relation

$$U = V + W \in W^{l+2m,N}(K^d) \cap H_a^{l+2m,N}(K), \qquad a < 1,$$

and from Sobolev's embedding theorem.

We can now construct an operator $\hat{\mathfrak{R}}$ with the desired properties.

Theorem 3.1. *Let Condition* 3.1 *hold. Then, for any* ε, $0<\varepsilon<1$, *there exist bounded operators*

$$\hat{\mathfrak{R}}\colon \{f\in\hat{\mathcal{S}}^{l,N}(K,\gamma):\operatorname{supp} f\subset\mathcal{O}_\varepsilon(0)\}\to\{U\in W^{l+2m,N}(K):\operatorname{supp}U\subset\mathcal{O}_{2\varepsilon_1}(0)\},$$
$$\hat{\mathfrak{M}},\hat{\mathfrak{T}}\colon \{f\in\hat{\mathcal{S}}^{l,N}(K,\gamma):\operatorname{supp} f\subset\mathcal{O}_\varepsilon(0)\}\to\{f\in\hat{\mathcal{S}}^{l,N}(K,\gamma):\operatorname{supp} f\subset\mathcal{O}_{2\varepsilon_1}(0)\},$$

with[4] $\varepsilon_1=\max\{\varepsilon,\varepsilon_0/\min\{\chi_{j\sigma ks},1\}\}$ *such that* $\|\hat{\mathfrak{M}}f\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)}\leqslant c\varepsilon_1\|f\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)}$, *where* $c>0$ *depends only on the coefficients of the operators* $\mathcal{P}_j(D_y)$ *and* $B_{j\sigma\mu ks}(D_y)$, *the operator* $\hat{\mathfrak{T}}$ *is compact, and*

$$\mathfrak{L}\hat{\mathfrak{R}}f=f+\hat{\mathfrak{M}}f+\hat{\mathfrak{T}}f. \tag{3.28}$$

Proof. Let us consider a function $\psi\in C_0^\infty(\mathbb{R}^2)$ satisfying the following conditions:

$$\psi(y)=1\quad\text{for}\quad |y|\leqslant\varepsilon_1=\max\{\varepsilon,\varepsilon_0/\min\{\chi_{j\sigma ks},1\}\},\quad \operatorname{supp}\psi\subset\mathcal{O}_{2\varepsilon_1}(0),$$

and such that ψ does not depend on the polar angle ω. Introduce the operator $\hat{\mathfrak{R}}$ by the formula

$$\hat{\mathfrak{R}}f=\psi U\quad \big(f\in\hat{\mathcal{S}}^{l,N}(K,\gamma),\ \operatorname{supp} f\subset\mathcal{O}_\varepsilon(0)\big),$$

where $U\in W^{l+2m,N}(K^{2\varepsilon_1})$ is a solution of problem (1.18), (1.19) with right-hand side f (see Lemma 3.5).

Let us prove (3.28). Relation 2.33 and Leibniz' formula imply that $\operatorname{supp}(\mathfrak{L}\hat{\mathfrak{R}}f-f)\subset\mathcal{O}_{2\varepsilon_1}(0)$ and

$$\|\mathfrak{L}\hat{\mathfrak{R}}f-f\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)}\leqslant k_1\varepsilon_1\|f\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)}+k_2(\varepsilon_1)\|\psi_1U\|_{H_0^{l+2m-1,N}(K)}), \tag{3.29}$$

where $\psi_1\in C_0^\infty(\mathbb{R}^2)$ is equal to 1 on the support of ψ. It follows from the proof of Lemma 3.2 that the operator $f\mapsto\psi U$ from $\{f\in\hat{\mathcal{S}}^{l,N}(K,\gamma):\operatorname{supp} f\subset\mathcal{O}_\varepsilon(0)\}$ to $H_a^{l+2m,N}(K)$, $0<a<1$, is bounded. Since the embedding

$$\{\psi_1V:V\in H_a^{l+2m,N}(K)\}\subset H_0^{l+2m-1,N}(K),\qquad a<1,$$

is compact (see Lemma 3.5 [21]), this implies that the operator $f\mapsto\psi_1U$ compactly maps the space $\{f\in\hat{\mathcal{S}}^{l,N}(K,\gamma):\operatorname{supp} f\subset\mathcal{O}_\varepsilon(0)\}$ into $H_0^{l+2m-1,N}(K)$. Thus, using Lemma 2.3 and the estimate (3.29), we complete the proof.

Let us state an analog of Theorem 2.2.

Theorem. *Let Condition* 3.1 *hold. Then, for any* ε, $0<\varepsilon<1$, *there exist bounded operators*

$$\hat{\mathfrak{R}}'\colon\{f':\{0,f'\}\in\hat{\mathcal{S}}^{l,N}(K,\gamma),\ \operatorname{supp} f'\subset\mathcal{O}_\varepsilon(0)\}\to\{U\in W^{l+2m,N}(K):\operatorname{supp}U\subset\mathcal{O}_{2\varepsilon}(0)\},$$
$$\hat{\mathfrak{M}}',\hat{\mathfrak{T}}'\colon\{f':\{0,f'\}\in\hat{\mathcal{S}}^{l,N}(K,\gamma),\ \operatorname{supp} f'\subset\mathcal{O}_\varepsilon(0)\}\to\{f\in\hat{\mathcal{S}}^{l,N}(K,\gamma):\operatorname{supp} f\subset\mathcal{O}_{2\varepsilon_2}(0)\},$$

$\varepsilon_2=\varepsilon/\min\{\chi_{j\sigma ks},1\}$, *such that* $\|\hat{\mathfrak{M}}'f'\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)}\leqslant c\varepsilon\|\{0,f'\}\|_{\hat{\mathcal{S}}^{l,N}(K,\gamma)}$, *where* $c>0$ *depends only on the coefficients of the operators* $\mathcal{P}_j(D_y)$ *and* $B_{j\sigma\mu ks}(D_y)$, *the operator* $\hat{\mathfrak{T}}'$ *is compact, and*

$$\mathfrak{L}\hat{\mathfrak{R}}'f'=\{0,f'\}+\hat{\mathfrak{M}}'f'+\hat{\mathfrak{T}}'f'.$$

The proof of Theorem 3.2 is similar to that of Theorem 2.2.

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[4]See footnote 3 on p. 16.

3.3. *Proof of Lemma 3.4*

Assume first that

$$W \in \prod_{j=1}^{N} C_0^\infty(\bar{K}_j \setminus \{0\});$$

then $f_j \in C_0^\infty(\bar{K}_j \setminus \{0\})$ and $f_{j\sigma\mu} \in C_0^\infty(\gamma_{j\sigma})$, where $f = \{f_j, f_{j\sigma\mu}\} = \mathcal{L}W$. We denote by $W_j(\omega, r)$ and $f_j(\omega, r)$ the functions $W_j(y)$ and $f_j(y)$, respectively, written in polar coordinates. Let $\tilde{W}_j(\omega, \lambda)$, $\tilde{f}_j(\omega, \lambda)$, and $\tilde{f}_{j\sigma\mu}(\lambda)$ be the Fourier transforms with respect to τ of the functions $W_j(\omega, e^\tau)$, $e^{2m\tau} f_j(\omega, e^\tau)$, and $e^{m_{j\sigma\mu}\tau} f_{j\sigma\mu}(e^\tau)$, respectively. Write $\tilde{f} = \{\tilde{f}_j, \tilde{f}_{j\sigma\mu}\}$. Under our assumptions, the function $\lambda \mapsto \tilde{f}(\lambda)$ is analytic in the entire complex plane; moreover, for $|\operatorname{Im}\lambda| \leqslant \text{const}$, this function tends to zero uniformly with respect to ω and λ, more rapidly than any power of $|\lambda|$ as $|\operatorname{Re}\lambda| \to \infty$.

By virtue of Lemma 2.1 in [15], there exists a finite-meromorphic operator-valued function $\tilde{\mathcal{R}}(\lambda)$ such that $\tilde{\mathcal{R}}(\lambda) = \big(\tilde{\mathcal{L}}(\lambda)\big)^{-1}$ for any λ which is not an eigenvalue of $\tilde{\mathcal{L}}(\lambda)$. Moreover, if the line $\operatorname{Im}\lambda = a + 1 - l - 2m$ contains no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$, then, by the proof of Theorem 2.1 in [15], the solution W is given by

$$W(\omega, e^\tau) = \int_{-\infty + i(a+1-l-2m)}^{+\infty + i(a+1-l-2m)} e^{i\lambda\tau} \tilde{\mathcal{R}}(\lambda) \tilde{f}(\lambda)\, d\lambda. \tag{3.30}$$

Let us consider an arbitrary derivative $D^{l+2m}W(y)$ of order $l + 2m$ of the function W with respect to y_1 and y_2. Suppose that the operator D^{l+2m} can be represented in polar coordinates in the form $r^{-(l+2m)}\tilde{M}(\omega, D_\omega, rD_r)$. After the substitution $r = e^\tau$, the operator D^{l+2m} becomes $e^{-(l+2m)\tau}\tilde{M}(\omega, D_\omega, D_\tau)$, where $D_\tau = -i\partial/\partial\tau$. Combining this fact with (3.30), we see that the function $D^{l+2m}W(y)$ can be obtained from the function

$$e^{-(l+2m)\tau} \int_{-\infty + i(a+1-l-2m)}^{+\infty + i(a+1-l-2m)} e^{i\lambda\tau} \tilde{M}(\omega, D_\omega, \lambda) \tilde{\mathcal{R}}(\lambda) \tilde{f}(\lambda)\, d\lambda \tag{3.31}$$

by the substitution $\tau = \log r$ followed by the passage from polar to Cartesian coordinates. Let us show that the operator-valued function $\tilde{M}(\omega, D_\omega, \lambda)\tilde{\mathcal{R}}(\lambda)$ is analytic near the point $\lambda_0 = i(1 - l - 2m)$. Since λ_0 is an eigenvalue of $\tilde{\mathcal{L}}(\lambda)$, it follows from [23] that

$$\tilde{\mathcal{R}}(\lambda) = \frac{A_{-1}}{\lambda - \lambda_0} + \Gamma(\lambda),$$

where $\Gamma(\lambda)$ is an analytic operator-valued function near λ_0 and the image of A_{-1} coincides with the linear span of the eigenvectors corresponding to λ_0. Therefore,

$$\tilde{M}(\omega, D_\omega, \lambda)\tilde{\mathcal{R}}(\lambda)\tilde{f} = \frac{\tilde{M}(\omega, D_\omega, \lambda)A_{-1}\tilde{f}}{\lambda - \lambda_0} + \tilde{M}(\omega, D_\omega, \lambda)\Gamma(\lambda)\tilde{f}$$

for any $\tilde{f} \in \mathcal{W}^{l,N}[-b, b]$. By the definition of a proper eigenvalue, the function $r^{l+2m-1}A_{-1}\tilde{f}$ is a vector $Q(y) = (Q_1(y), \dots, Q_N(y))$, where $Q_j(y)$ are some polynomials of degree $l + 2m - 1$ in y_1 and y_2. Hence,

$$\tilde{M}(\omega, D_\omega, \lambda)A_{-1}\tilde{f} = r^{1-l-2m}\tilde{M}(\omega, D_\omega, rD_r)(r^{l+2m-1}A_{-1}\tilde{f}) = rD^{l+2m}Q(y) = 0.$$

Thus, the operator-valued function $\tilde{M}(\omega, D_\omega, \lambda)\tilde{\mathcal{R}}(\lambda)$ is analytic near $\lambda_0 = i(1 - l - 2m)$, and therefore in the closed strip $1 - l - 2m \leqslant \operatorname{Im}\lambda \leqslant a + 1 - l - 2m$.

Moreover, for $|\operatorname{Im}\lambda| \leqslant \text{const}$, the growth of the norm $\|\tilde{M}(\omega, D_\omega, \lambda)\tilde{\mathcal{R}}(\lambda)\|_{\mathcal{W}^{l,N}[-b,b]\to W^{0,N}(-b,b)}$ is at most power-law with respect to $|\lambda|$ (see Lemma 2.1 in [15]), whereas $\|\tilde{f}(\lambda)\|_{\mathcal{W}^{l,N}[-b,b]}$ tends to zero more rapidly than any power of $|\lambda|$ as $|\operatorname{Re}\lambda| \to \infty$. Therefore, we can replace the integration line $\operatorname{Im}\lambda = a + 1 - l - 2m$ in (3.31) by the line $\operatorname{Im}\lambda = 1 - l - 2m$. Thus, the function $D^{l+2m}W(y)$ can be obtained from the function

$$e^{-(l+2m)\tau} \int_{-\infty+i(1-l-2m)}^{+\infty+i(1-l-2m)} e^{i\lambda\tau} \tilde{M}(\omega, D_\omega, \lambda)\tilde{\mathcal{R}}(\lambda)\tilde{f}(\lambda)\, d\lambda \tag{3.32}$$

by the substitution $\tau = \log r$ followed by the passage from the polar coordinates to the Cartesian coordinates. Let us estimate the norm of $D^{l+2m}W$,

$$\begin{aligned} \|D^{l+2m}W\|^2_{H_0^{0,N}(K)} &= \sum_j \int_{K_j} |D^{l+2m}W_j|^2 dy \\ &= \sum_j \int_{-b_j}^{b_j} d\omega \int_{-\infty}^{+\infty} e^{-2(l+2m-1)\tau} \left| \int_{-\infty+i(1-l-2m)}^{+\infty+i(1-l-2m)} e^{i\lambda\tau} \tilde{M}(\omega, D_\omega, \lambda)\tilde{\mathcal{R}}(\lambda)\tilde{f}(\lambda)\, d\lambda \right|^2 d\tau. \end{aligned}$$

Combining this with the complex analog of Parseval's equality, we obtain

$$\|D^{l+2m}W\|^2_{H_0^{0,N}(K)} = \int_{-\infty+i(1-l-2m)}^{+\infty+i(1-l-2m)} \|\tilde{M}(\omega, D_\omega, \lambda)\tilde{\mathcal{R}}(\lambda)\tilde{f}(\lambda)\|^2_{W^{0,N}(-b,b)} d\lambda. \tag{3.33}$$

Let us estimate the norm in the integrand on the right-hand side. To do this, we introduce equivalent norms depending on parameter $\lambda \neq 0$ as follows:

$$\begin{aligned} |||\tilde{U}_j|||^2_{W^k(-b_j,b_j)} &= \|\tilde{U}_j\|^2_{W^k(-b_j,b_j)} + |\lambda|^{2k}\|\tilde{U}_j\|^2_{L_2(-b_j,b_j)}, \\ |||\tilde{f}|||^2_{\mathcal{W}^{l,N}[-b,b]} &= \sum_j \{|||\tilde{f}_j|||^2_{W^l(-b_j,b_j)} + \sum_{\sigma,\mu} |\lambda|^{2(l+2m-m_{j\sigma\mu}-1/2)}|\tilde{f}_{j\sigma\mu}|^2\}. \end{aligned}$$

By virtue of the interpolation inequality

$$|\lambda|^{l+2m-k}\|\tilde{U}_j\|_{W^k(-b_j,b_j)} \leqslant c_k |||\tilde{U}_j|||_{W^{l+2m}(-b_j,b_j)}, \quad 0 < k < l+2m$$

(see. [25, Ch. 1]) and by Lemma 2.1 in [15], there exists $C > 0$ such that the following estimate holds for all $\lambda \in \mathbb{C}$ satisfying $\operatorname{Im}\lambda = 1 - l - 2m$ and $|\operatorname{Re}\lambda| > C$:

$$\|\tilde{M}(\omega, D_\omega, \lambda)\tilde{\mathcal{R}}(\lambda)\tilde{f}(\lambda)\|^2_{W^{0,N}(-b,b)} \leqslant k_1 |||\tilde{f}(\lambda)|||^2_{\mathcal{W}^{l,N}[-b,b]}. \tag{3.34}$$

Since the operator-valued function

$$\tilde{M}(\omega, D_\omega, \lambda)\tilde{\mathcal{R}}(\lambda) : \mathcal{W}^{l,N}[-b,b] \to W^{0,N}(-b,b)$$

is analytic on the segment $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = 1-l-2m,\ |\operatorname{Re}\lambda| \leqslant C\}$, it follows that inequality (3.34) holds on the entire line $\operatorname{Im}\lambda = 1 - l - 2m$. By (3.33) and (3.34) we obtain

$$\|D^{l+2m}W\|^2_{H_0^{0,N}(K)} \leqslant k_1 \int_{-\infty+i(1-l-2m)}^{+\infty+i(1-l-2m)} |||\tilde{f}(\lambda)|||^2_{\mathcal{W}^{l,N}[-b,b]} d\lambda.$$

Combining this fact with inequalities (1.9) and (1.10) in [21] yields the estimate (3.22). Since $C_0^\infty(\bar{K}_j \setminus \{0\})$ is dense in $H_a^k(K_j)$ for any a and k, it follows that the estimate (3.22) holds for any $W \in H_a^{l+2m,N}(K)$ and $f \in \mathcal{H}_0^{l,N}(K,\gamma)$.

To be continued.

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