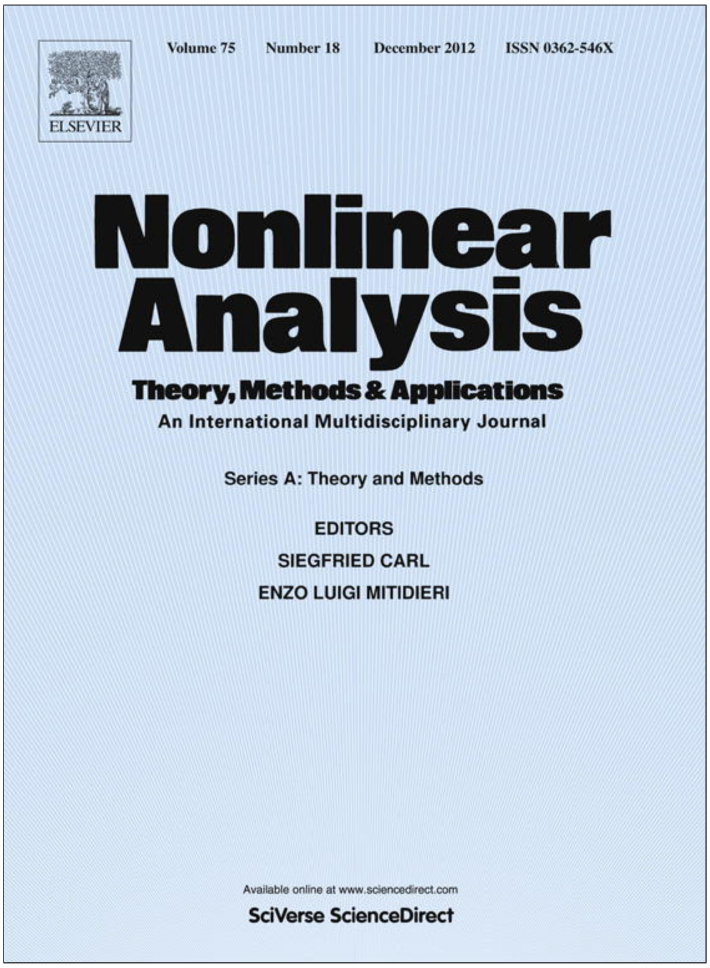
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Uniqueness of transverse solutions for reaction–diffusion equations with spatially distributed hysteresis

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ABSTRACT

The paper deals with reaction–diffusion equations involving a hysteretic discontinuity in the source term, which is defined at each spatial point. Such problems describe biological processes and chemical reactions in which diffusive and nondiffusive substances interact according to hysteresis law. Under the assumption that the initial data are spatially transverse, we prove a theorem on the uniqueness of solutions. The theorem covers the case of non-Lipschitz hysteresis branches arising in the theory of slow–fast systems.

1. Introduction

We consider reaction–diffusion equations with right-hand sides involving a discontinuous hysteresis defined at each spatial point. Such problems describe biological processes and chemical reactions in which diffusive and nondiffusive substances interact according to hysteresis law. As a result, various spatial and spatio-temporal patterns may appear (see, e.g., [1,2]).

First rigorous results about the existence of solutions of parabolic equations with hysteresis in the source term have been obtained in [3–5] for multi-valued hysteresis. Formal asymptotic expansions of solutions were recently obtained for some special case in [6]. However, the uniqueness of solutions and their continuous dependence on initial data as well as a thorough analysis of pattern formation remained open questions.

In [7], a new approach was suggested. It allowed us to find a broad class of initial data (transverse functions, see Section 2) for which a solution exists and, if unique, continuously depends on initial data. The approach is based on tracking the so-called free boundary which defines the hysteresis topology. The hysteresis topology is, in the simplest case, related to the structure of subdomains in space where hysteresis takes the same value. The main advantage of this approach is that, tracking the hysteretic free boundary, one gets necessary information on the precise form of emerging spatio-temporal patterns.

The transversality assumption roughly speaking means that the initial function has a nonvanishing derivative on the boundary between the above-mentioned subdomains (see Condition 2.2). Under this assumption, we prove in the present paper that the solution for the reaction–diffusion equation with discontinuous spatially distributed hysteresis is unique.

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More precisely, we assume that there exist two transverse solutions with the same initial data and prove that they coincide. In such a formulation, the proof does not use any results from [7].

We consider a one-dimensional domain and a scalar reaction–diffusion equation. The uniqueness result can be easily generalized to a wide class of systems of reaction–diffusion equations with spatially distributed hysteresis. As for the multi-dimensional domain, we do not know at the moment if a similar result holds.

The paper is organized as follows. In Section 2, we define functional spaces, introduce spatially distributed hysteresis and set the prototype problem. In the end of Section 2, we formulate the main result of the paper: Theorem 2.2 on the uniqueness of transverse solutions.

Section 3 is devoted to the proof of Theorem 2.2.

Interestingly, the uniqueness of solutions holds for some classes of non-Lipschitz hysteresis branches, too. In particular, we prove the uniqueness for hysteresis branches arising in bistable slow–fast reaction–diffusion systems (see, e.g., [8] and the references therein). In Appendix, we briefly describe such systems and show that the arising hysteresis branches satisfy assumptions of our main theorem.

2. Setting of the problem

2.1. Functional spaces

We denote by $L_q = L_q(0,1)$, $q > 1$, the standard Lebesgue space and by $W_q^l = W_q^l(0,1)$ with natural l the standard Sobolev space. For a noninteger $l > 0$, denote by $W_q^l = W_q^l(0,1)$ the Sobolev space with the norm

$$\|v\|_{W_q^l} = \|v\|_{W_q^{[l]}} + \left(\int_0^1 dx \int_0^1 \frac{|v^{([l])}(x) - v^{([l])}(y)|^q}{|x-y|^{1+q(l-[l])}} dy \right)^{1/q},$$

where $[l]$ is the integer part of l.

Let $Q_T = (0,1) \times (0,T)$. We introduce the Hölder space $C^\gamma(\overline{Q}_T)$, $0 < \gamma < 1$, and the anisotropic Sobolev space $W_q^{2,1}(Q_T)$ with the norm

$$\|u\|_{W_q^{2,1}(Q_T)} = \left(\int_0^T \|u(\cdot,t)\|_{W_q^2}^q \, dt + \int_0^T \|u_t(\cdot,t)\|_{L_q}^q \, dt \right)^{1/q}.$$

Throughout the paper, we fix q and γ such that

$$q > 3, \quad 0 < \gamma < 1 - 3/q.$$

Then $u, u_x \in C^\gamma(\overline{Q}_T)$ whenever $u \in W_q^{2,1}(Q_T)$ (see Lemma 3.3 in [9, Chapter 2]).

In what follows, we will consider solutions of parabolic problems in the space $W_q^{2,1}(Q_T)$ (see a motivation after the definition of solution — Definition 2.2). To define the space of initial data, we will use the fact that if $u \in W_q^{2,1}(Q_T)$, then the trace $u|_{t=t_0}$ is well defined and belongs to $W_q^{2-2/q}$ for all $t_0 \in [0,T]$ (see Lemma 2.4 in [9, Chapter 2]). Moreover, one can define the space $W_{q,N}^{2-2/q}$ as the subspace of functions from $W_q^{2-2/q}$ with the zero Neumann boundary conditions.

2.2. Hysteresis

In this section, we introduce a hysteresis operator defined for functions of time variable t. Then we extend the definition to a spatially distributed hysteresis acting on a space of functions of time variable t and space variable x.

We fix two numbers α and β such that $\alpha < \beta$. The numbers α and β will play a role of thresholds for the hysteresis operator. Next, we introduce continuous functions (*hysteresis branches*)

$$H_1 : (-\infty, \beta] \mapsto \mathbb{R}, \qquad H_2 : [\alpha, \infty) \mapsto \mathbb{R}.$$

We assume throughout that the following condition holds.

Condition 2.1. *There is a number $\sigma \in [0,1)$ such that, for any $U > 0$, there exists $M = M(U) > 0$ with the properties*

$$|H_1(u) - H_1(\hat{u})| \le \frac{M}{(\beta-u)^\sigma + (\beta-\hat{u})^\sigma} |u-\hat{u}|, \quad \forall u, \hat{u} \in [-U, \beta), \tag{2.1}$$

$$|H_2(u) - H_2(\hat{u})| \le \frac{M}{(u-\alpha)^\sigma + (\hat{u}-\alpha)^\sigma} |u-\hat{u}|, \quad \forall u, \hat{u} \in (\alpha, U]. \tag{2.2}$$

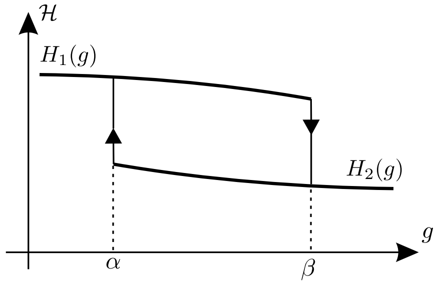

Fig. 2.1. The hysteresis operator $\mathscr{H}$.

Remark 2.1. 1. Any locally Lipschitz continuous functions $H_1(u)$ and $H_2(u)$ satisfy Condition 2.1. Moreover, this condition covers the important case where the hysteresis branches $H_1(u)$ and $H_2(u)$ are the stable parts of the curve $g(u, v) = 0$ in the slow–fast system (A.1). They are not Lipschitz continuous near the points $u = \beta$ and α, respectively. But they still satisfy Condition 2.1 (see Appendix for details).

2. On the other hand, any $H_1(u)$ and $H_2(u)$ satisfying Condition 2.1 are locally Hölder continuous with exponent $1 - \sigma$ on $(-\infty, \beta]$ and $[\alpha, \infty)$, respectively. Furthermore, $H_1(u)$ and $H_2(u)$ are locally Lipschitz on the open intervals $(-\infty, \beta)$ and (α, ∞), respectively. Therefore, they satisfy the assumptions in [7], where existence of solutions and their continuous dependence on initial data were proved.

We fix $T > 0$ and denote by $C_r[0, T)$ the space of functions which are continuous on the right in $[0, T)$. For any $\zeta_0 \in \{1, 2\}$ (*initial configuration of hysteresis*) and $g \in C[0, T]$ (*input*), we introduce the *configuration* function

$$\zeta : \{1, 2\} \times C[0, T] \to C_r[0, T), \qquad \zeta(t) = \zeta(\zeta_0, g)(t)$$

as follows. Let $X_t = \{t' \in (0, t] : g(t') = \alpha \text{ or } \beta\}$. Then

$$\zeta(0) = \begin{cases} 1 & \text{if } g(0) \le \alpha, \\ 2 & \text{if } g(0) \ge \beta, \\ \zeta_0 & \text{if } g(0) \in (\alpha, \beta) \end{cases}$$

and for $t \in (0, T]$

$$\zeta(t) = \begin{cases} \zeta(0) & \text{if } X_t = \varnothing, \\ 1 & \text{if } X_t \ne \varnothing \text{ and } g(\max X_t) = \alpha, \\ 2 & \text{if } X_t \ne \varnothing \text{and } g(\max X_t) = \beta. \end{cases}$$

Now we introduce the *hysteresis operator* (cf. [10,11])

$$\mathscr{H} : \{1, 2\} \times C[0, T] \to C_r[0, T)$$

by the following rule. For any initial configuration $\zeta_0 \in \{1, 2\}$ and input $g \in C[0, T]$, the function $\mathscr{H}(\zeta_0, g) : [0, T] \to \mathbb{R}$ (*output*) is given by

$$\mathscr{H}(\zeta_0, g)(t) = H_{\zeta(t)}(g(t)),$$

where $\zeta(t)$ is the configuration function defined above (see Fig. 2.1). Note that this hysteresis operator maps continuous functions to functions which are in general discontinuous.

Now we introduce a *spatially distributed hysteresis*. Assume that the initial configuration and the input function depend on spatial variable $x \in [0, 1]$. Denote them by $\xi_0(x)$ and $u(x, t)$, where

$$\xi_0 : [0, 1] \mapsto \{1, 2\}, \qquad u : [0, 1] \times [0, T] \mapsto \mathbb{R}.$$

Let $u(x, \cdot) \in C[0, T]$. Denote $\varphi(x) = u(x, 0)$. We say that a function $\varphi(x)$ and a hysteresis configuration $\xi_0(x)$ are *consistent* if, for any $x \in [0, 1]$,

$$\xi_0(x) \in \begin{cases} \{1\} & \text{if } \varphi(x) \le \alpha, \\ \{2\} & \text{if } \varphi(x) \ge \beta, \\ \{1, 2\} & \text{if } \varphi(x) \in (\alpha, \beta). \end{cases}$$

Then we can define the function

$$v(x, t) = \mathscr{H}(\xi_0(x), u(x, \cdot))(t), \tag{2.3}$$

which is called *spatially distributed hysteresis*. The function $\xi(x, t) = \zeta(\xi_0(x), u(x, \cdot))(t)$ is said to be its *spatial configuration*.

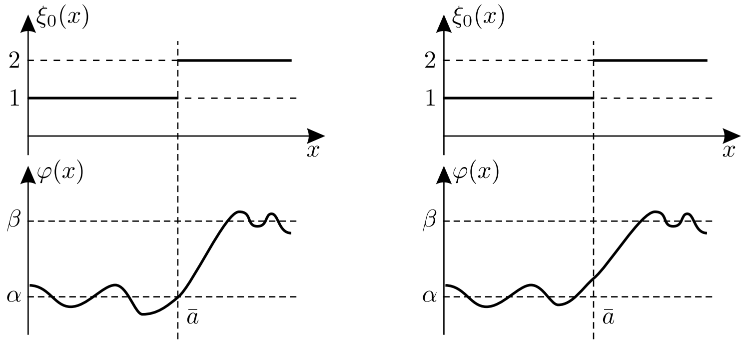

Fig. 2.2. Initial data satisfying Condition 2.2.

2.3. *Reaction–diffusion equations with hysteresis*

The main object of this paper is the initial boundary-value problem for the reaction–diffusion equation

$$u_t = u_{xx} + v, \quad (x, t) \in Q_T, \tag{2.4}$$

$$u_x|_{x=0} = u_x|_{x=1} = 0, \tag{2.5}$$

$$u|_{t=0} = \varphi(x), \quad x \in (0, 1), \tag{2.6}$$

where $v = v(x, t)$ represents the spatially distributed hysteresis given by (2.3).

Remark 2.2. The more general equation

$$u_t = u_{xx} + f(u, v)$$

with locally Lipschitz continuous right-hand side f can be reduced to Eq. (2.4). Indeed, it suffices to replace the hysteresis branches $H_j(u)$ in the definition of hysteresis $\mathcal{H}$ by $F_j(u) = f(u, H_j(u))$, $j = 1, 2$. One can check that the functions $F_j(u)$ also satisfy Condition 2.1.

The general assumption on the initial data $\varphi(x)$ under which the uniqueness result holds is that $\varphi(x)$ is transverse with respect to the initial configuration $\xi_0(x)$.

Definition 2.1. We say that a function $\varphi \in C^1[0, 1]$ is *transverse* (with respect to a spatial configuration $\xi_0(x)$) if it is consistent with $\xi_0(x)$ and the following holds:

1. if $\varphi(\bar{x}) = \alpha$ and $\varphi'(\bar{x}) = 0$ for some $\bar{x} \in [0, 1]$, then $\xi_0(\bar{x}) = 1$ in a neighborhood of $\bar{x}$;
2. if $\varphi(\bar{x}) = \beta$ and $\varphi'(\bar{x}) = 0$ for some $\bar{x} \in [0, 1]$, then $\xi_0(\bar{x}) = 2$ in a neighborhood of $\bar{x}$.

For the clarity of exposition, we now restrict ourselves to the following prototype situation (see Fig. 2.2).

Condition 2.2. *The function $\varphi(x)$ belongs to $C^1[0, 1]$, and there exists $\bar{a} \in (0, 1)$ such that the following holds.*

1. *One has*
$$\xi_0(x) = \begin{cases} 1, & x \le \bar{a}, \\ 2, & x > \bar{a}. \end{cases}$$
2. $\varphi(x) < \beta$ *for all* $x \in [0, \bar{a}]$.
3. $\varphi(x) > \alpha$ *for all* $x \in (\bar{a}, 1]$.
4. *If* $\varphi(\bar{a}) = \alpha$, *then* $\varphi'(\bar{a}) > 0$.

It follows from this condition that the hysteresis (2.3) at the initial moment is given by

$$v|_{t=0} = \begin{cases} H_1(\varphi(x)), & x \le \bar{a}, \\ H_2(\varphi(x)), & x > \bar{a}. \end{cases}$$

We give the definition of a solution of problem (2.4)–(2.6), assuming that $\varphi \in W_{q,N}^{2-2/q}$ (hence $\varphi \in C^1[0, 1]$).

Definition 2.2. A function $u(x, t)$ is called a *solution of problem* (2.4)–(2.6) (*in* Q_T) if $u \in W_q^{2,1}(Q_T)$, $v(x, t)$ is measurable, u and v satisfy Eq. (2.4) for a.e. $(x, t) \in Q_T$, and conditions (2.5) and (2.6) are satisfied in the sense of traces.

The space $W_q^{2,1}(Q_T)$ has been chosen because the right hand-side $v(x, t)$ of the parabolic equation (2.4), given by hysteresis, is a discontinuous function in x and t, which excludes the framework of Hölder spaces. On the other hand, the freedom to take large q yields the continuity of solutions and of their spatial derivative. Thus, since $u \in C(\overline{Q}_T)$, the function $v(x, t)$ is well defined by (2.3) and belongs to $L_\infty(Q_T)$. Furthermore, since $u_x \in C(\overline{Q}_T)$, we can formulate the (spatial) transversality condition (see below), which ensures well-posedness of problem (2.4)–(2.6).

2.4. Main result

In what follows, we always assume that Conditions 2.1 and 2.2 hold.

Definition 2.3. A function $u \in C(\overline{Q}_T)$ such that $u_x \in C(\overline{Q}_T)$ is called *transverse on* $[0, T]$ (with respect to a spatial configuration $\xi(x, t)$) if, for every fixed $t \in [0, T]$, the function $u(\cdot, t)$ is transverse with respect to the spatial configuration $\xi(\cdot, t)$.

In [7, Theorems 2.1 and 2.2], the following existence result is proved.

Theorem 2.1. *Let Conditions 2.1 and 2.2 be satisfied, and let $q > 3$. Then the following statements are true.*

1. *There is a number $T > 0$ such that*
   (a) *there is at least one solution of problem (2.4)–(2.6) in Q_T,*
   (b) *any solution of problem (2.4)–(2.6) in Q_T is transverse.*
2. *Let $u \in W_q^{2,1}(Q_{t_0})$ be a transverse solution of problem (2.4)–(2.6) on some time interval $[0, t_0]$. Then there is a number $T_{\max} \in (t_0, \infty]$ such that either*
   (a) *$T_{\max} = \infty$ and u can be extended to a transverse solution $u \in W_q^{2,1}(Q_T)$ for any $T > 0$, or*
   (b) *$T_{\max} < \infty$ and u can be extended to a solution $u \in W_q^{2,1}(Q_{T_{\max}})$, which is transverse on any time interval $[0, T]$ with $T < T_{\max}$, but the function $u(\cdot, T_{\max})$ is not transverse.*

In this paper, we show that there exists no more than one transverse solution of problem (2.4)–(2.6). We formulate the main result as follows.

Theorem 2.2. *Let Conditions 2.1 and 2.2 hold, and let $q > 3$. Assume that $u, \hat{u} \in W_q^{2,1}(Q_{T_0})$ are two transverse solutions of problem (2.4)–(2.6) in Q_{T_0} for some T_0. Then $u = \hat{u}$.*

3. Proof of Theorem 2.2

3.1. Local uniqueness

In this subsection, we prove the local uniqueness of solution of problem (2.4)–(2.6). In the next subsection, we deduce the global uniqueness from it.

Thus, let us prove in this subsection the following result.

Lemma 3.1. *Under the assumptions of Theorem 2.2, there exists a (sufficiently small) $T = T(\varphi, \xi_0) > 0$ such that any two transverse solutions u and $\hat{u}$ of problem (2.4)–(2.6) with the same initial data φ and ξ_0 coincide on the time interval $[0, T]$.*

We begin with one observation concerning the initial data. It follows from Condition 2.2 that either $\varphi(\overline{a}) = \alpha$ or $\varphi(\overline{a}) > \alpha$ (see Fig. 2.2). We will concentrate on the first case. It will be clear from the proof that the second case is much simpler (see a comment at the end of this subsection), and we shall leave the details to the reader.

Thus, we assume throughout this subsection that

$$\varphi(\overline{a}) = \alpha \tag{3.1}$$

and prove the uniqueness on a sufficiently small time interval $[0, T]$, $T \le T_0$.

We denote

$$\overline{\varphi} = \frac{\varphi'(\overline{a})}{2}.$$

By Condition 2.2 and assumption (3.1), we have $\overline{\varphi} > 0$.

Lemma 3.2. *Let $u \in W_q^{2,1}(Q_{T_0})$ be a solution of problem (2.4)–(2.6) for some $T_0 > 0$, and let assumption (3.1) hold. Then there exist $T \in (0, T_0)$ and $\delta > 0$ such that the following statements hold for each $t \in [0, T]$:*

1. *$u_x(x, t) \ge \overline{\varphi}$ for all $x \in [\overline{a} - \delta, \overline{a} + \delta]$;*
2. *the equation $u(x, t) = \alpha$ on the interval $x \in [\overline{a} - \delta, 1]$ has a unique root $x = a(t)$;*
3. *$a(t)$ is continuous;*
4. *the function*
$$b(t) = \max_{s \in [0,t]} a(s) \tag{3.2}$$
   satisfies $b(t) \in [\overline{a}, \overline{a} + \delta]$;
5. *the equation $u(x, t) = \beta$ on the interval $x \in [0, b(t)]$ has no roots.*

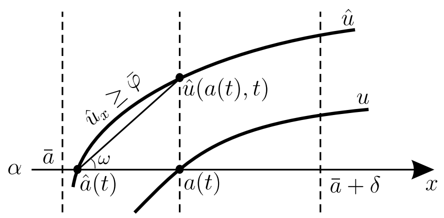

Fig. 3.1. The mean-value theorem for u and $\hat{u}$: $\tan\omega \geq \overline{\varphi}$.

Proof. The assertions of the lemma follow from the fact that $u, u_x \in C^\gamma(\overline{Q}_T)$ for all sufficiently small T (with the norms in $C^\gamma(\overline{Q}_T)$ bounded uniformly with respect to small T), $u(x,0) = \varphi(x)$, and $\varphi(x)$ is transverse (see Condition 2.2). □

By possibly decreasing T and δ, we see that Lemma 3.2 holds for $\hat{u}$ with some functions $\hat{a}(t)$ and $\hat{b}(t) = \max_{s\in[0,t]} \hat{a}(s)$ instead of $a(t)$ and $b(t)$.

Now the key observation is that the hysteresis $\mathcal{H}$ acting on transverse functions u and $\hat{u}$ on the time interval $[0,T]$ can be represented in terms of the *free boundaries* $b(t)$ and $\hat{b}(t)$ as follows:

$$\mathcal{H}(\xi_0(x), u(x,\cdot))(t) = \begin{cases} H_1(u(x,t)), & 0 \leq x \leq b(t), \\ H_2(u(x,t)), & b(t) < x \leq 1, \end{cases}$$
$$\mathcal{H}(\xi_0(x), \hat{u}(x,\cdot))(t) = \begin{cases} H_1(\hat{u}(x,t)), & 0 \leq x \leq \hat{b}(t), \\ H_2(\hat{u}(x,t)), & \hat{b}(t) < x \leq 1. \end{cases} \tag{3.3}$$

The following lemma allows us to estimate the distance between the free boundaries $b(t)$ and $\hat{b}(t)$ in terms of the difference between u and $\hat{u}$.

Lemma 3.3. *Let T be the number from Lemma 3.2. Then*

$$\|b - \hat{b}\|_{C[0,T]} \leq \frac{1}{\overline{\varphi}} \|u - \hat{u}\|_{C(\overline{Q}_T)}.$$

Proof. It follows from the definition of $b(t)$ and $\hat{b}(t)$ that

$$\|b - \hat{b}\|_{C[0,T]} \leq \|a - \hat{a}\|_{C[0,T]}.$$

On the other hand, using Lemma 3.2 (in particular, the inequalities $u_x(x,t) \geq \overline{\varphi}$ and $\hat{u}_x(x,t) \geq \overline{\varphi}$ for x between $a(t)$ and $\hat{a}(t)$), we obtain for any $t \in [0,T]$

$$|a(t) - \hat{a}(t)| \leq \frac{1}{\overline{\varphi}} |u(a(t),t) - \hat{u}(a(t),t)| \leq \frac{1}{\overline{\varphi}} \|u - \hat{u}\|_{C(\overline{Q}_T)}$$

(see Fig. 3.1). This completes the proof. □

Now we can prove the local uniqueness of the solution of problem (2.4)–(2.6).

Proof of Lemma 3.1. 1. Denote $w = u - \hat{u}$. The function w satisfies the linear parabolic equation

$$w_t = w_{xx} + h(x,t), \quad (x,t) \in Q_T, \tag{3.4}$$

where $h(x,t) = \mathcal{H}(\xi_0(x), u(x,\cdot)) - \mathcal{H}(\xi_0(x), \hat{u}(x,\cdot))$, and the zero boundary and initial conditions. Obviously, $h \in L_\infty(Q_T)$, and the function w can be represented via the Green function $G(x,y,t,s)$ of the heat equation with the Neumann boundary conditions:

$$w(x,t) = \int_0^t \int_0^1 G(x,y,t,s) h(y,s)\, dyds.$$

Therefore, using the estimate

$$|G(x,y,t,s)| \leq \frac{k_1}{\sqrt{t-s}} e^{-(x-y)^2/(4(t-s))} \leq \frac{k_1}{\sqrt{t-s}}, \quad 0 < s < t,$$

with $k_1 > 0$ not depending on $(x,t) \in Q_T$ and $T > 0$ (see, e.g., [12]), we obtain

$$|w(x,t)| \leq k_1 \int_0^t \frac{ds}{\sqrt{t-s}} \int_0^1 h(y,s)\, dy. \tag{3.5}$$

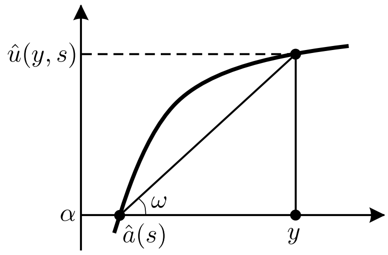

Fig. 3.2. The mean-value theorem for $\hat{u}(y,s)$: $\tan\omega \ge \overline{\varphi}$.

2. Let us estimate the interior integral in (3.5) for a fixed s. We assume that $b(s) < \hat{b}(s)$ (the case $b(s) \ge \hat{b}(s)$ is treated analogously). Then, due to (3.3),

$$h(y,s) = \begin{cases} H_1(u) - H_1(\hat{u}), & 0 < y < b(s), \\ H_2(u) - H_1(\hat{u}), & b(s) < y < \hat{b}(s), \\ H_2(u) - H_2(\hat{u}), & \hat{b}(s) < y < 1. \end{cases}$$

2.1. Assertion 5 in Lemma 3.2 implies that

$$u(y,s) < \beta, \qquad \hat{u}(y,s) < \beta$$

on the closed set $\{(y,s) : y \in [0, b(s)],\ s \in [0,T]\}$. Hence, the values $\beta - u(y,s)$ and $\beta - \hat{u}(y,s)$ are separated from 0. Therefore, using Condition 2.1, we obtain

$$\int_0^{b(s)} |h(y,s)|\,dy \le \int_0^{b(s)} \frac{M}{(\beta - u(y,s))^\sigma + (\beta - \hat{u}(y,s))^\sigma} |u(y,s) - \hat{u}(y,s)|\,dy$$
$$\le k_2 \int_0^{b(s)} |u(y,s) - \hat{u}(y,s)|\,dy \le k_2 \|w\|_{C(\overline{Q}_T)}, \tag{3.6}$$

where $k_2 > 0$ and the constants $k_3, k_4, k_5 > 0$ below do not depend on $s \in [0,T]$.

2.2. By the boundedness of $H_1(\hat{u})$ and $H_2(u)$ for $(y,s) \in \overline{Q}_T$, we have

$$\int_{b(s)}^{\hat{b}(s)} |h(y,s)|\,dy \le k_3 \int_{b(s)}^{\hat{b}(s)} dy \le k_3 \|b - \hat{b}\|_{C[0,T]}.$$

Applying Lemma 3.3 yields

$$\int_{b(s)}^{\hat{b}(s)} |h(y,s)|\,dy \le \frac{k_3}{\overline{\varphi}} \|w\|_{C(\overline{Q}_T)}. \tag{3.7}$$

2.3. Let δ be the number from Lemma 3.2, and let $\hat{b}(s) < y < \overline{a} + \delta$. Then, using assertion 1 in Lemma 3.2 and the mean-value theorem, we have (see Fig. 3.2)

$$|\hat{u}(y,s) - \alpha| = \hat{u}(y,s) - \hat{u}(\hat{a}(s),s) \ge (y - \hat{a}(s))\overline{\varphi} \ge (y - \hat{b}(s))\overline{\varphi}.$$

Similarly,

$$|u(y,s) - \alpha| \ge (y - \hat{b}(s))\overline{\varphi}.$$

Taking into account these two inequalities and using Condition 2.1, we obtain

$$\int_{\hat{b}(s)}^{\overline{a}+\delta} |h(y,s)|\,dy \le \frac{M}{2} \int_{\hat{b}(s)}^{\overline{a}+\delta} \frac{|u(y,s) - \hat{u}(y,s)|}{(y - \hat{b}(s))^\sigma}\,dy$$
$$\le \frac{M\|w\|_{C(\overline{Q}_T)}}{2} \int_{\hat{b}(s)}^{\overline{a}+\delta} \frac{1}{(y - \hat{b}(s))^\sigma}\,dy \le k_4 \|w\|_{C(\overline{Q}_T)}. \tag{3.8}$$

2.4. Finally, assertions 1 and 2 in Lemma 3.2 imply that

$$u(y,s) > \alpha, \qquad \hat{u}(y,s) > \alpha$$

on the closed set $[\overline{a}+\delta, 1]\times[0,T]$. Hence, the values $u(y,s)-\alpha$ and $\hat{u}(y,s)-\alpha$ are separated from 0. Therefore, due to Condition 2.1,

$$\int_{\overline{a}+\delta}^{1}|h(y,s)|\,dy \le \int_{\overline{a}+\delta}^{1}\frac{M}{(u(y,s)-\alpha)^{\sigma}+(\hat{u}(y,s)-\alpha)^{\sigma}}|u(y,s)-\hat{u}(y,s)|\,dy$$
$$\le k_5\int_{\overline{a}+\delta}^{1}|u(y,s)-\hat{u}(y,s)|dy \le k_5\|w\|_{C(\overline{Q}_T)}. \tag{3.9}$$

3. Combining estimate (3.5) with inequalities (3.6)–(3.9), we obtain

$$|w(x,t)| \le k_6\|w\|_{C(\overline{Q}_T)}\int_0^t\frac{ds}{\sqrt{t-s}} = 2k_6T^{1/2}\|w\|_{C(\overline{Q}_T)}.$$

Since k_6 does not depend on x and t, the latter inequality is equivalent to $\|w\|_{C(\overline{Q}_T)} \le 2k_6T^{1/2}\|w\|_{C(\overline{Q}_T)}$. Since k_6 does not depend on (small) T either, it follows that $w=0$ provided that $T>0$ is small enough. □

In the conclusion of the proof of local uniqueness, we comment on our assumption $\varphi(\overline{a})=\alpha$. If $\varphi(\overline{a})>\alpha$, then the continuity of the solutions u and $\hat{u}$ implies that $u(\overline{a},t)>\alpha$ and $\hat{u}(\overline{a},t)>\alpha$ on a sufficiently small time interval $[0,T]$. Therefore, formulas (3.3) reduce to the following:

$$\mathcal{H}(\xi_0(x),u(x,\cdot))(t)=\begin{cases}H_1(u(x,t)), & 0\le x\le\overline{a},\\ H_2(u(x,t)), & \overline{a}<x\le 1,\end{cases}$$
$$\mathcal{H}(\xi_0(x),\hat{u}(x,\cdot))(t)=\begin{cases}H_1(\hat{u}(x,t)), & 0\le x\le\overline{a},\\ H_2(\hat{u}(x,t)), & \overline{a}<x\le 1.\end{cases} \tag{3.10}$$

This means that the free boundary is actually fixed at $x=\overline{a}$, in which case the above proof of the local uniqueness becomes much simpler.

3.2. Global uniqueness

Let us complete the proof of Theorem 2.2, deducing the global uniqueness of solutions of problem (2.4)–(2.6) from the local uniqueness (Lemma 3.1).

Assume that u and $\hat{u}$ are two *different* transverse solutions of problem (2.4)–(2.6) on the interval $[0,T_0]$ (for some T_0) with the same initial data. Denote

$$M=\{t\in[0,T_0]: u(x,t)=\hat{u}(x,t)\text{ in }\overline{Q}_t\}.$$

Since $0\in M$, the set M is nonempty. Let $t_0=\sup M$. Since u and $\hat{u}$ are assumed to be different on $\overline{Q}_{T_0}$, it follows that $t_0<T_0$. By the continuity of u and $\hat{u}$ in $\overline{Q}_{T_0}$, we have $u(\cdot,t)=\hat{u}(\cdot,t)$ for all $t\in[0,t_0]$. Therefore, the corresponding spatial-configuration functions also coincide for all $t\in[0,t_0]$. Now, using Lemma 3.1 and the semigroup property of the hysteresis operator

$$\mathcal{H}(\xi_0(x),u(x,\cdot))(t_0+t)=\mathcal{H}(\xi(x,t_0),u(x,t_0+\cdot))(t),\quad t\ge t_0,$$

we see that there exists $\tau>0$ such that $u(\cdot,t)=\hat{u}(\cdot,t)$ for $t\in[t_0,t_0+\tau]$. Hence, t_0 cannot be the supremum of M.

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Appendix. Connection with slow–fast systems

It is known [14,15] that hysteresis may approximate solutions of ordinary differential equations with a small parameter $\varepsilon>0$ and a "bistable" right-hand side. Combining such an ordinary differential equation with the reaction–diffusion equation (2.4), one obtains a slow–fast system of the form

$$\begin{cases}u_t=u_{xx}+f(u,v),\\ \varepsilon v_t=g(u,v),\end{cases} \tag{A.1}$$

where $\varepsilon>0$, $f(u,v)$ and $g(u,v)$ are smooth functions of class C^∞, and $g(u,v)$ satisfies the following condition (see Fig. A.1).

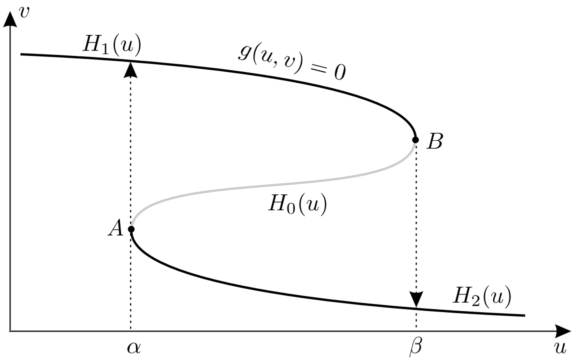

Fig. A.1. The nullcline of $g(u, v)$.

Condition A.1. 1. *There are two numbers* $\alpha < \beta$ *such that the equation* $g(u, v) = 0$ *has a unique root* $v = H_1(u)$ *for* $u < \alpha$ *and* $v = H_2(u)$ *for* $u > \beta$; *three distinct roots* $v = H_1(u)$, $v = H_2(u)$, *and* $v = H_0(u)$ *for* $u \in (\alpha, \beta)$; *two distinct roots* $v = H_1(\alpha)$ *and* $v = H_2(\alpha) = H_0(\alpha)$; *two distinct roots* $v = H_1(\beta) = H_0(\beta)$ *and* $v = H_2(\beta)$.

2. *The functions* $H_1(u)$ *and* $H_2(u)$ *are locally Lipschitz continuous for* $u < \beta$ *and* $u > \alpha$, *respectively.*
3. *Let* $A = (\alpha, H_2(\alpha))$ *and* $B = (\beta, H_1(\beta))$. *Then* $\frac{\partial g(A)}{\partial u} \neq 0$ *and* $\frac{\partial g(B)}{\partial u} \neq 0$.
4. $\frac{\partial g(A)}{\partial v} = \cdots = \frac{\partial^{n-1} g(A)}{\partial v^{n-1}} = 0$, $\frac{\partial^n g(A)}{\partial v^n} \neq 0$ *for some even* $n \geq 2$ *and the same holds at the point B.*
5. $g(u, v) > 0 (< 0)$ *if a point* (u, v) *lies to the left (right) from the curve* $g(u, v) = 0$ *on the plane* (u, v).

The simplest example of such a bistable nonlinearity typical, e.g., for the FitzHugh–Nagumo system is given by

$$g(u, v) = u + v - v^3.$$

As we have mentioned before, the natural conjecture is that the solutions of the slow–fast system (A.1) approximate as $\varepsilon \to 0$ the solution of Eq. (2.4) with hysteresis $\mathcal{H}$ defined by the curves $H_1(u)$ and $H_2(u)$ from Condition A.1. We refer to [8], where singular limit analysis of traveling waves in bistable systems is done, and to [16,13], where equations of the form $u_t = \Delta \Phi(u)$ with a cubic nonlinearity Φ are studied.

The proof of the above conjecture in the general situation is an open question. However, it is clear that the proof would deal with hysteresis defined by non-Lipschitz curves. The following result assures that the non-Lipschitz curves $H_1(u)$ and $H_2(u)$ from Condition A.1 satisfy Condition 2.1 and thus fit into our theory.

Lemma A.1. *Let* $g(u, v)$ *satisfy* Condition A.1. *Then the functions* $H_1(u)$ *and* $H_2(u)$ *from* Condition A.1 *satisfy* Condition 2.1 *with* $\sigma = (n-1)/n$.

Proof. 1. Without loss of generality, we assume that $A = 0$ and prove the lemma for the function $H_2(u)$. Due to item 2 in Condition A.1, it suffices to prove inequality (2.2) for u in a small neighborhood of $\alpha = 0$.

Since $\frac{\partial g(A)}{\partial u} \neq 0$, it follows from the implicit function theorem that there is a unique function $G_2(v)$ defined in a small neighborhood of 0 such that $g(G_2(v), v) \equiv 0$ and

$$G_2(0) = G_2'(0) = \cdots = G_2^{(n-1)}(0) = 0, \qquad G_2^{(n)}(0) > 0. \tag{A.2}$$

Obviously, $G_2(v)$ and $H_2(u)$ are inverse to each other for all negative v and positive u close to 0. It is convenient to introduce the function $G(w) = G_2(-w)$ defined for small positive w. Due to (A.2), it satisfies

$$G(0) = G'(0) = \cdots G^{(n-1)}(0) = 0, \qquad G^{(n)}(0) > 0 \tag{A.3}$$

since n is even. Now if we denote $w = -H_2(u)$ (>0), then $u = G_2(H_2(u)) = G_2(-w) = G(w)$. Therefore, the inequality for H_2 in Condition 2.1, which we have to prove, is equivalent to the following:

$$\frac{G(w) - G(\hat{w})}{w - \hat{w}} \geq M \left([G(w)]^{\frac{n-1}{n}} + [G(\hat{w})]^{\frac{n-1}{n}} \right) \tag{A.4}$$

for all $0 < \hat{w} < w \leq \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small and $M > 0$ does not depend on $w, \hat{w}$.

2. Expanding $G(w)$ by the Taylor formula about $w = \hat{w}$, we have

$$\frac{G(w) - G(\hat{w})}{w - \hat{w}} = G'(\hat{w}) + \sum_{k=2}^{n-1} \frac{G^{(k)}(\hat{w})}{k!} (w - \hat{w})^{k-1} + \frac{G^{(n)}(\xi)}{n!} (w - \hat{w})^{n-1}, \tag{A.5}$$

where $0 \leq \xi = \xi(w, \hat{w}) \leq \varepsilon_0$.

Further, we expand $G^{(k)}(\hat{w})$, $k = 1, \dots, n-1$ by the Taylor formula about the origin, using (A.3):

$$G^{(k)}(\hat{w}) = \frac{G^{(n)}(\xi_k)}{(n-k)!}\hat{w}^{n-k},$$

where $0 \le \xi_k = \xi_k(\hat{w}) \le \varepsilon_0$. Substituting these expressions into (A.5) yields

$$\begin{aligned} \frac{G(w)-G(\hat{w})}{w-\hat{w}} &\ge \frac{G^{(n)}(\xi_1)}{(n-1)!}\hat{w}^{n-1} + \frac{G^{(n)}(\xi)}{n!}(w-\hat{w})^{n-1} \\ &\ge \frac{G^{(n)}(0)}{2(n-1)!}\left(\hat{w}^{n-1} + \frac{1}{n}(w-\hat{w})^{n-1}\right), \end{aligned} \tag{A.6}$$

where we assume ε_0 so small that $G^{(n)}(\xi_k) \ge \frac{G^{(n)}(0)}{2}$ and $G^{(n)}(\xi) \ge \frac{G^{(n)}(0)}{2}$.

It is easy to show that

$$\hat{w}^{n-1} + \frac{1}{n}(w-\hat{w})^{n-1} \ge M_1\left(w^{n-1} + \hat{w}^{n-1}\right)$$

for some $M_1 > 0$ not depending on w and $\hat{w}$, $0 < \hat{w} < w$. Hence, inequality (A.6) implies

$$\frac{G(w)-G(\hat{w})}{w-\hat{w}} \ge M_2\left(w^{n-1} + \hat{w}^{n-1}\right), \tag{A.7}$$

where $M_2 > 0$ does not depend on w and $\hat{w}$, $0 < \hat{w} < w \le \varepsilon$. Combining (A.7) and the relations

$$G(w) = \frac{G^{(n)}(\zeta)}{n!}w^n \le \frac{2G^{(n)}(0)}{n!}w^n$$

with $0 \le \zeta = \zeta(w) \le \varepsilon_0$ yields (A.4). □

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