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# Non-local elliptic problems with non-linear argument transformations near the points of conjugation

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**Abstract.** We consider elliptic equations of order $2m$ in a domain $G \subset \mathbb{R}^n$ with non-local conditions that connect the values of the unknown function and its derivatives on $(n-1)$-dimensional submanifolds $\overline{\Upsilon}_i$ (where $\bigcup_i \overline{\Upsilon}_i = \partial G$) with the values on $\omega_{is}(\overline{\Upsilon}_i) \subset \overline{G}$. Non-local elliptic problems in dihedral angles arise as model problems near the conjugation points $g \in \overline{\Upsilon}_i \cap \overline{\Upsilon}_j \neq \varnothing$, $i \neq j$. We study the case when the transformations $\omega_{is}$ correspond to *non-linear* transformations in the model problems. It is proved that the operator of the problem remains Fredholm and its index does not change as we pass from linear argument transformations to non-linear ones.

## Introduction

The first mathematicians who studied ordinary differential equations with non-local conditions were Sommerfeld [1], Tamarkin [2], Picone [3]. In 1932, Carleman [4] considered the problem of finding a holomorphic function in a bounded domain $G$ satisfying the following condition: the value of the unknown function at each point $x$ of the boundary is connected with the value at $\omega(x)$, where $\omega(\omega(x)) = x$, $\omega(\partial G) = \partial G$. Such a statement of the problem led to further investigations of non-local elliptic problems with shifts that map the boundary of the domain onto itself. In 1969 Bitsadze and Samarskii [5] considered an essentially different type of non-local problems. They studied the Laplace equation in a bounded domain $G$ with a boundary condition connecting the values of the unknown function on a manifold $\Upsilon_1 \subset \partial G$ with its values on some manifold lying inside $G$, assuming that a Dirichlet condition is imposed on $\partial G \setminus \Upsilon_1$. In the general case, this problem was stated as an unsolved one.

The most difficult situation appears when the support of non-local terms intersects the boundary of the domain. We consider the following example. Let $G \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded domain with boundary $\partial G = \Upsilon_1 \cup \Upsilon_2 \cup \mathcal{K}_1$, where $\Upsilon_i$ are connected open (in the topology of $\partial G$) $(n-1)$-dimensional $C^\infty$-manifolds, and $\mathcal{K}_1 = \overline{\Upsilon}_1 \cap \overline{\Upsilon}_2$ is an $(n-2)$-dimensional connected $C^\infty$ manifold without boundary. (If $n=2$, then $\mathcal{K}_1 = \{g_1, g_2\}$, where $g_1, g_2$ are the ends of the curves $\overline{\Upsilon}_1$, $\overline{\Upsilon}_2$.)

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Suppose that, in a neighbourhood of each point $g \in \mathcal{K}_1$, the domain $G$ is diffeomorphic to an $n$-dimensional dihedral angle (a planar angle if $n = 2$). Consider the following non-local problem in $G$:

$$\Delta u = f_0(y), \qquad y \in G, \tag{0.1}$$

$$u|_{\Upsilon_i} - b_i u(\omega_i(y))|_{\Upsilon_i} = 0, \qquad i = 1, 2. \tag{0.2}$$

Here $b_1, b_2 \in \mathbb{R}$, and $\omega_i$ is an infinitely differentiable non-degenerate transformation that maps a neighbourhood $\mathcal{O}_i$ of $\Upsilon_i$ onto the set $\omega(\mathcal{O}_i)$ such that $\omega_i(\Upsilon_i) \subset G$ and $\overline{\omega_i(\Upsilon_i)} \cap \partial G \neq \varnothing$ (See Fig. 0.1, $a$, $b$).

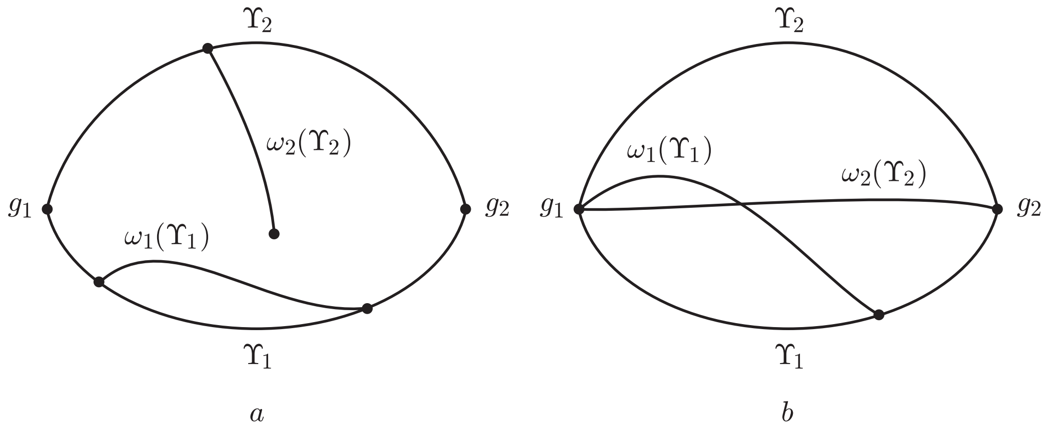


FIGURE 0.1. The domain $G$ with boundary $\partial G = \overline{\Upsilon}_1 \cup \overline{\Upsilon}_2$ for $n = 2$. Here $g_1$ and $g_2$ are points of conjugation of non-local conditions.

Problems of type (0.1), (0.2) were considered by many mathematicians (see [6]–[8] and others). The most complete theory of such problems is developed by Skubachevskii and his pupils [9]–[14]. In particular, they proved Fredholm solubility of higher-order elliptic equations with general non-local conditions, determined asymptotic behaviour of solutions near the points of conjugation of non-local conditions, and studied the smoothness of generalized solutions. It is shown [15] that the index of a non-local problem is equal to that of the corresponding local problem if the support of non-local terms contains no points of conjugation (see Fig. 0.1, $a$). This is not generally true in the opposite case (see Fig. 0.1, $b$).

Properties of non-local problems in bounded domains are essentially determined by the properties of model non-local problems in dihedral (or planar if $n = 2$) angles $\Omega = \{x = (y, z) \in \mathbb{R}^n : b' < \varphi < b'',\ z \in \mathbb{R}^{n-2}\}$ (with $(\varphi, r)$ being the polar coordinates of $y$) corresponding to the points of conjugation of non-local conditions. The previous works [9]–[11] considered only the case when the transformations $\omega_{is}$ correspond to *linear* transformations (that is, compositions of rotations and dilations) in the $y$-plane. This restriction is quite unnatural in many applications. Let us explain this on examples. Problems of type (0.1), (0.2) arise as mathematical models of some plasma processes in a bounded domain [16]. The non-local conditions connect the plasma temperature on the boundary with the temperature inside the domain and at other points of the boundary.

Another important application arises in the theory of diffusion processes. Such processes describe, for example, the Brownian motion of a particle in a membrane

$G \subset \mathbb{R}^n$. It is known [17]–[19] that every diffusion process generates some Feller semigroup. By the Hille–Iosida theorem, investigation of this semigroup is reduced to the study of an elliptic operator with boundary conditions that contain an integral over $\overline{G}$ with respect to a non-negative Borel measure [20]. In the most difficult case when the measure is atomic, the non-local conditions take the form (0.2). Their probabilistic meaning is as follows: once the particle gets to a point $y \in \Upsilon_i$, it either jumps to the point $\omega_i(y)$ with probability $b_i$, $0 \leqslant b_i \leqslant 1$, or "dies" with probability $1 - b_i$ (and then the process terminates). Thus the argument transformations are generally non-linear in both the plasma theory and the theory of diffusion processes.

Let us mention one more application of non-local problems. As shown in [21], one can reduce some boundary-value problems for elliptic differential-difference equations (in particular, those arising in the modern aircraft technology as models of sandwich shells and plates [22], [21]) to elliptic equations with non-local conditions on some shifts of the boundary. Thus we again obtain non-linear argument transformations in the non-local terms. (These transformations happen to be linear only when the boundary of the domain coincides with $(n-1)$-dimensional hyperplanes on certain sets.) One can consult [21] for other applications and references to papers devoted to non-local problems.

In this paper we consider an elliptic equation of order $2m$ in a domain $G \subset \mathbb{R}^n$ with non-local conditions that connect the values of the unknown function and its derivatives on $(n-1)$-dimensional submanifolds $\Upsilon_i$ (where $\bigcup_i \overline{\Upsilon}_i = \partial G$) with the vales on $\omega_{is}(\Upsilon_i) \subset G$. As mentioned before, essential difficulties arise when the the support $\bigcup_{i,s} \overline{\omega_{is}(\Upsilon_i)}$ of non-local terms intersects the boundary of the domain. Then generalized solutions may have power singularities near some set [9]. (For example, in the case of the problem (0.1), (0.2), such singularities may appear near the points $g_1$, $g_2$.) Therefore it is natural to consider such problems in weighted spaces. This enables us to investigate higher-order elliptic equations with general non-local conditions. We study the case when the transformations $\omega_{is}$ correspond to *non-linear* transformations in the model problems. The problem with non-linear transformations turns out to be neither small nor compact perturbation of the corresponding local problem. Nevertheless, we shall show that the operator of the problem remains Fredholm and its index does not change as we pass from linear transformations to non-linear ones.

We note that a more general structure of the conjugation point set and non-local terms was considered in [8] in the case of second-order elliptic equations with non-local perturbations of the Dirichlet conditions. This also justifies the importance of studying non-linear transformations $\omega_{is}$. From our viewpoint, the advantage of our approach is that it enables us to study equations of order $2m$ with general boundary conditions, whose non-local perturbations may be arbitrarily large. On the other hand, this approach also enables us to investigate the asymptotic behaviour of solutions near the conjugation points [9], [14].

The paper is organized as follows. In § 1 we consider the statement of the problem and discuss conditions that are imposed on the argument transformations in the non-local terms. We also introduce the main functional spaces (weighted Sobolev spaces) and obtain the model problems in dihedral and planar angles. In § 2 we give an example of a non-local problem with a non-linear argument transformation and show that the operator corresponding to this problem is neither small nor compact perturbation of the operator corresponding to the problem with linearized transformations. In § 3 we study properties of non-linear transformations near the points of conjugation of non-linear conditions and prove several lemmas, which are used in § 4 to get a priori estimates of solutions. In § 5 we construct a right regularizer which, along with a priori estimates, guarantees the Fredholm solubility of the non-local problem. Finally, in § 6 we show that the index of the problem with non-linear argument transformation is equal to that of the problem with transformations linearized near the points of conjugation of non-local conditions.

## § 1. Statement of the problem in a bounded domain

**1.** Let $G \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded domain with boundary $\partial G = \bigcup_{i=1}^{N_0} \overline{\Upsilon}_i$, where $\Upsilon_i$ are connected open (in the topology of $\partial G$) $(n-1)$-dimensional $C^\infty$-manifolds. We assume that, in a neighbourhood of each point $g \in \partial G \setminus \bigcup_{i=1}^{N_0} \Upsilon_i$, the domain $G$ is diffeomorphic to some $n$-dimensional dihedral (or planar if $n = 2$) angle $\Omega = \{x = (y, z) \in \mathbb{R}^n : 0 < b' < \varphi < b'' < 2\pi,\ z \in \mathbb{R}^{n-2}\}$, where $(\varphi, r)$ are the polar coordinates of $y$.

We denote by $\mathbf{P}(x, D)$ and $B_{i\mu s}(x, D)$ differential operators of order $2m$ and $m_{i\mu}$ respectively with complex-valued $C^\infty$-coefficients $(i = 1, \dots, N_0,\ \mu = 1, \dots, m,$ $s = 0, \dots, S_i)$. Suppose that $\mathbf{P}(x, D)$ and $B_{i\mu 0}(x, D)$ satisfy the following conditions (see, for example, [23], Ch. 2, § 1).

**Condition 1.1.** *The operator* $\mathbf{P}(x, D)$ *is properly elliptic for all* $x \in \overline{G}$.

**Condition 1.2.** *The system* $\{B_{i\mu 0}(x, D)\}_{\mu=1}^m$ *covers the operator* $\mathbf{P}(x, D)$ *for all* $i = 1, \dots, N_0$ *and* $x \in \overline{\Upsilon}_i$.

Let $\omega_{is}$ $(i = 1, \dots, N_0,\ \ s = 1, \dots, S_i)$ be an infinitely differentiable transformation that maps some neighbourhood $\mathcal{O}_i$ of the manifold $\Upsilon_i$ onto the set $\omega_{is}(\mathcal{O}_i)$ such that $\omega_{is}(\Upsilon_i) \subset G$. We assume that the set

$$\mathcal{K} = \Big\{\bigcup_i (\overline{\Upsilon}_i \setminus \Upsilon_i)\Big\} \cup \Big\{\bigcup_{i,s} \omega_{is}(\overline{\Upsilon}_i \setminus \Upsilon_i)\Big\} \cup \Big\{\bigcup_{j,p} \bigcup_{i,s} \omega_{jp}\big(\omega_{is}(\overline{\Upsilon}_i \setminus \Upsilon_i) \cap \Upsilon_j\big)\Big\}$$

can be represented as $\mathcal{K} = \bigcup_{j=1}^3 \mathcal{K}_j$, where

$$\mathcal{K}_1 = \bigcup_{p=1}^{N_1} \mathcal{K}_{1p} = \partial G \setminus \bigcup_{i=1}^{N_0} \Upsilon_i, \qquad \mathcal{K}_2 = \bigcup_{p=1}^{N_2} \mathcal{K}_{2p} \subset \bigcup_{i=1}^{N_0} \Upsilon_i, \qquad \mathcal{K}_3 = \bigcup_{p=1}^{N_3} \mathcal{K}_{3p} \subset G. \tag{1.1}$$

Here $\mathcal{K}_{jp}$ are disjoint $(n-2)$-dimensional connected $C^\infty$-manifold without boundary (points if $n = 2$).

We consider the non-local boundary-value problem

$$\mathbf{P}(x, D)u = f_0(x), \qquad x \in G, \tag{1.2}$$

$$\mathbf{B}_{i\mu}(x, D)u \equiv \sum_{s=0}^{S_i} (B_{i\mu s}(x, D)u)(\omega_{is}(x))|_{\Upsilon_i} = g_{i\mu}(x), \tag{1.3}$$
$$x \in \Upsilon_i, \qquad i = 1, \dots, N_0, \qquad \mu = 1, \dots, m,$$

where $\big(B_{i\mu s}(x,D)u\big)(\omega_{is}(x)) = B_{i\mu s}(x',D_{x'})u(x')|_{x'=\omega_{is}(x)}$, $\omega_{i0}(x)\equiv x$.

**Example 1.1.** Let us consider problem (0.1), (0.2) in the two-dimensional case with the transformations $\omega_i$ corresponding to Fig. 1.1. Then we have $\mathcal{K}_1=\{g_1,g_2\}$, $\mathcal{K}_2=\{\omega_1(g_2)\}$, $\mathcal{K}_3=\{\omega_2(g_2),\omega_1(\omega_1(g_2))\}$.

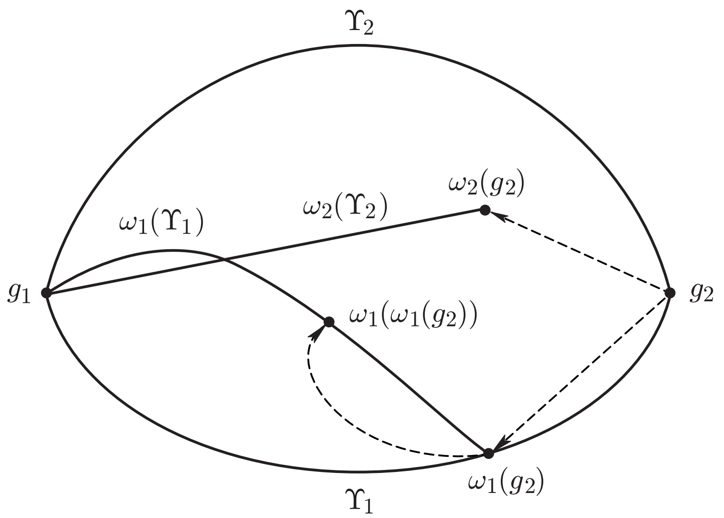


FIGURE 1.1. The domain $G$ with boundary $\partial G=\overline{\Upsilon}_1\cup\overline{\Upsilon}_2$ for $n=2$

It is shown in [9] that solutions of the problem (1.2), (1.3) may have power singularities near the points of $\mathcal{K}_1$. Therefore it is natural to consider (1.2), (1.3) in weighted spaces. We introduce the space $H_b^l(Q)$ as the completion of $C_0^\infty(\overline{Q}\setminus M)$ with respect to the norm

$$\|u\|_{H_b^l(Q)} = \Big(\sum_{|\alpha|\leqslant l}\int_Q \rho^{2(b-l+|\alpha|)}|D^\alpha u|^2\,dx\Big)^{1/2}.$$

Here $Q$ is either the domain $G$, the angle $\Omega$, or $\mathbb{R}^n$; $M=\mathcal{K}_1$ if $Q=G$, and $M=\{x=(y,z)\in\mathbb{R}^n: y=0,\ z\in\mathbb{R}^{n-2}\}$ if $Q=\Omega$ or $Q=\mathbb{R}^n$; $C_0^\infty(\overline{Q}\setminus M)$ is the set of infinitely differentiable functions with compact supports contained in $\overline{Q}\setminus M$; $l\geqslant 0$ is an integer; $b\in\mathbb{R}$; $\rho=\rho(x)\in C^\infty(\mathbb{R}^n\setminus\mathcal{K}_1)$ is a function[1] satisfying $c_1\operatorname{dist}(x,\mathcal{K}_1)\leqslant\rho(x)\leqslant c_2\operatorname{dist}(x,\mathcal{K}_1)$ ($x\in G$, $c_1,c_2>0$, and $\operatorname{dist}(x,\mathcal{K}_1)$ is the distance from $x$ to $\mathcal{K}_1$) if $Q=G$, and $\rho(x)=|y|$ if $Q=\Omega$ or $Q=\mathbb{R}^n$. For $l\geqslant 1$, we denote by $H_b^{l-1/2}(\Upsilon)$ the space of traces on a smooth $(n-1)$-dimensional manifold $\Upsilon\subset\overline{Q}$ with the norm

$$\|\psi\|_{H_b^{l-1/2}(\Upsilon)} = \inf\|u\|_{H_b^l(Q)},\qquad u\in H_b^l(Q):\ u|_\Upsilon=\psi.$$

We assume that $l+2m-m_{i\mu}-1\geqslant 0$ for all $i$, $\mu$ and introduce the following bounded operator corresponding to the non-local problem (1.2), (1.3):

$$\mathbf{L}=\{\mathbf{P}(x,D),\mathbf{B}_{i\mu}(x,D)\}:$$
$$H_b^{l+2m}(G)\to\mathcal{H}_b^l(G,\Upsilon)=H_b^l(G)\times\prod_{i=1}^{N_0}\prod_{\mu=1}^{m}H_b^{l+2m-m_{i\mu}-1/2}(\Upsilon_i).$$

[1] The existence of $\rho(x)$ follows from Theorem 2 in [24], Ch. 6, § 2.

From now on we suppose that $b > l+2m-1$ unless otherwise specified.

Let us explain the restriction on the exponent $b$. Suppose that the transformation $\omega_{is}$ takes a point $g \in \overline{\Upsilon}_i \cap \mathcal{K}_1$ to the point $\omega_{is}(g)$ such that $\omega_{is}(g) \in \mathcal{K}_2$ or $\omega_{is}(g) \in \mathcal{K}_3$. Since the function $u$ belongs to the Sobolev space $W_2^{l+2m}$ near $\omega_{is}(g)$, we see that the function $u(\omega_{is}(x))$ belongs to $W_2^{l+2m}$ near $g$. However, if $b \leqslant l+2m-1$, then $u(\omega_{is}(x))$ does not generally belong to the weighted space $H_b^{l+2m}$. Therefore the trace $(B_{i\mu s}(x,D)u)(\omega_{is}(x))|_{\Upsilon_i}$ may not belong to $H_b^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$, so the operator $\mathbf{L}$ is not well defined. But if $b > l+2m-1$, then we have $W_2^{l+2m}(G) \subset H_b^{l+2m}(G)$ by Lemma 5.2 of [12], and thus $\mathbf{L}$ is well defined.

We note that in the two-dimensional case one can consider (1.2), (1.3) in weighted spaces with arbitrary exponent $b$ (see [9]). To do this, one should impose some consistency conditions generated by the transformations $\omega_{is}$. Namely, one must assume that the solution $u$ and the right-hand side $\{f_0, g_{i\mu}\}$ belong to the corresponding weighted spaces not only near $\mathcal{K}_1$ but also near $\mathcal{K}_2$ and $\mathcal{K}_3$. On the one hand, this situation is thoroughly studied in [9] for transformations that are linear near $\mathcal{K}_1$. On the other hand, the changes described have nothing to do with the transformations $\omega_{is}$ near $\mathcal{K}_1$. Therefore we omit the proofs of the results for arbitrary $b$ in the two-dimensional case (see the end of § 5).

**2.** We now consider the structure of $\omega_{is}$ near $\mathcal{K}_1$ in more detail. We denote the transformation $\omega_{is}\colon \mathcal{O}_i \to \omega_{is}(\mathcal{O}_i)$ by $\omega_{is}^{+1}$, and let $\omega_{is}^{-1}\colon\omega_{is}(\mathcal{O}_i) \to \mathcal{O}_i$ be the inverse transformation. Consider a point $g \in \mathcal{K}_1$. The set of all points $\omega_{i_p s_p}^{\pm 1}(\dots\omega_{i_1 s_1}^{\pm 1}(g)) \in \mathcal{K}_1$ with $1 \leqslant s_j \leqslant S_{i_j}$, $j = 1,\dots,p$ (that is, all points which are obtained from $g$ by successive transformations $\omega_{i_j s_j}^{+1}$ or $\omega_{i_j s_j}^{-1}$ taking points of $\mathcal{K}_1$ to $\mathcal{K}_1$) is called the *orbit of* $g$ and is denoted by $\operatorname{Orb}(g)$.

We introduce the set $\mathcal{S}_{i1} = \{0 \leqslant s \leqslant S_i\colon \omega_{is}(\overline{\Upsilon}_i) \cap \mathcal{K}_1 \ne \varnothing\}$. Clearly, $0 \in \mathcal{S}_{i1}$. Suppose that the following conditions hold.

**Condition 1.3.** *For each* $g \in \mathcal{K}_1$,

a) *the set* $\operatorname{Orb}(g)$ *consists of finitely many points* $g^j$, $j = 1,\dots,N = N(g)$;

b) *the points* $g^j$ *have neighbourhoods*

$$\widehat{\mathcal{V}}(g^j) \subset \mathcal{V}(g^j) \subset \mathbb{R}^n \setminus \Big\{\bigcup_{i,s} \omega_{is}(\overline{\Upsilon}_i) \cup \mathcal{K}_2 \cup \mathcal{K}_3\Big\}, \qquad s \notin \mathcal{S}_{i1}$$

*such that* $\mathcal{V}(g^j) \cap \mathcal{V}(g^k) = \varnothing$ *for* $j \ne k$, *and if* $g^j \in \overline{\Upsilon}_i$ *and* $\omega_{is}(g^j) = g^k$, *then* $\mathcal{V}(g^j) \subset \mathcal{O}_i$ *and* $\omega_{is}(\widehat{\mathcal{V}}(g^j)) \subset \mathcal{V}(g^k)$.

**Condition 1.4.** *For each* $g \in \mathcal{K}_1$ *and each* $j = 1,\dots,N(g)$ *there is a non-degenerate smooth transformation* $x \mapsto x'(g,j)$ *mapping* $\mathcal{V}(g^j)$ $(\widehat{\mathcal{V}}(g^j))$ *onto a neighbourhood* $\mathcal{V}_j(0)$ $(\widehat{\mathcal{V}}_j(0))$ *of the origin such that the following properties hold.*

a) *The images of* $G \cap \mathcal{V}(g^j)$ $(G \cap \widehat{\mathcal{V}}(g^j))$ *and* $\Upsilon_i \cap \mathcal{V}(g^j)$ $(\Upsilon_i \cap \widehat{\mathcal{V}}(g^j))$ *are given respectively by the intersection of the dihedral angle* $\Omega_j = \{x = (y,z) \in \mathbb{R}^n : 0 < b_j' < \varphi < b_j'' < 2\pi,\ z \in \mathbb{R}^{n-2}\}$ *with* $\mathcal{V}_j(0)$ $(\widehat{\mathcal{V}}_j(0))$ *and the intersection of the side of* $\Omega_j$ *with* $\mathcal{V}_j(0)$ $(\widehat{\mathcal{V}}_j(0))$.

b) *For* $x \in \widehat{\mathcal{V}}(g^j)$ *the transformation* $\omega_{is}(x)$ *with* $s \in \mathcal{S}_{i1} \setminus \{0\}$ *is given in the new coordinates by* $(y', z') \mapsto (\omega'_{is}(y', z'), z')$, *where* $\omega'_{is}(y', z') = \mathcal{G}'_{is}y' + o(|x'|)$ *with* $\mathcal{G}'_{is}$ *being the operator of rotation by an angle* $\varphi'_{is}$ *followed by a dilation with coefficient* $\chi'_{is} > 0$ *in the* $y'$*-plane. We also assume that* $\omega'_{is}(0, z) \equiv 0$.

c) *In the new coordinate system, the operator* $\mathcal{G}'_{is}$ *maps the side of the corresponding angle* $\Omega_j$ $(j = j(i))$ *onto an* $(n-1)$*-dimensional half-plane lying strictly inside an angle* $\Omega_k$ (*where* $k = k(i, s)$ *may be different from* $j$).

Conditions 1.3 and 1.4 are analogous to those in [9], [11], where one studied transformations that are linear near $\mathcal{K}_1$ (and arbitrary outside a neighbourhood of $\mathcal{K}_1$).

Condition 1.3, a) is in a sense equivalent to Carleman's condition [4], which is used in the theory of non-local problems with transformations mapping the boundary of the domain onto itself.

Condition 1.4 means in particular that if $g \in \omega_{is}(\overline{\Upsilon}_i \setminus \Upsilon_i) \cap \overline{\Upsilon}_j \cap \mathcal{K}_1 \neq \varnothing$, then the surfaces $\omega_{is}(\overline{\Upsilon}_i)$ and $\overline{\Upsilon}_j$ have different tangent planes at $g$. The requirement $\omega'_{is}(0, z) \equiv 0$ is necessary for (1.1) to be possible. If $\omega_{is}(\overline{\Upsilon}_i \setminus \Upsilon_i) \subset \overline{G} \setminus \mathcal{K}_1$, then (similarly to [9], [11]) there are no restrictions on the geometrical structure of $\omega_{is}(\overline{\Upsilon}_i)$ near $\partial G$.

*Remark* 1.1. One can consider the more general case when, for $x \in \widehat{\mathcal{V}}(g^j)$, the transformation $\omega_{is}(x)$ with $s \in \mathcal{S}_{i1} \setminus \{0\}$ is given in the new coordinates by $(y', z') \mapsto (\omega'_{is}(y', z'), \omega''_{is}(y', z'))$, where $\omega'_{is}(y', z')$ is the same as before while $\omega''_{is}(y', z') = z' + o(|x'|)$ and $\omega''_{is}(0, z') \equiv z'$ (the latter condition guarantees that Condition 1.3 a) holds). However, for simplicity we study the transformations described by Condition 1.4.

**3.** Let us write model problems corresponding to the points of $\mathcal{K}_1$.

We fix a point $g \in \mathcal{K}_1$. Suppose that $\operatorname{supp} u \subset \big(\bigcup_{j=1}^{N(g)} \widehat{\mathcal{V}}(g^j)\big) \cap \overline{G}$. We denote the function $u(x)$ for $x \in \mathcal{V}(g^j) \cap G$ by $u_j(x)$. If $g^j \in \overline{\Upsilon}_i$, $x \in \widehat{\mathcal{V}}(g^j)$, and $\omega_{is}(x) \in \mathcal{V}(g^k)$, then we denote $u(\omega_{is}(x))$ by $u_k(\omega_{is}(x))$. Clearly, $u(\omega_{i0}(x)) \equiv u(x) \equiv u_j(x)$. The non-local problem (1.2), (1.3) takes the form

$$\begin{gathered}
\mathbf{P}(x, D)u_j = f_0(x), \qquad x \in \widehat{\mathcal{V}}(g^j) \cap G, \\
\sum_{s \in \mathcal{S}_{i1}} (B_{i\mu s}(x, D)u_k)(\omega_{is}(x))|_{\Upsilon_i} = g_{i\mu}(x), \\
x \in \widehat{\mathcal{V}}(g^j) \cap \Upsilon_i, \qquad i \in \{1 \leqslant i \leqslant N_0 \colon\ \widehat{\mathcal{V}}(g^j) \cap \Upsilon_i \neq \varnothing\}, \\
j = 1, \dots, N = N(g), \qquad \mu = 1, \dots, m.
\end{gathered}$$

By Condition 1.4, in the new coordinates, the linear part $\mathcal{G}'_{is}$ of the transformation $\omega'_{is}$ maps one of the sides of $\Omega_j$ (where $j = j(i)$) onto an $(n-1)$-dimensional half-plane that lies strictly inside $\Omega_k$, where $k = k(i, s)$ may be different from $j$. We denote all these $(n-1)$-dimensional half-planes by $\Gamma_{k2}, \dots, \Gamma_{k,R_k} \subset \Omega_k$. (If none of the sides of the angles $\Omega_1, \dots, \Omega_N$ is mapped inside $\Omega_k$, then we put $R_k = 1$.) We also put $b_{k1} = b'_k$, $b_{k,R_k+1} = b''_k$. Then the sets

$$\Gamma_{k\sigma} = \{x = (y, z) \in \mathbb{R}^n \colon\ \varphi = b_{k\sigma},\ z \in \mathbb{R}^{n-2}\}, \qquad \sigma = 1, R_k + 1,$$

are the sides of $\Omega_k$ while the half-planes $\Gamma_{kq}$ are given by

$$\Gamma_{kq} = \{x = (y, z) \in \mathbb{R}^n : \ \varphi = b_{kq}, \ z \in \mathbb{R}^{n-2}\}, \qquad q = 2, \dots, R_k,$$

where $0 < b_{k1} < \dots < b_{k,R_k+1} < 2\pi$.

Let us introduce the function $U_j(x') = u_j(x(x'))$ and denote $x'$ again by $x$. By Conditions 1.3 and 1.4, the problem (1.2), (1.3) takes the following final form:

$$\mathcal{P}_j(x, D_y, D_z)U_j = f_j(x), \qquad x \in \Omega_j, \tag{1.4}$$

$$\begin{aligned}\mathcal{B}_{j\sigma\mu}(x, D_y, D_z)U \equiv{}& B_{j\sigma\mu}(x, D_y, D_z)U_j|_{\Gamma_{j\sigma}} \\ &+ \sum_{k,q,s} (B_{j\sigma\mu kqs}(x, D_y, D_z)U_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} \\ ={}& g_{j\sigma\mu}(x), \qquad x \in \Gamma_{j\sigma}.\end{aligned} \tag{1.5}$$

Here and in what follows (unless otherwise stated) we have $j, k = 1, \dots, N$, $\sigma = 1, R_j + 1$, $q = 2, \dots, R_k$, $\mu = 1, \dots, m$, $s = 1, \dots, S_{j\sigma kq}$, $\mathcal{P}_j(x, D_y, D_z)$, $B_{j\sigma\mu}(x, D_y, D_z)$, and $B_{j\sigma\mu kqs}(x, D_y, D_z)$ are operators of orders $2m$, $m_{j\sigma\mu}$, and $m_{j\sigma\mu}$ respectively with variable $C^\infty$-coefficients, $\omega'_{j\sigma kqs}(y, z) = \mathcal{G}_{j\sigma kqs}y + o(|x|)$ with $\mathcal{G}_{j\sigma kqs}$ being the operator of rotation by an angle $\varphi_{j\sigma kq}$ and dilation by a number $\chi_{j\sigma kqs} > 0$ in the $y$-plane. Furthermore, $\omega'_{j\sigma kqs}(0, z) \equiv 0$ and $b_{k1} < b_{j\sigma} + \varphi_{j\sigma kq} = b_{kq} < b_{k,R_k+1}$.

We define the following spaces of vector-valued functions:

$$H_b^{l+2m,N}(\Omega) = \prod_j H_b^{l+2m}(\Omega_j), \qquad \mathcal{H}_b^{l,N}(\Omega, \Gamma) = \prod_j \mathcal{H}_b^l(\Omega_j, \Gamma_j),$$

$$\mathcal{H}_b^l(\Omega_j, \Gamma_j) = H_b^l(\Omega_j) \times \prod_{\sigma,\mu} H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma}).$$

We introduce bounded operators

$$\mathcal{L}^\omega = \{\mathcal{P}_j(D_y, D_z), \mathcal{B}^\omega_{j\sigma\mu}(D_y, D_z)\} : \ H_b^{l+2m,N}(\Omega) \to \mathcal{H}_b^{l,N}(\Omega, \Gamma),$$
$$\mathcal{L}^\mathcal{G} = \{\mathcal{P}_j(D_y, D_z), \mathcal{B}^\mathcal{G}_{j\sigma\mu}(D_y, D_z)\} : \ H_b^{l+2m,N}(\Omega) \to \mathcal{H}_b^{l,N}(\Omega, \Gamma).$$

Here[2]

$$\begin{aligned}\mathcal{B}^\omega_{j\sigma\mu}(D_y, D_z)U ={}& B_{j\sigma\mu}(D_y, D_z)U_j|_{\Gamma_{j\sigma}} \\ &+ \sum_{k,q,s} (B_{j\sigma\mu kqs}(D_y, D_z)U_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}},\\ \mathcal{B}^\mathcal{G}_{j\sigma\mu}(D_y, D_z)U ={}& B_{j\sigma\mu}(D_y, D_z)U_j|_{\Gamma_{j\sigma}} \\ &+ \sum_{k,q,s} (B_{j\sigma\mu kqs}(D_y, D_z)U_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}},\end{aligned}$$

[2]In what follows we consider functions $U_k$ that are compactly supported in a neighbourhood of the origin and we assume that $(\omega'_{j\sigma kqs}(y, z), z) \in \Omega_k$ for $x \in \operatorname{supp} U_k$. This guarantees that the operators $\mathcal{B}^\omega_{j\sigma\mu}(D_y, D_z)$ are well defined.

with $\mathcal{P}_j(D_y, D_z)$, $B_{j\sigma\mu}(D_y, D_z)$, and $B_{j\sigma\mu kqs}(D_y, D_z)$ being the principal homogeneous parts of the operators $\mathcal{P}_j(0, D_y, D_z)$, $B_{j\sigma\mu}(0, D_y, D_z)$, and $B_{j\sigma\mu kqs}(0, D_y, D_z)$ respectively.

In what follows we use $\mathcal{P}_j$, $B_{j\sigma\mu}$, $B_{j\sigma\mu kqs}$, $\mathcal{B}^{\omega}_{j\sigma\mu}$, and $\mathcal{B}^{\mathcal{G}}_{j\sigma\mu}$ as a short notation for $\mathcal{P}_j(D_y, D_z)$, $B_{j\sigma\mu}(D_y, D_z)$, $B_{j\sigma\mu kqs}(D_y, D_z)$, $\mathcal{B}^{\omega}_{j\sigma\mu}(D_y, D_z)$, and $\mathcal{B}^{\mathcal{G}}_{j\sigma\mu}(D_y, D_z)$ respectively.

We note that the non-local terms of the operator $\mathcal{B}^{\omega}_{j\sigma\mu}$ contain the non-linear transformations $\omega'_{j\sigma kqs}$ while the non-local terms of $\mathcal{B}^{\mathcal{G}}_{j\sigma\mu}$ contain the linear transformations $\mathcal{G}_{j\sigma kqs}$. Thus $\mathcal{L}^{\omega}$ and $\mathcal{L}^{\mathcal{G}}$ correspond to model problems with non-linear and linearized transformations respectively.

As mentioned above, the problem with transformations linear near $\mathcal{K}_1$ was studied in [9]–[11]. In particular, its Fredholm solubility was proved. In § 2 of the present paper we shall show that the operator $\mathcal{L}^{\omega}$ is neither small nor compact perturbation of $\mathcal{L}^{\mathcal{G}}$ even if we consider functions $U$ with arbitrarily small supports. Therefore, to prove the Fredholm solubility of the problem (1.2), (1.3) with non-linear transformations, we shall obtain new a priori estimates and construct a right regularizer. This will be done in § 4, 5.

**4.** The proof of a priori estimates and the construction of the right regularizer is based on invertibility of the model operator $\mathcal{L}^{\mathcal{G}}$. Let us formulate the conditions under which $\mathcal{L}^{\mathcal{G}}$ is an isomorphism. Along with the model operator in dihedral angles for $n \geqslant 3$, we also consider a model operator with a parameter $\theta$ in planar angles. For any angle $K = \{y \in \mathbb{R}^2 \colon 0 < b' < \varphi < b'' < 2\pi\}$ we define a space $E^l_b(K)$ as the completion of $C^{\infty}_0(\overline{K} \setminus \{0\})$ with respect to the norm

$$\|u\|_{E^l_b(K)} = \left( \sum_{|\alpha| \leqslant l} \int_K |y|^{2b}(|y|^{2(|\alpha|-l)} + 1)|D^{\alpha}_y u(y)|^2 \, dy \right)^{1/2}.$$

For $l \geqslant 1$ we denote by $E^{l-1/2}_b(\gamma)$ the space of traces on a ray $\gamma \subset \overline{K}$ with the norm

$$\|\psi\|_{E^{l-1/2}_b(\gamma)} = \inf \|u\|_{E^l_b(K)}, \qquad u \in E^l_b(K) \colon \ u|_{\gamma} = \psi.$$

(Equivalent constructive definitions of the trace spaces $H^{l-1/2}_b(\Upsilon)$ and $E^{l-1/2}_b(\gamma)$ are given in [25], § 1.)

We introduce the following spaces of vector-valued functions:

$$E^{l+2m,N}_b(K) = \prod_j E^{l+2m}_b(K_j), \qquad \mathcal{E}^{l,N}_b(K,\gamma) = \prod_j \mathcal{E}^l_b(K_j,\gamma_j),$$
$$\mathcal{E}^l_b(K_j,\gamma_j) = E^l_b(K_j) \times \prod_{\sigma,\mu} E^{l+2m-m_{j\sigma\mu}-1/2}_b(\gamma_{j\sigma}),$$

where $K_j = \{y \in \mathbb{R}^2 \colon b_{j1} < \varphi < b_{j,R_j+1}\}$ and $\gamma_{j\sigma} = \{y \in \mathbb{R}^2 \colon \varphi = b_{j\sigma}\}$.

We consider the bounded operator

$$\mathcal{L}^{\mathcal{G}}(\theta) = \{\mathcal{P}_j(D_y,\theta), \mathcal{B}^{\mathcal{G}}_{j\sigma\mu}(D_y,\theta)\} \colon E^{l+2m,N}_b(K) \to \mathcal{E}^{l,N}_b(K,\gamma),$$

where $\theta$ is an arbitrary point of the unit sphere $S^{n-3} = \{\theta \in \mathbb{R}^{n-2} \colon |\theta| = 1\}$.

**5.** Let us write the operators $\mathcal{P}_j(D_y,0)$, $B_{j\sigma\mu}(D_y,0)$, and $B_{j\sigma\mu kqs}(D_y,0)$ in the polar coordinates:

$$\mathcal{P}_j(D_y,0)=r^{-2m}\widetilde{\mathcal{P}}_j(\varphi,D_\varphi,rD_r),$$
$$B_{j\sigma\mu}(D_y,0)=r^{-m_{j\sigma\mu}}\widetilde{B}_{j\sigma\mu}(\varphi,D_\varphi,rD_r),$$
$$B_{j\sigma\mu kqs}(D_y,0)=r^{-m_{j\sigma\mu}}\widetilde{B}_{j\sigma\mu kqs}(\varphi,D_\varphi,rD_r),$$

where $D_\varphi=-i\frac{\partial}{\partial\varphi}$, $D_r=-i\frac{\partial}{\partial r}$. We consider an operator-valued function $\widetilde{\mathcal{L}}(\lambda)\colon W_2^{l+2m,N}(b_1,b_2)\to\mathcal{W}_2^{l,N}[b_1,b_2]$ given by

$$\widetilde{\mathcal{L}}^{\mathcal{G}}(\lambda)\widetilde{U}=\Big\{\widetilde{\mathcal{P}}_j(\varphi,D_\varphi,\lambda)\widetilde{U}_j,\ \widetilde{B}_{j\sigma\mu}(\varphi,D_\varphi,\lambda)\widetilde{U}_j(\varphi)|_{\varphi=b_{j\sigma}}$$
$$+\sum_{k,q,s}e^{(i\lambda-m_{j\sigma\mu})\ln\chi_{j\sigma kqs}}\widetilde{B}_{j\sigma\mu kqs}(\varphi,D_\varphi,\lambda)\widetilde{U}_k(\varphi+\varphi_{j\sigma kqs})|_{\varphi=b_{j\sigma}}\Big\},$$

where

$$W_2^{l+2m,N}(b_1,b_2)=\prod_j W_2^{l+2m}(b_{j1},b_{j,R_j+1}),$$
$$\mathcal{W}_2^{l,N}[b_1,b_2]=\prod_j \mathcal{W}_2^{l}[b_{j1},b_{j,R_j+1}],$$
$$\mathcal{W}_2^{l}[b_{j1},b_{j,R_j+1}]=W_2^{l}(b_{j1},b_{j,R_j+1})\times\mathbb{C}^{2m}.$$

By Lemmas 2.1, 2.2 of [10], there is a finite-meromorphic operator-valued function $(\widetilde{\mathcal{L}}^{\mathcal{G}})^{-1}(\lambda)$ such that $(\widetilde{\mathcal{L}}^{\mathcal{G}})^{-1}(\lambda)$ is inverse to $\widetilde{\mathcal{L}}^{\mathcal{G}}(\lambda)$ if $\lambda$ is not a pole of $(\widetilde{\mathcal{L}}^{\mathcal{G}})^{-1}(\lambda)$ and, furthermore, for every pole $\lambda_0$ there is $\delta>0$ such that the set $\{\lambda\in\mathbb{C}\colon 0<|\operatorname{Im}\lambda-\operatorname{Im}\lambda_0|<\delta\}$ contains no poles of $(\widetilde{\mathcal{L}}^{\mathcal{G}})^{-1}(\lambda)$.

If $n=2$, then Theorem 2.1 of [10] shows that $\mathcal{L}^{\mathcal{G}}$ is an isomorphism if and only if the line $\operatorname{Im}\lambda=b+1-l-2m$ contains no poles of $(\widetilde{\mathcal{L}}^{\mathcal{G}})^{-1}(\lambda)$.

Suppose that $n\geqslant 3$ and assume that the system $\{B_{j\sigma\mu}(D_y,D_z)\}_{\mu=1}^{m}$ is normal on $\Gamma_{j\sigma}$ and the orders $m_{j\sigma\mu}$ of the operators $B_{j\sigma\mu}(D_y,D_z)$, $B_{j\sigma\mu kqs}(D_y,D_z)$ are less than or equal to $2m-1$. Then Theorem 9.1 of [13] shows that the operator $\mathcal{L}^{\mathcal{G}}(\theta)$ is Fredholm if and only if the line $\operatorname{Im}\lambda=b+1-l-2m$ contains no poles of $(\widetilde{\mathcal{L}}^{\mathcal{G}})^{-1}(\lambda)$. By Theorem 3.3 of [10], if we also have $\dim\ker(\mathcal{L}^{\mathcal{G}}(\theta))=\operatorname{codim}\mathcal{R}(\mathcal{L}^{\mathcal{G}}(\theta))=0$ for $b$ replaced by $b-l$, $l$ replaced by 0, and for all $\theta\in S^{n-3}$, then the operator $\mathcal{L}^{\mathcal{G}}$ is an isomorphism for any $l$ (see the corresponding example in [13], § 10). We notice that if $\mathcal{L}^{\mathcal{G}}$ is not an isomorphism, then $\mathcal{L}^{\mathcal{G}}(\theta)$ is not Fredholm (see [13], Theorem 9.3).

Since the operators $\mathcal{L}^{\omega}$, $\mathcal{L}^{\mathcal{G}}$, $\mathcal{L}^{\mathcal{G}}(\theta)$, and $\widetilde{\mathcal{L}}^{\mathcal{G}}(\lambda)$ corresponding to the problem (1.4), (1.5) depend on the choice of $g\in\mathcal{K}_1$, we denote them by $\mathcal{L}^{\omega}_g$, $\mathcal{L}^{\mathcal{G}}_g$, $\mathcal{L}^{\mathcal{G}}_g(\theta)$, and $\widetilde{\mathcal{L}}^{\mathcal{G}}_g(\lambda)$ respectively.

## § 2. An example of non-local problem with non-linear argument transformation

In this section we show on a simple example that a problem having a transformation which is non-linear in a neighbourhood of $\mathcal{K}_1$ is neither small nor compact perturbation of the problem with the linearized transformation.

**1.** For simplicity we consider the problem (1.2), (1.3) in a planar domain. Let the model problem (1.4), (1.5) corresponding to some point of $\mathcal{K}_1$ have the form

$$\begin{aligned} \Delta u &= f(y), && y \in K, \\ u|_{\gamma_1} + u(\omega'(y))|_{\gamma_1} &= g_1(y), && y \in \gamma_1, \\ u|_{\gamma_2} &= g_2(y), && y \in \gamma_2. \end{aligned}$$

Here $K = \{y \in \mathbb{R}^2 : r > 0,\ |\varphi| < \pi/2\}$ is a planar angle (of opening $\pi$) with the sides $\gamma_i = \{y \in \mathbb{R}^2 : r > 0,\ \varphi = (-1)^i\pi/2\}$, $i = 1, 2$. We suppose that $\omega'(y) = \mu(\mathcal{G}y)$, where $\mathcal{G}$ is the operator of rotation by $\pi/2$ mapping $\gamma_1$ onto a ray $\gamma = \{y \in \mathbb{R}^2 : r > 0,\ \varphi = 0\}$, and

$$\mu\colon (y_1, y_2) \mapsto \left(\frac{y_1}{\sqrt{1+y_1^2}}, y_2 + \frac{y_1^2}{\sqrt{1+y_1^2}}\right)$$

is an infinitely differentiable transformation mapping $\gamma$ onto the curve $\mu(\gamma)$, which is tangent to $\gamma$ at the origin (see Fig. 2.1).

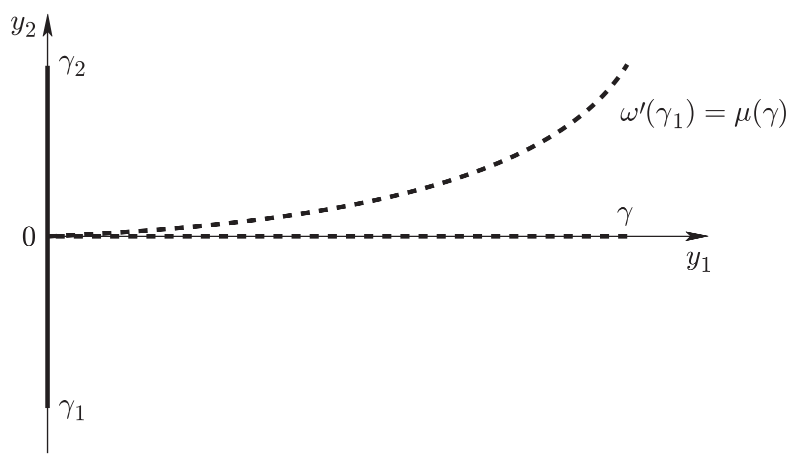


FIGURE 2.1. The angle $K$ of opening $\pi$

The operators $\mathcal{L}^\omega, \mathcal{L}^\mathcal{G} : H_b^{l+2}(K) \to H_b^l(K) \times \prod_{i=1}^2 H_b^{l+3/2}(\gamma_i)$ corresponding to the model problems with non-linear and linearized transformations have the form

$$\begin{aligned} \mathcal{L}^\omega u &= \big\{\Delta u,\ u|_{\gamma_1} + u(\omega'(y))|_{\gamma_1},\ u|_{\gamma_2}\big\}, \\ \mathcal{L}^\mathcal{G} u &= \big\{\Delta u,\ u|_{\gamma_1} + u(\mathcal{G}y)|_{\gamma_1},\ u|_{\gamma_2}\big\}. \end{aligned}$$

Clearly, the non-zero component of the difference $\mathcal{L}^\mathcal{G} u - \mathcal{L}^\omega u$ is

$$u(\mathcal{G}y)|_{\gamma_1} - u(\omega'(y))|_{\gamma_1} = u(y)|_\gamma - u(\mu(y))|_\gamma.$$

We introduce the operator $A_\varepsilon\colon H_b^{l+2}(K) \to H_b^{l+3/2}(\gamma)$ with domain $D(A_\varepsilon) = \big\{u \in H_b^{l+2}(K)\colon \operatorname{supp} u \subset \{r < \varepsilon\} \cap \overline{K}\big\}$ by the formula

$$A_\varepsilon u(y) = u(y)|_\gamma - u(\mu(y))|_\gamma.$$

Let us prove that one cannot make the operator $A_\varepsilon$ small or compact by choosing $\varepsilon$ sufficiently small. We shall do this in the case when $A_\varepsilon$ acts from $H_b^1(K)$ to $H_b^{1/2}(\gamma)$. The general case can be considered in the same way. We shall construct a sequence $u_\varepsilon \in D(A_\varepsilon)$, $\varepsilon \to 0$ such that

$$\big\|u_\varepsilon|_\gamma - u_\varepsilon(\mu(\,\cdot\,))|_\gamma\big\|_{H_b^{1/2}(\gamma)} \geqslant c\|u_\varepsilon\|_{H_b^1(K)},$$

where $c > 0$ is independent of $\varepsilon$.

We write the restriction of $\mu$ onto $\gamma$ in the polar coordinates $(\varphi, r)$ as

$$\mu|_\gamma\colon (0, r) \mapsto (\Phi(r), r),$$

where $\Phi(r) = \operatorname{arctg} r$. Clearly, $\Phi(0) = 0$, $\Phi(1) = \frac{\pi}{4}$, and $\frac{1}{\sqrt{2}} \leqslant \frac{\Phi}{r}, \frac{d\Phi}{dr} \leqslant 1$ on $[0, 1]$.

Let us consider the transformation

$$\widetilde{\mu}\colon (\varphi, r) \mapsto (\varphi + \Phi(r), r).$$

We see that $u(\mu(y))|_\gamma = u(\widetilde{\mu}(y))|_\gamma$ since $\mu|_\gamma = \widetilde{\mu}|_\gamma$. Therefore we may assume without loss of generality that $\mu$ is given by

$$\mu\colon (\varphi, r) \mapsto (\varphi + \Phi(r), r).$$

Notice that the the norm of any function $u \in H_b^1(K)$ written in the polar coordinates is equivalent to

$$\bigg(\sum_{|\alpha|\leqslant 1} \int_0^\infty \int_{-\pi/2}^{\pi/2} r^{2b-1}\big|(rD_r)^{\alpha_1} D_\varphi^{\alpha_2} u(\varphi, r)\big|^2\, d\varphi\, dr\bigg)^{1/2}.$$

Set $r = e^{-t}$. Then $\mu$ is given in the new coordinates $(\varphi, t)$ by

$$\mu\colon (\varphi, t) \mapsto (\varphi + \Phi(e^{-t}), t).$$

Putting $v(\varphi, t) = u(\varphi, e^{-t})$, we see that the norm $\|u\|_{H_b^1(K)}$ is equivalent to the norm

$$\|v\|_{W_{2,b}^1(Q)} = \bigg(\sum_{|\alpha|\leqslant 1} \int_{-\infty}^\infty \int_{-\pi/2}^{\pi/2} e^{-2bt}|D_t^{\alpha_1} D_\varphi^{\alpha_2} v(\varphi, t)|^2\, d\varphi\, dt\bigg)^{1/2}, \tag{2.1}$$

where $Q = \{t \in \mathbb{R}, |\varphi| < \pi/2\}$ and $W_{2,b}^1(Q)$ is the space with norm (2.1). Clearly, $W_{2,0}^1(Q)$ coincides with the Sobolev space $W_2^1(Q)$.

Since the norms $\|v\|_{W_{2,b}^1(Q)}$ and $\|e^{-bt}v\|_{W_2^1(Q)}$ are equivalent, it suffices to study the case when $b = 0$. In what follows we consider functions $v(\varphi, t)$ whose support is contained in the strip $\{|\varphi| < \pi/2\}$. Putting $v = 0$ for $|\varphi| \geqslant \pi/2$, we obtain $\|v\|_{W_2^1(Q)} = \|v\|_{W_2^1(\mathbb{R}^2)}$.

Our task is thus reduced to constructing a sequence $v_s \in W_2^1(\mathbb{R}^2)$ such that $\operatorname{supp} v_s \subset \{t > 2s, |\varphi| < \pi/2\}$ and

$$\|v_s(0,t) - v_s(\Phi(e^{-t}),t)\|_{W_2^{1/2}(\mathbb{R})} \geqslant c\|v_s\|_{W_2^1(\mathbb{R}^2)},$$

where $c > 0$ is independent of $s$.

To this end, we pass from the variables $(\varphi, t)$ to $(\varphi, \tau)$: we introduce the sets

$$Q_s = \left\{|\theta| \leqslant \frac{\pi}{2},\ 2s \leqslant \tau \leqslant 2s+1\right\}, \qquad s = 0, 1, 2, \dots,$$

and put

$$\varphi = F(\theta, \tau), \qquad t = \tau. \tag{2.2}$$

Here $F(\theta,\tau) = \theta e^{2s}\Phi(e^{-\tau})$ for $(\theta,\tau) \in Q_s$, $s = 0,1,2,\dots$, and $F(\theta,\tau)$ is extended onto $\mathbb{R}^2 \setminus \bigcup_{s=0}^{\infty} Q_s$ in such a way that the transformation (2.2) remains continuously differentiable and the Jacobian $\frac{\partial F}{\partial \theta}$ satisfies

$$0 < c_1 \leqslant \left|\frac{\partial F}{\partial \theta}\right| \leqslant c_2 \quad \text{on} \quad \mathbb{R}^2. \tag{2.3}$$

Such an extension does exist. Indeed,

$$\frac{\partial F}{\partial \theta} = e^{2s}\Phi(e^{-\tau}), \qquad \frac{\partial F}{\partial \tau} = -\theta e^{-\tau+2s}\left.\frac{d\Phi}{dr}\right|_{r=e^{-\tau}}, \qquad (\theta,\tau) \in Q_s.$$

Therefore the properties of $\Phi$ above show that the function $F(\theta,\tau)$ is continuously differentiable on $\bigcup_{s=0}^{\infty} Q_s$ with respect to $\theta$ and $\tau$ and inequalities (2.3) hold.

One easily sees that the change of variables (2.2) represents the interval $Q_s \cap \{\theta = 0\}$ by an interval of the line $\{\varphi = 0\}$. Furthermore, the transformation $\mu$ has the following form on $Q_s$:

$$\mu\colon (\theta,\tau) \mapsto (\theta + e^{-2s}, \tau), \qquad (\theta,\tau) \in Q_s. \tag{2.4}$$

We consider functions $f, g \in C^\infty(\mathbb{R})$ such that $\operatorname{supp} f \subset \{|\theta| < \frac{\pi}{2}\}$, $f(0) \neq f(1)$, $\operatorname{supp} g \subset \{0 < \tau < 1\}$, $g(\tau) \not\equiv 0$ and define a sequence $w_s(\theta,\tau) = f_s(\theta)g_s(\tau)$, where

$$f_s(\theta) = f(\theta e^{2s}), \qquad g_s(\tau) = g((\tau - 2s)e^{2s}), \qquad s = 0,1,2,\dots.$$

Clearly, $\operatorname{supp} w_s \subset Q_s$ (see Fig. 2.2).

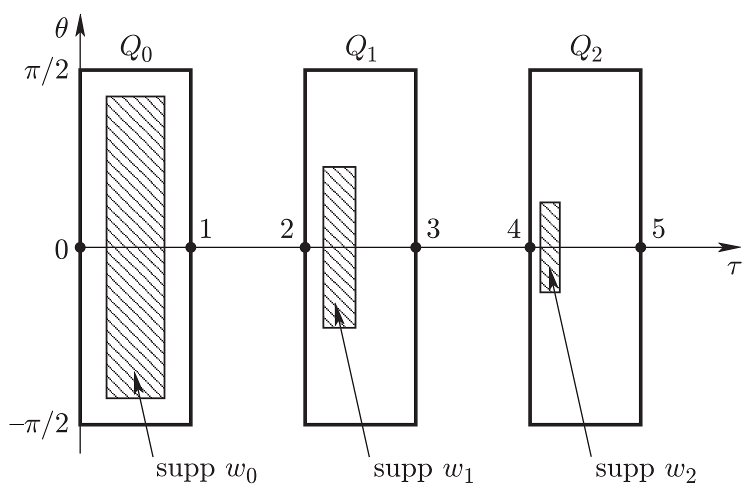


FIGURE 2.2. The supports of $w_s$ are contained in the hatched domains

We have

$$\|w_s\|^2_{W_2^1(\mathbb{R}^2)} = \|f_s\|^2_{L_2(\mathbb{R})}\|g_s\|^2_{L_2(\mathbb{R})} + \left\|\frac{df_s}{d\theta}\right\|^2_{L_2(\mathbb{R})}\|g_s\|^2_{L_2(\mathbb{R})} + \|f_s\|^2_{L_2(\mathbb{R})}\left\|\frac{dg_s}{d\tau}\right\|^2_{L_2(\mathbb{R})}$$
$$= e^{-4s}\|f\|^2_{L_2(\mathbb{R})}\|g\|^2_{L_2(\mathbb{R})} + \left\|\frac{df}{d\theta}\right\|^2_{L_2(\mathbb{R})}\|g\|^2_{L_2(\mathbb{R})} + \|f\|^2_{L_2(\mathbb{R})}\left\|\frac{dg}{d\tau}\right\|^2_{L_2(\mathbb{R})}. \qquad (2.5)$$

Since the norm is $W_2^{1/2}(\mathbb{R})$ is given by

$$\|g\|_{W_2^{1/2}(\mathbb{R})} = \left(\|g\|^2_{L_2(\mathbb{R})} + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|g(\tau_1)-g(\tau_2)|^2}{|\tau_1-\tau_2|^2}\,d\tau_1\,d\tau_2\right)^{1/2}$$

(see [26]) and $\mu$ takes the form (2.4) in the coordinates $(\theta,\tau)$, we get

$$\big\|w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}\big\|^2_{W_2^{1/2}(\mathbb{R})} = \big|f_s(0) - f_s(e^{-2s})\big|^2\|g_s\|^2_{W_2^{1/2}(\mathbb{R})}$$
$$\geqslant |f(0)-f(1)|^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|g(\tau_1)-g(\tau_2)|^2}{|\tau_1-\tau_2|^2}\,d\tau_1\,d\tau_2. \qquad (2.6)$$

It follows from (2.5) and (2.6) that

$$\big\|w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}\big\|^2_{W_2^{1/2}(\mathbb{R})} \geqslant c\|w_s\|^2_{W_2^1(\mathbb{R}^2)}.$$

**2.** Using the sequence $w_s$, one can easily show that, for any $\varepsilon$, the operator $A_\varepsilon$ is not compact. Indeed, the sequence $w_s$ is bounded in $W_2^1(\mathbb{R}^2)$. However the sequence $w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}$ contains no subsequences convergent in $W_2^{1/2}(\mathbb{R})$ because (2.6) shows that the expression

$$\big\|\big[w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}\big] - \big[w_h|_{\theta=0} - w_h(\mu(\cdot))|_{\theta=0}\big]\big\|_{W_2^{1/2}(\mathbb{R})}$$
$$= \big\|w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}\big\|_{W_2^{1/2}(\mathbb{R})} + \big\|w_h|_{\theta=0} - w_h(\mu(\cdot))|_{\theta=0}\big\|_{W_2^{1/2}(\mathbb{R})}$$

is bounded from below by a positive constant for all positive integers $s \neq h$.

## § 3. Argument transformations near the set $\mathcal{K}_1$

The results of § 2 show that proving the Fredholm solubility of problems with transformations non-linear near $\mathcal{K}_1$ requires obtaining new a priori estimates and constructing the right regularizer. To do this, we start by studying some properties of the transformations $\omega_{is}$ near the set $\mathcal{K}_1$.

We fix a point $g \in \mathcal{K}_1$, make the changes of variables $x \mapsto x'(g,j)$ for each $j = 1,\dots,N$, $N = N(g)$, and consider the transformations $\omega'_{j\sigma kqs}(y,z)$ for $(y,z) \in \mathcal{V}_{\varepsilon_0}(0) = \{x \in \mathbb{R}^n : |x| < \varepsilon_0\}$. The number $\varepsilon_0$ is supposed to be small so that $\overline{\mathcal{V}_{\varepsilon_0}(0)} \subset \widehat{\mathcal{V}}_j(0)$, $j = 1,\dots,N$. Some additional conditions on $\varepsilon_0$ will be imposed below.

**1.** Before we proceed to study the transformations $\omega_{is}$, let us establish an auxiliary result which will be used to prove the lemma on representation of $\omega_{is}$ in the polar coordinates (see Lemma 3.2).

**Lemma 3.1.** *Let $h = h(r, z)$ be a function such that $|D_r^k D_z^\alpha h| \leqslant c_{k\alpha}$ for $r \geqslant 0$, $z \in \mathbb{R}^{n-2}$, $(r^2 + |z|^2)^{1/2} \leqslant \varepsilon_0$. Set $f(r, z) = r^{-l} h(r, z)$ for some $l \in \mathbb{N}$ and assume that $|f| \leqslant c$. Then $|D_r^k f| \leqslant c_k$ for $r \geqslant 0$, $z \in \mathbb{R}^{n-2}$, $(r^2 + |z|^2)^{1/2} \leqslant \varepsilon_0$ and for all $k = 1, 2, \dots$ .*

*Proof.* 1) Consider the case when $l = 1$, that is, $f(r, z) = r^{-1} h(r, z)$. By Leibnitz' formula,

$$\frac{\partial^k f(r,z)}{\partial r^k} = \sum_{s=0}^{k} \frac{(-1)^s k!}{(k-s)!} r^{-s-1} \frac{\partial^{k-s} h(r,z)}{\partial r^{k-s}}.$$

Expanding $\frac{\partial^{k-s} h}{\partial r^{k-s}}$ by the Taylor formula near $r = 0$ and using the boundedness of the derivatives of $h$, we obtain

$$\begin{aligned}\frac{\partial^k f(r,z)}{\partial r^k} &= \sum_{s=0}^{k} \frac{(-1)^s k!}{(k-s)!} r^{-s-1} \left[ \sum_{p=0}^{s} \frac{1}{p!} \frac{\partial^{k-s+p} h}{\partial r^{k-s+p}}(0, z) r^p + \frac{\partial^{k+1} h}{\partial r^{k+1}}(\varkappa_{rz} r, z) r^{s+1} \right] \\ &= \sum_{s=0}^{k} \sum_{p=0}^{s} \frac{(-1)^s k!}{(k-s)!\, p!} \frac{\partial^{k-s+p} h}{\partial r^{k-s+p}}(0, z) r^{-s-1+p} + O(1),\end{aligned}$$

where $\varkappa_{rz} \in (0, 1)$.

Putting $p' = s - p$ in the last sum and denoting $p'$ again by $p$, we get

$$\frac{\partial^k f(r,z)}{\partial r^k} = \sum_{s=0}^{k} \sum_{p=0}^{s} \frac{(-1)^s k!}{(k-s)!\,(s-p)!} \frac{\partial^{k-p} h}{\partial r^{k-p}}(0, z) r^{-p-1} + O(1).$$

Consider the coefficient $a_p(z)$ at $r^{-p-1}$ on the right-hand side of the last identity:

$$\begin{aligned} a_p(z) &= \frac{\partial^{k-p} h}{\partial r^{k-p}}(0, z) \sum_{s=p}^{k} \frac{(-1)^s k!}{(k-s)!\,(s-p)!} \\ &= \frac{\partial^{k-p} h}{\partial r^{k-p}}(0, z)(-1)^p \sum_{s=0}^{k-p} k(k-1)\cdots(k-(s+p)+1)\frac{1}{s!}(-1)^s, \\ & p = 0, \dots, k. \end{aligned}$$

Since $|r^{-1} h(r, z)| \leqslant c$ by assumption, we have $h(0, z) \equiv 0$, whence $a_k(z) \equiv 0$. On the other hand, notice that, for $0 \leqslant p < k$, we have

$$\begin{aligned} 0 = \frac{d^p}{dt^p}(t+1)^k\Big|_{t=-1} &= \left( \sum_{s=0}^{k-p} k(k-1)\cdots(k-(s+p)+1)\frac{1}{s!} t^s \right)\Bigg|_{t=-1} \\ &= \sum_{s=0}^{k-p} k(k-1)\cdots(k-(s+p)+1)\frac{1}{s!}(-1)^s. \end{aligned}$$

Thus $a_p(z) \equiv 0$ for all $p = 0, \dots, k$, and the lemma is proved for $l = 1$.

2) If $l \geqslant 2$, we use the induction. Let the lemma be true for $l = 1, \dots, l_1 - 1$. We claim that it is true for $l = l_1$. Indeed, we have $f = r^{-1} f_1$, where $f_1 = r^{-(l_1-1)} h$. Since $|f| \leqslant c$, it follows that $|f_1| \leqslant c$ and, therefore, the estimate $|D_r^k D_z^\alpha f_1| \leqslant c_{k\alpha}$ holds by the inductive assumption (for $l = l_1 - 1$). Applying the inductive assumption once more (now with $l = 1$), we get the conclusion of the lemma for $r^{-1} f_1$, that is, for $f = r^{-l_1} h$. The lemma is proved.

We now proceed to study the transformations $\omega$. The following lemma describes the structure of $\omega'_{j\sigma kqs}$ in the cylindrical coordinates. This representation turns out to be convenient when we study non-local problems in weighted spaces.

**Lemma 3.2.** *For sufficiently small $\varepsilon_0$, the transformation*

$$\omega'_{j\sigma kqs}(y, z)|_{\Gamma_{j\sigma} \cap \mathcal{V}_{\varepsilon_0}(0)}$$

*can be represented in the polar coordinates as*

$$(b_{j\sigma}, r) \mapsto \big(b_{kq} + \Phi_{j\sigma kqs}(r, z),\ \chi_{j\sigma kqs} r + R_{j\sigma kqs}(r, z)\big), \quad (r^2 + |z|^2)^{1/2} \leqslant \varepsilon_0, \tag{3.1}$$

*where $\Phi_{j\sigma kqs}(r, z)$ and $R_{j\sigma kqs}(r, z)$ are infinitely differentiable functions such that*

$$|\Phi_{j\sigma kqs}| \leqslant c\varepsilon_0, \qquad |R_{j\sigma kqs}| \leqslant c\varepsilon_0 r, \tag{3.2}$$

$$|D_r^k D_z^\alpha \Phi_{j\sigma kqs}| \leqslant c_{k\alpha}, \qquad |D_r^k D_z^\alpha (R_{j\sigma kqs}/r)| \leqslant c_{k\alpha}. \tag{3.3}$$

*Here $k + |\alpha| \geqslant 1$, and $c, c_{k\alpha} > 0$ are independent of $\varepsilon_0$.*

*Proof.* Write $\omega'_{j\sigma kqs}(y, z) = \big(\omega^1_{j\sigma kqs}(y, z), \omega^2_{j\sigma kqs}(y, z)\big)$. By Condition 1.4, we have $\omega^i_{j\sigma kqs}(0, z) \equiv 0$, $i = 1, 2$. Therefore the Taylor formula near $r = 0$ implies that

$$\begin{aligned} &\omega^i_{j\sigma kqs}(r\cos b_{j\sigma}, r\sin b_{j\sigma}, z) \\ &\qquad = \left(\frac{\partial \omega^i_{j\sigma kqs}}{\partial y_1}(0, z)\cos b_{j\sigma} + \frac{\partial \omega^i_{j\sigma kqs}}{\partial y_2}(0, z)\sin b_{j\sigma}\right) r + O(r^2). \end{aligned} \tag{3.4}$$

Here $O(r^2)$ is a function whose absolute value is majorized by $cr^2$, where $c$ is independent of $r$ and $z$. (To verify this, one should write the remainder of the Taylor formula in Lagrange's form and use the smoothness of $\omega^i_{j\sigma kqs}$.) Expanding $\frac{\partial \omega^i_{j\sigma kqs}}{\partial y_1}(0, z)$ and $\frac{\partial \omega^i_{j\sigma kqs}}{\partial y_2}(0, z)$ by the Taylor formula near $z = 0$, we see from (3.4) that

$$\omega^i_{j\sigma kqs} = \left(\frac{\partial \omega^i_{j\sigma kqs}}{\partial y_1}(0)\cos b_{j\sigma} + \frac{\partial \omega^i_{j\sigma kqs}}{\partial y_2}(0)\sin b_{j\sigma}\right) r + O(|z|) r + O(r^2). \tag{3.5}$$

Notice that

$$\frac{\partial \omega^1_{j\sigma kqs}}{\partial y_1}(0)\cos b_{j\sigma} + \frac{\partial \omega^1_{j\sigma kqs}}{\partial y_2}(0)\sin b_{j\sigma} \quad \text{and} \quad \frac{\partial \omega^2_{j\sigma kqs}}{\partial y_1}(0)\cos b_{j\sigma} + \frac{\partial \omega^2_{j\sigma kqs}}{\partial y_2}(0)\sin b_{j\sigma}$$

are not simultaneously equal to zero. (This follows from the non-degeneracy of the Jacobian of the transformation $(y,z)\mapsto(\omega'_{j\sigma kqs}(y,z),z)$ at the origin.) To be definite, we assume that

$$\frac{\partial\omega^1_{j\sigma kqs}}{\partial y_1}(0)\cos b_{j\sigma}+\frac{\partial\omega^1_{j\sigma kqs}}{\partial y_2}(0)\sin b_{j\sigma}\neq 0. \tag{3.6}$$

Hence, by (3.5), we have

$$\omega^1_{j\sigma kqs}\neq 0\quad\text{for}\quad (r^2+|z|^2)^{1/2}\leqslant\varepsilon_0 \tag{3.7}$$

with $\varepsilon_0$ small enough, and the transformation $\omega'_{j\sigma kqs}|_{\Gamma_{j\sigma}\cap\mathcal{V}_{\varepsilon_0}(0)}$ is given in the polar coordinates by

$$(b_{j\sigma},r)\mapsto\left(\operatorname{arctg}\frac{\omega^2_{j\sigma kqs}}{\omega^1_{j\sigma kqs}}+\pi l,\ \sqrt{\sum_{i=1}^{2}(\omega^i_{j\sigma kqs})^2}\right), \tag{3.8}$$

where $l=0$ if $\omega^1_{j\sigma kqs}>0$ and $\omega^2_{j\sigma kqs}\geqslant 0$, $l=1$ if $\omega^1_{j\sigma kqs}<0$, $l=2$ if $\omega^1_{j\sigma kqs}>0$ and $\omega^2_{j\sigma kqs}<0$.

It follows from (3.5) and the Taylor formula that

$$\operatorname{arctg}\frac{\omega^2_{j\sigma kqs}}{\omega^1_{j\sigma kqs}}=\operatorname{arctg}\frac{\frac{\partial\omega^2_{j\sigma kqs}}{\partial y_1}(0)\cos b_{j\sigma}+\frac{\partial\omega^2_{j\sigma kqs}}{\partial y_2}(0)\sin b_{j\sigma}}{\frac{\partial\omega^1_{j\sigma kqs}}{\partial y_1}(0)\cos b_{j\sigma}+\frac{\partial\omega^1_{j\sigma kqs}}{\partial y_2}(0)\sin b_{j\sigma}}+O(|z|)+O(r),$$

$$\sqrt{\sum_{i=1}^{2}(\omega^i_{j\sigma kqs})^2}=r\sqrt{\sum_{i=1}^{2}\left(\frac{\partial\omega^i_{j\sigma kqs}}{\partial y_1}(0)\cos b_{j\sigma}+\frac{\partial\omega^i_{j\sigma kqs}}{\partial y_2}(0)\sin b_{j\sigma}\right)^2}$$
$$+O(|z|)r+O(r^2).$$

Setting

$$b_{kq}=\operatorname{arctg}\frac{\frac{\partial\omega^2_{j\sigma kqs}}{\partial y_1}(0)\cos b_{j\sigma}+\frac{\partial\omega^2_{j\sigma kqs}}{\partial y_2}(0)\sin b_{j\sigma}}{\frac{\partial\omega^1_{j\sigma kqs}}{\partial y_1}(0)\cos b_{j\sigma}+\frac{\partial\omega^1_{j\sigma kqs}}{\partial y_2}(0)\sin b_{j\sigma}}+\pi l,$$

$$\chi_{j\sigma kqs}=\sqrt{\sum_{i=1}^{2}\left(\frac{\partial\omega^i_{j\sigma kqs}}{\partial y_1}(0)\cos b_{j\sigma}+\frac{\partial\omega^i_{j\sigma kqs}}{\partial y_2}(0)\sin b_{j\sigma}\right)^2},$$

we get formula (3.1) and inequalities (3.2).

Let us prove the first inequality in (3.3). Using (3.7), we have

$$\left|\frac{\omega^2_{j\sigma kqs}}{\omega^1_{j\sigma kqs}}\right|\leqslant c\quad\text{for}\quad (r^2+|z|^2)^{1/2}\leqslant\varepsilon_0.$$

Therefore, by (3.1) and (3.8), it suffices to prove that the derivatives $D_r^k D_z^\alpha \frac{\omega^2_{j\sigma kqs}}{\omega^1_{j\sigma kqs}}$ are bounded. Clearly, we have

$$\frac{\omega^2_{j\sigma kqs}}{\omega^1_{j\sigma kqs}} = \frac{r^{-1}\omega^2_{j\sigma kqs}}{r^{-1}\omega^1_{j\sigma kqs}}.$$

It follows from (3.5) and (3.6) that $r^{-1}\omega^1_{j\sigma kqs} \neq 0$ for $(r^2+|z|^2)^{1/2} \leqslant \varepsilon_0$. Hence it suffices to prove that

$$\left|D_r^k D_z^\alpha (r^{-1}\omega^i_{j\sigma kqs})\right| = \left|D_r^k (r^{-1} D_z^\alpha \omega^i_{j\sigma kqs})\right| \leqslant c_{k\alpha}, \qquad i = 1, 2.$$

But the function $D_z^\alpha \omega^i_{j\sigma kqs}$ is infinitely differentiable for $(r^2+|z|^2)^{1/2} \leqslant \varepsilon_0$. Since $\omega^i_{j\sigma kqs}(0, z) \equiv 0$, we have $D_z^\alpha \omega^i_{j\sigma kqs} = O(r)$. Therefore $|r^{-1} D_z^\alpha \omega^i_{j\sigma kqs}| \leqslant c_\alpha$. Now the conclusion of the lemma follows from Lemma 3.1.

One can similarly prove the second inequality in (3.3). It follows from (3.1) and (3.8) that

$$\frac{R_{j\sigma kqs}(r,z)}{r} = \sqrt{\sum_{i=1}^2 \frac{(\omega^i_{j\sigma kqs})^2}{r^2}} - \chi_{j\sigma kqs}.$$

Using (3.5) and (3.6), we see that $\sum_{i=1}^2 (\omega^i_{j\sigma kqs})^2/r^2 \neq 0$ for $(r^2+|z|^2)^{1/2} \leqslant \varepsilon_0$. Therefore it suffices to prove that

$$\left| D_r^k D_z^\alpha \sum_{i=1}^2 \frac{(\omega^i_{j\sigma kqs})^2}{r^2} \right| \leqslant c_{k\alpha}.$$

But the function $D_z^\alpha \sum_{i=1}^2 (\omega^i_{j\sigma kqs})^2$ is infinitely differentiable for $(r^2+|z|^2)^{1/2} \leqslant \varepsilon_0$. Since $\omega^i_{j\sigma kqs}(0, z) \equiv 0$, we get $D_z^\alpha \sum_{i=1}^2 (\omega^i_{j\sigma kqs})^2 = O(r^2)$. Hence $\left|D_z^\alpha \sum_{i=1}^2 (\omega^i_{j\sigma kqs})^2/r^2\right| \leqslant c_\alpha$, and the conclusion of the lemma follows from Lemma 3.1. The lemma is proved.

**2.** We put $\delta = \min\{b_{j,q+1} - b_{jq}\}/2$, $j = 1, \dots, N$, $q = 1, \dots, R_j$, $d_1 = \min\{1, \chi_{j\sigma kqs}\}/2$, and $d_2 = 2\max\{1, \chi_{j\sigma kqs}\}$. Let $\varepsilon_0$ be small such that

$$|\Phi_{j\sigma kqs}| \leqslant \delta/2, \qquad |R_{j\sigma kqs}| \leqslant \chi_{j\sigma kqs} r/2 \quad \text{for} \quad (r^2+|z|^2)^{1/2} \leqslant \varepsilon_0/d_1. \tag{3.9}$$

The existence of such an $\varepsilon_0$ follows from Lemma 3.2.

We introduce infinitely differentiable functions $\zeta_{j\sigma,i}(\varphi)$, $\zeta_{kq,i}(\varphi)$ such that

$$\begin{aligned} &\zeta_{j\sigma,i}(\varphi) = 1 \quad \text{for} \quad |b_{j\sigma} - \varphi| \leqslant \delta/2^{i+1}, \\ &\zeta_{j\sigma,i}(\varphi) = 0 \quad \text{for} \quad |b_{j\sigma} - \varphi| \geqslant \delta/2^{i}, \\ &\zeta_{kq,i}(\varphi) = \zeta_{j\sigma,i}(\varphi - \varphi_{j\sigma kq}), \qquad i = 0, \dots, 4. \end{aligned} \tag{3.10}$$

Clearly, $\zeta_{kq,i}(\varphi) = 1$ for $|b_{kq} - \varphi| \leqslant \delta/2^{i+1}$, and $\zeta_{kq,i}(\varphi) = 0$ for $|b_{kq} - \varphi| \geqslant \delta/2^{i}$.

We consider the transformations $\widetilde{\omega}'_{j\sigma kqs}(y,z)$ that are given in the polar coordinates by

$$(\varphi, r) \mapsto \big(\varphi + \varphi_{j\sigma kq} + \Phi_{j\sigma kqs}(r,z),\ \chi_{j\sigma kqs} r + R_{j\sigma kqs}(r,z)\big). \tag{3.11}$$

Lemma 3.2 implies that

$$\widetilde{\omega}'_{j\sigma kqs}(y,z)|_{\Gamma_{j\sigma}\cap\mathcal{V}_{\varepsilon_0}(0)} = \omega'_{j\sigma kqs}(y,z)|_{\Gamma_{j\sigma}\cap\mathcal{V}_{\varepsilon_0}(0)}.$$

Hence we can assume in what follows that the transformation $\omega'_{j\sigma kqs}(y,z)$ is given by (3.11). We notice that $\omega'_{j\sigma kqs}(y,z)$ may now have a singularity at the origin since the new transformation $\omega'_{j\sigma kqs}(y,z)$ coincides with the old one $\omega'_{j\sigma kqs}(y,z)$ only on $\Gamma_{j\sigma}\cap\mathcal{V}_{\varepsilon_0}(0)$.

For any function $W(y,z)$ we put $\widehat{W}(y,z) = W(\omega'_{j\sigma kqs}(\mathcal{G}^{-1}_{j\sigma kqs}y, z), z)$. By Lemma 3.2, the transformation $\omega'_{j\sigma kqs}(\mathcal{G}^{-1}_{j\sigma kqs}y,z)$ is given in the polar coordinates by

$$(\varphi, r) \mapsto \big(\varphi + \Phi'_{j\sigma kqs}(r,z),\ r + R'_{j\sigma kqs}(r,z)\big), \tag{3.12}$$

where $\Phi'_{j\sigma kqs}(r,z) = \Phi_{j\sigma kqs}(\chi^{-1}_{j\sigma kqs}r, z)$ and $R'_{j\sigma kqs}(r,z) = R_{j\sigma kqs}(\chi^{-1}_{j\sigma kqs}r,z)$. It is easy to see that $\Phi'_{j\sigma kqs}$ and $R'_{j\sigma kqs}$ also satisfy (3.2), (3.3).

**Lemma 3.3.** *For all sufficiently small $\varepsilon_0$ and any $W \in H^l_b(\Omega_k)$ with* $\operatorname{supp} W \subset \overline{\Omega}_k \cap \mathcal{V}_{\varepsilon_0}(0)$ *we have $\zeta_{kq,1}\widehat{W} \in H^l_b(\Omega_k)$ and*

$$\|\zeta_{kq,1}\widehat{W}\|_{H^l_b(\Omega_k)} \leqslant c\|W\|_{H^l_b(\Omega_k)},$$

*where $q = 2, \dots, R_k$. Here $c > 0$ is independent of $W$ and $\varepsilon_0$.*

*Proof.* We shall use the following obvious assertion:

$$W \in H^l_b(\Omega_k) \qquad \Longleftrightarrow \qquad D^\alpha W \in H^0_{b+|\alpha|-l}(\Omega_k), \quad |\alpha| \leqslant l. \tag{3.13}$$

We see from formula (3.12) and inequalities (3.9) that the transformation (3.12) maps $\overline{\mathcal{V}_{\varepsilon_0}(0)} \cap \{x\colon |\varphi - b_{kq}| < \delta\} \cap \Omega_k$ into $\Omega_k$ for $q = 2, \dots, R_k$. Furthermore, inequalities (3.2) and (3.3) imply that, for small $\varepsilon_0$, the absolute value of the Jacobian of (3.12) is bounded and does not vanish in $\overline{\mathcal{V}_{\varepsilon_0}(0)} \cap \{x\colon |\varphi - b_{kq}| < \delta\} \cap \Omega_k$. This proves the lemma for $l = 0$ and with $\zeta_{kq,1}$ replaced by $\zeta_{kq,0}$.

Let us consider functions $\zeta^p_{kq,0} \in C^\infty_0(\mathbb{R})$ $(p = 0, \dots, l)$ such that $\zeta^0_{kq,0} = \zeta_{kq,0}$, $\zeta^l_{kq,0} = \zeta_{kq,1}$, and $\zeta^{p-1}_{kq,0}(\varphi) = 1$ for $\varphi \in \operatorname{supp}\zeta^p_{kq,0}$, $p = 1, \dots, l$. We assume that the lemma holds for $l = p - 1$ with $\zeta_{kq,1}$ replaced by $\zeta^{p-1}_{kq,0}$. We claim that it holds for $l = p$ with $\zeta_{kq,1}$ replaced by $\zeta^p_{kq,0}$, $p \geqslant 1$.

Indeed, suppose that $W \in H_b^p(\Omega_k)$. Then

$$\frac{1}{r}\frac{\partial W}{\partial \varphi},\ \frac{\partial W}{\partial r},\ \frac{\partial W}{\partial z_\xi} \in H_b^{p-1}(\Omega_k), \qquad \xi = 1, \dots, n-2.$$

Hence the induction assumption yields that

$$\zeta_{kq,0}^{p-1}\left(\widehat{\frac{1}{r}\frac{\partial W}{\partial \varphi}}\right),\ \zeta_{kq,0}^{p-1}\widehat{\frac{\partial W}{\partial r}},\ \zeta_{kq,0}^{p-1}\widehat{\frac{\partial W}{\partial z_\xi}} \in H_b^{p-1}(\Omega_k).$$

Combining this with the formulas

$$\begin{aligned}
\frac{1}{r}\frac{\partial \widehat{W}_k}{\partial \varphi} &= \left(\widehat{\frac{1}{r}\frac{\partial W}{\partial \varphi}}\right)\left(1 + \frac{R'_{j\sigma qks}}{r}\right),\\
\frac{\partial \widehat{W}_k}{\partial r} &= \left(\widehat{\frac{1}{r}\frac{\partial W}{\partial \varphi}}\right)\left(1 + \frac{R'_{j\sigma kqs}}{r}\right) r\frac{\partial \Phi'_{j\sigma kqs}}{\partial r} + \widehat{\frac{\partial W}{\partial r}}\left(1 + \frac{\partial R'_{j\sigma kqs}}{\partial r}\right),\\
\frac{\partial \widehat{W}_k}{\partial z_\xi} &= \left(\widehat{\frac{1}{r}\frac{\partial W}{\partial \varphi}}\right)\left(1 + \frac{R'_{j\sigma kqs}}{r}\right) r\frac{\partial \Phi'_{j\sigma kqs}}{\partial z_\xi} + \widehat{\frac{\partial W}{\partial r}}\frac{\partial R'_{j\sigma kqs}}{\partial z_\xi} + \widehat{\frac{\partial W}{\partial z_\xi}},
\end{aligned} \tag{3.14}$$

inequalities (3.2), (3.3) and Lemma 2.1 of [27] , we get[3]

$$\zeta_{kq,0}^{p-1}\frac{1}{r}\frac{\partial \widehat{W}}{\partial \varphi},\ \zeta_{kq,0}^{p-1}\frac{\partial \widehat{W}}{\partial r},\ \zeta_{kq,0}^{p-1}\frac{\partial \widehat{W}}{\partial z_\xi} \in H_b^{p-1}(\Omega_k). \tag{3.15}$$

Using the inclusion $W \in H_b^p(\Omega_k)$, the embedding $H_b^p(\Omega_k) \subset H_{b-p}^0(\Omega_k)$, and the conclusion of the lemma for $l = 0$, we see that $\zeta_{kq,0}^p \widehat{W} \in H_{b-p}^0(\Omega_k)$. Together with (3.13) and (3.15), this implies that $D^\alpha(\zeta_{kq,0}^p\widehat{W}) \in H_{b+|\alpha|-p}^0(\Omega_k)$, $|\alpha| \leqslant p$. Using (3.13) again, we prove the lemma.

Thus we have proved that the operator $W \mapsto \zeta_{kq,1}\widehat{W}$ is bounded in $H_b^l(\Omega_k)$.

**Lemma 3.4.** *The following inequality holds for any $W \in H_b^l(\Omega_k)$ with* $\operatorname{supp} W \subset \overline{\Omega}_k \cap \mathcal{V}_{\varepsilon_0}(0)$ *and all multi-indices $\gamma$ with $1 \leqslant |\gamma| \leqslant l$:*

$$\|\zeta_{kq,2}D^\gamma\widehat{W} - \zeta_{kq,2}\widehat{D^\gamma W}\|_{H_b^{l-|\gamma|}(\Omega_k)} \leqslant c\varepsilon_0\|W\|_{H_b^l(\Omega_k)}, \tag{3.16}$$

*where $q = 2, \dots, R_k$, and $c > 0$ is independent of $W$ and $\varepsilon_0$.*

*Proof.* We introduce functions $\zeta_{kq,1}^p \in C_0^\infty(\mathbb{R})$ $(p = 1, \dots, l)$ such that $\zeta_{kq,1}^1 = \zeta_{kq,1}$, $\zeta_{kq,1}^l = \zeta_{kq,2}$, and $\zeta_{kq,1}^{p-1}(\varphi) = 1$ for $\varphi \in \operatorname{supp}\zeta_{kq,1}^p$, $p = 2, \dots, l$.

[3] Lemma 2.1 of [27] and Lemmas 2.2, 3.5, 3.6 of [27], which are used below, are proved by Kondrat'ev for domains with angular or conical points. However, it is easy to see that they remain valid for domains with edges that are considered here.

Suppose that $|\gamma|=1$. Then it suffices to prove (3.16) with $D^\gamma$ replaced by any of the operators $\frac{1}{r}\frac{\partial}{\partial\varphi}$, $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial z_\xi}$. We consider the operator $\frac{1}{r}\frac{\partial}{\partial\varphi}$. (The other choices are treated in the same way.) Combining the first formula (3.14) with Leibnitz' formula, we get

$$\begin{aligned}\Big\|\zeta^1_{kq,1}\frac{1}{r}\frac{\partial\widehat{W}}{\partial\varphi}-\zeta^1_{kq,1}\Big(\widehat{\frac{1}{r}\frac{\partial W}{\partial\varphi}}\Big)\Big\|^2_{H_b^{l-1}(\Omega_k)}&=\Big\|\zeta^1_{kq,1}\Big(\widehat{\frac{1}{r}\frac{\partial W}{\partial\varphi}}\Big)\frac{R'_{j\sigma qks}}{r}\Big\|^2_{H_b^{l-1}(\Omega_k)}\\ &\leqslant k_1\sum_{|\alpha|\leqslant l-1}\sum_{|\beta|\leqslant|\alpha|}\int_{\Omega_k}r^{2(b+|\alpha|-(l-1))}\Big|D^{\alpha-\beta}\frac{R'_{j\sigma qks}}{r}\Big|^2\\ &\qquad\times\Big|D^\beta\Big(\zeta^1_{kq,1}\Big(\widehat{\frac{1}{r}\frac{\partial W}{\partial\varphi}}\Big)\Big)\Big|^2\,dx.\end{aligned}$$

Using this along with the last inequalities of (3.2) and (3.3), we obtain

$$\Big\|\zeta^1_{kq,1}\frac{1}{r}\frac{\partial\widehat{W}}{\partial\varphi}-\zeta^1_{kq,1}\Big(\widehat{\frac{1}{r}\frac{\partial W}{\partial\varphi}}\Big)\Big\|^2_{H_b^{l-1}(\Omega_k)}\leqslant k_2\varepsilon_0^2\Big\|\zeta^1_{kq,1}\Big(\widehat{\frac{1}{r}\frac{\partial W}{\partial\varphi}}\Big)\Big\|^2_{H_b^{l-1}(\Omega_k)}. \tag{3.17}$$

Then (3.17) and Lemma 3.3 prove the lemma for $|\gamma|=1$ with $\zeta_{kq,2}$ replaced by $\zeta^1_{kq,1}$.

Assume that the lemma holds for $1\leqslant|\gamma|\leqslant p-1$ with $\zeta_{kq,2}$ replaced by $\zeta^{p-1}_{kq,1}$. We claim that it holds for $|\gamma|=p$ with $\zeta_{kq,2}$ replaced by $\zeta^p_{kq,1}$, $p\geqslant 2$. Indeed, we have

$$\begin{aligned}&\big\|\zeta^p_{kq,1}D^\gamma\widehat{W}-\zeta^p_{kq,1}\widehat{D^\gamma W}\big\|_{H_b^{l-|\gamma|}(\Omega_k)}\\ &\quad\leqslant\big\|\zeta^p_{kq,1}D^{|\gamma|-1}(D^1\widehat{W})-\zeta^p_{kq,1}D^{|\gamma|-1}\widehat{D^1W}\big\|_{H_b^{l-|\gamma|}(\Omega_k)}\\ &\qquad+\big\|\zeta^p_{kq,1}D^{|\gamma|-1}\widehat{D^1W}-\zeta^p_{kq,1}\widehat{D^{|\gamma|-1}(D^1W)}\big\|_{H_b^{l-|\gamma|}(\Omega_k)}\\ &\quad\leqslant k_3\big(\big\|\zeta^{p-1}_{kq,1}D^1\widehat{W}-\zeta^{p-1}_{kq,1}\widehat{D^1W}\big\|_{H_b^{l-1}(\Omega_k)}\\ &\qquad+\big\|\zeta^p_{kq,1}D^{|\gamma|-1}\widehat{D^1W}-\zeta^p_{kq,1}\widehat{D^{|\gamma|-1}(D^1W)}\big\|_{H_b^{l-|\gamma|}(\Omega_k)}\big),\end{aligned} \tag{3.18}$$

where $D^{|\gamma|-1}$ and $D^1$ are some derivatives of order $|\gamma|-1$ and 1 respectively. By the inductive assumption, the following estimate holds for each of the two norms in the right-hand side of (3.18):

$$\begin{aligned}\big\|\zeta^{p-1}_{kq,1}D^1\widehat{W}-\zeta^{p-1}_{kq,1}\widehat{D^1W}\big\|_{H_b^{l-1}(\Omega_k)}&\leqslant k_4\varepsilon_0\|W\|_{H_b^l(\Omega_k)},\\ \big\|\zeta^p_{kq,1}D^{|\gamma|-1}\widehat{D^1W}-\zeta^p_{kq,1}\widehat{D^{|\gamma|-1}(D^1W)}\big\|_{H_b^{l-|\gamma|}(\Omega_k)}&\leqslant k_5\varepsilon_0\|D^1W\|_{H_b^{l-1}(\Omega_k)}\\ &\leqslant k_6\varepsilon_0\|W\|_{H_b^l(\Omega_k)}.\end{aligned}$$

This and (3.18) yield the conclusion of the lemma.

We note that $\varepsilon_0$ appears in (3.16) because both terms in the left-hand side contain the same transformation $\omega'_{j\sigma kqs}(\mathcal{G}^{-1}_{j\sigma kqs}y, z)$, but the first term is the derivative $D^\gamma$ of the transformed function $\widehat{W}$ while the second term is the transformation of the derivative $D^\gamma W$.

**Lemma 3.5.** *The following inequality holds for any function* $U_k \in H_b^{l+2m}(\Omega_k)$ *with* $\operatorname{supp} U_k \subset \overline{\Omega}_k \cap \mathcal{V}_{\varepsilon_0}(0)$:

$$\begin{aligned}
&\big\|(B_{j\sigma\mu kqs}U_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}}\\
&\qquad - (B_{j\sigma\mu kqs}U_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}}\big\|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})}\\
&\qquad \leqslant c\big(\varepsilon_0\|U_k\|_{H_b^{l+2m}(\Omega_k)} + \|\zeta_{kq,3}U_k - \zeta_{kq,3}\widehat{U}_k\|_{H_b^{l+2m}(\Omega_k)}\big),
\end{aligned} \tag{3.19}$$

*where* $c > 0$ *is independent of* $U$ *and* $\varepsilon_0$.

*Proof.* Since the trace operator is bounded in weighted spaces, we get

$$\begin{aligned}
&\big\|(B_{j\sigma\mu kqs}U_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}}\\
&\qquad - (B_{j\sigma\mu kqs}U_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}}\big\|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})}\\
&\quad \leqslant k_1\|\zeta_{kq,4}B_{j\sigma\mu kqs}U_k - \zeta_{kq,4}\widehat{B_{j\sigma\mu kqs}U_k}\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)}\\
&\quad \leqslant k_1\big(\|\zeta_{kq,4}B_{j\sigma\mu kqs}U_k - \zeta_{kq,4}B_{j\sigma\mu kqs}\widehat{U}_k\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)}\\
&\qquad + \|\zeta_{kq,4}B_{j\sigma\mu kqs}\widehat{U}_k - \zeta_{kq,4}\widehat{B_{j\sigma\mu kqs}U_k}\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)}\big).
\end{aligned} \tag{3.20}$$

We estimate the first norm in the right-hand side of (3.20) as

$$\begin{aligned}
&\|\zeta_{kq,4}B_{j\sigma\mu kqs}U_k - \zeta_{kq,4}B_{j\sigma\mu kqs}\widehat{U}_k\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)}\\
&\qquad\qquad \leqslant k_2\|\zeta_{kq,3}U_k - \zeta_{kq,3}\widehat{U}_k\|_{H_b^{l+2m}(\Omega_k)}.
\end{aligned} \tag{3.21}$$

The second norm in the right-hand side of (3.20) is estimated with the help of Lemma 3.4:

$$\|\zeta_{kq,4}B_{j\sigma\mu kqs}\widehat{U}_k - \zeta_{kq,4}\widehat{B_{j\sigma\mu kqs}U_k}\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)} \leqslant k_3\varepsilon_0\|U_k\|_{H_b^{l+2m}(\Omega_k)}. \tag{3.22}$$

The lemma follows from (3.20)–(3.22).

We note that the right-hand side of (3.19) contains the norm of the difference between a function and its transform. We use the following result to estimate such differences.

**Lemma 3.6.** *The following inequality holds for all* $W \in H^1_{b+1}(\Omega_k)$ *with* $\operatorname{supp} W \subset \overline{\Omega}_k \cap \mathcal{V}_{\varepsilon_0}(0)$:

$$\|\zeta_{kq,1}W - \zeta_{kq,1}\widehat{W}\|_{H_b^0(\Omega_k)} \leqslant c\varepsilon_0\|W\|_{H^1_{b+1}(\Omega_k)}, \tag{3.23}$$

*where $c>0$ is independent of $W$ and $\varepsilon_0$.*

*Proof.* Writing the arguments of $W$ and $\widehat{W}$ in the cylindrical coordinates, we get

$$\begin{aligned}
&\|\zeta_{kq,1}W-\zeta_{kq,1}\widehat{W}\|_{H_b^0(\Omega_k)}\\
&\qquad\leqslant \big\|\zeta_{kq,1}W(\varphi,r,z)-\zeta_{kq,1}W(\varphi+\Phi'_{j\sigma kqs}(r,z),r,z)\big\|_{H_b^0(\Omega_k)}\\
&\qquad\quad+\big\|\zeta_{kq,1}W(\varphi+\Phi'_{j\sigma kqs}(r,z),r,z)\\
&\qquad\qquad-\zeta_{kq,1}W(\varphi+\Phi'_{j\sigma kqs}(r,z),r+R'_{j\sigma kqs}(r,z),z)\big\|_{H_b^0(\Omega_k)}.
\end{aligned}\tag{3.24}$$

Using the Schwartz inequality, we estimate the square of the first norm in the right-hand side of (3.24):

$$\begin{aligned}
&\big\|\zeta_{kq,1}W(\varphi,r,z)-\zeta_{kq,1}W(\varphi+\Phi'_{j\sigma kqs}(r,z),r,z)\big\|^2_{H_b^0(\Omega_k)}\\
&\qquad=\int_{\mathbb{R}^{n-2}}dz\int_0^\infty r^{2b}r\,dr\int_{b_{k1}}^{b_{k2}}\left|\zeta_{kq,1}\int_\varphi^{\varphi+\Phi'_{j\sigma kqs}(r,z)}\frac{\partial W}{\partial\varphi'}\,d\varphi'\right|^2d\varphi\\
&\qquad\leqslant\int_{\mathbb{R}^{n-2}}dz\int_0^\infty r^{2b}r\,dr\int_{b_{k1}}^{b_{k2}}|\zeta_{kq,1}|^2|\Phi'_{j\sigma kqs}(r,z)|\\
&\qquad\quad\times\left|\int_\varphi^{\varphi+\Phi'_{j\sigma kqs}(r,z)}\left|\frac{\partial W}{\partial\varphi'}\right|^2d\varphi'\right|d\varphi.
\end{aligned}$$

Taking the conditions on the supports of $W$, $\zeta_{kq,1}$ into account and using (3.9), we can change the order of integration with respect to $\varphi$ and $\varphi'$. As a result, using (3.2), we get

$$\begin{aligned}
&\big\|\zeta_{kq,1}W(\varphi,r,z)-\zeta_{kq,1}W(\varphi+\Phi'_{j\sigma kqs}(r,z),r,z)\big\|^2_{H_b^0(\Omega_k)}\\
&\qquad\leqslant k_1\int_{\mathbb{R}^{n-2}}dz\int_0^\infty r^{2b}r|\Phi'_{j\sigma kqs}(r,z)|^2\,dr\int_{b_{k1}}^{b_{k2}}\left|\frac{\partial W}{\partial\varphi}\right|^2d\varphi\\
&\qquad\leqslant k_2\varepsilon_0^2\int_{\mathbb{R}^{n-2}}dz\int_0^\infty r^{2(b+1)}r\,dr\int_{b_{k1}}^{b_{k2}}\left|\frac{1}{r}\frac{\partial W}{\partial\varphi}\right|^2d\varphi\\
&\qquad\leqslant k_3\varepsilon_0^2\|W\|^2_{H^1_{b+1}(\Omega_k)}.
\end{aligned}$$

One can similarly estimate the square of the second norm in the right-hand side of (3.24). The lemma is proved.

Thus the factor $\varepsilon_0$ appears in (3.23) when the order of differentiation is increased by 1. (There is an $H_b^0(\Omega_k)$-norm in the left-hand side of (3.23) and an $H^1_{b+1}(\Omega_k)$-norm in the right-hand side.) This can be explained as follows. In contrast to (3.16), we now estimate the difference of functions one of which does not contain a transformation while the second does.

## § 4. A priori estimates of solutions

In this section we prove an a priori estimate for the operator $\mathbf{L}$, which implies that its kernel is finite-dimensional and its range is closed.

**1.** We first prove an a priori estimate for functions supported in a neighbourhood of $\mathcal{K}_1$. To do this, we use the invertibility of the model operators $\mathcal{L}_g^{\mathcal{G}}$ $(g \in \mathcal{K}_1)$ with linear transformations as well as Lemmas 3.3–3.6. In subsection 2 we use the results of [11] and Lemma 5.2 of [12] to obtain a priori estimates for functions supported in the closure of $G$.

We put $\mathcal{O}_\varepsilon(\mathcal{K}_1) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \mathcal{K}_1) < \varepsilon\}$.

**Lemma 4.1.** *Suppose that Conditions 1.1–1.4 hold, and the operators $\mathcal{L}_g^{\mathcal{G}}$ are isomorphisms*[4] *for all $g \in \mathcal{K}$. Then there is $\varepsilon$ with $0 < \varepsilon < \operatorname{dist}(\mathcal{K}_1, \mathcal{K}_2 \cup \mathcal{K}_3)/2$ such that the following estimate holds for all $u \in \{u \in H_b^{l+2m}(G) \colon \operatorname{supp} u \subset \overline{G} \cap \mathcal{O}_\varepsilon(\mathcal{K}_1)\}$:*

$$\|u\|_{H_b^{l+2m}(G)} \leqslant c\big(\|\mathbf{L}u\|_{\mathcal{H}_b^l(G,\Upsilon)} + \|u\|_{H^0_{b+1-l-2m}(G)}\big),$$

*where $c > 0$ is independent of $u$.*

Using partitions of unity, Leibnitz' formula, Lemma 2.1 of [27] and Lemma 1.2 of [9], we reduce the proof of Lemma 4.1 to the proof of the following result.

**Lemma 4.2.** *Suppose that the hypotheses of Lemma 4.1 hold. Then for each $g \in \mathcal{K}_1$ there is $\varepsilon_0 = \varepsilon_0(g) > 0$ such that the following inequality holds for all $U \in \{U \in H_b^{l+2m,N}(\Omega) \colon \operatorname{supp} U_j \subset \overline{\Omega}_j \cap \mathcal{V}_{\varepsilon_0}(0),\ j = 1, \dots, N,\ N = N(g)\}$:*

$$\|U\|_{H_b^{l+2m,N}(\Omega)} \leqslant c\|\mathcal{L}_g^{\omega} U\|_{H_b^{l,N}(\Omega)},$$

*where $\mathcal{V}_{\varepsilon_0}(0) = \{x \in \mathbb{R}^n : |x| < \varepsilon_0\}$, and $c > 0$ is independent of $U$.*

*Proof.* Using the invertibility of $\mathcal{L}_g^{\mathcal{G}}$ and Lemma 3.5, we get the following inequality for all $U \in H_b^{l+2m,N}(\Omega)$ with $\operatorname{supp} U_j \subset \overline{\Omega}_j \cap \mathcal{V}_{\varepsilon_0}(0)$:

$$\begin{aligned} \|U\|_{H_b^{l+2m,N}(\Omega)} &\leqslant k_1 \|\mathcal{L}_g^{\mathcal{G}} U\|_{H_b^{l,N}(\Omega)} \\ &\leqslant k_2 \Bigg( \|\mathcal{L}_g^{\omega} U\|_{H_b^{l,N}(\Omega)} + \varepsilon_0 \|U\|_{H_b^{l+2m,N}(\Omega)} \\ &\qquad + \sum_{k=1}^{N} \sum_{q=2}^{R_k} \|\zeta_{qk,3} U_k - \zeta_{qk,3} \widehat{U}_k\|_{H_b^{l+2m}(\Omega_k)} \Bigg). \end{aligned} \tag{4.1}$$

Let us estimate the last norm in (4.1). By Theorem 4.1 of [25], we have

$$\begin{aligned} \|\zeta_{qk,3} U_k - \zeta_{qk,3} \widehat{U}_k\|_{H_b^{l+2m}(\Omega_k)} &\leqslant k_3 \big( \|\mathcal{P}_k(\zeta_{kq,3} U_k - \zeta_{kq,3} \widehat{U}_k)\|_{H_b^l(\Omega_k)} \\ &\qquad + \|\zeta_{kq,3} U_k - \zeta_{kq,3} \widehat{U}_k\|_{H^0_{b-l-2m}(\Omega_k)} \big). \end{aligned} \tag{4.2}$$

Using Lemma 3.6 and the continuity of the embedding $H_b^{l+2m}(\Omega_k) \subset H^1_{b-l-2m+1}(\Omega_k)$, we get

$$\|\zeta_{kq,3} U_k - \zeta_{kq,3} \widehat{U}_k\|_{H^0_{b-l-2m}(\Omega_k)} \leqslant k_4 \varepsilon_0 \|U_k\|_{H_b^{l+2m}(\Omega_k)}. \tag{4.3}$$

[4] Subsection 5 of § 1 contains a necessary and sufficient condition for $\mathcal{L}_g^{\mathcal{G}}$ to be an isomorphism.

To estimate the first norm in the right-hand side of (4.2), we apply Leibnitz' formula and Lemmas 3.3, 3.4:

$$
\begin{aligned}
&\|\mathcal{P}_k(\zeta_{kq,3}U_k - \zeta_{kq,3}\widehat{U}_k)\|_{H_b^l(\Omega_k)} \\
&\quad\leqslant k_5\Bigg(\|\zeta_{kq,3}\mathcal{P}_kU_k\|_{H_b^l(\Omega_k)} + \|\zeta_{kq,3}\mathcal{P}_k\widehat{U}_k\|_{H_b^l(\Omega_k)} \\
&\qquad + \sum_{|\beta|\leqslant 2m-1}\ \sum_{|\gamma|=2m-|\beta|}\|D^\gamma\zeta_{kq,3}D^\beta U_k - D^\gamma\zeta_{kq,3}D^\beta\widehat{U}_k\|_{H_b^l(\Omega_k)}\Bigg) \\
&\quad\leqslant k_6\Bigg(\|\mathcal{P}_kU_k\|_{H_b^l(\Omega_k)} + \varepsilon_0\|U_k\|_{H_b^{l+2m}(\Omega_k)} \\
&\qquad + \sum_{|\beta|\leqslant 2m-1}\ \sum_{|\gamma|=2m-|\beta|}\|D^\gamma\zeta_{kq,3}D^\beta U_k - D^\gamma\zeta_{kq,3}D^\beta\widehat{U}_k\|_{H_b^l(\Omega_k)}\Bigg).
\end{aligned}
\tag{4.4}
$$

Since $|D^\gamma\zeta_{kq,3}| \leqslant k_7r^{-|\gamma|}|\zeta_{kq,2}|$, it follows that

$$
\begin{aligned}
&\sum_{|\beta|\leqslant 2m-1}\ \sum_{|\gamma|=2m-|\beta|}\|D^\gamma\zeta_{kq,3}D^\beta U_k - D^\gamma\zeta_{kq,3}D^\beta\widehat{U}_k\|_{H_b^l(\Omega_k)} \\
&\quad\leqslant k_8\sum_{|\alpha|\leqslant l+2m-1}\|\zeta_{kq,2}D^\alpha U_k - \zeta_{kq,2}D^\alpha\widehat{U}_k\|_{H^0_{b+|\alpha|-l-2m}(\Omega_k)} \\
&\quad\leqslant k_9\sum_{|\alpha|\leqslant l+2m-1}\big\{\|\zeta_{kq,2}D^\alpha U_k - \zeta_{kq,2}\widehat{D^\alpha U_k}\|_{H^0_{b+|\alpha|-l-2m}(\Omega_k)} \\
&\qquad\qquad + \|\zeta_{kq,2}\widehat{D^\alpha U_k} - \zeta_{kq,2}D^\alpha\widehat{U}_k\|_{H^0_{b+|\alpha|-l-2m}(\Omega_k)}\big\}.
\end{aligned}
\tag{4.5}
$$

Using Lemma 3.6 and the continuity of the embedding

$$
H_b^{l+2m}(\Omega_k) \subset H^{1+|\alpha|}_{b+1+|\alpha|-l-2m}(\Omega_k)
$$

for $|\alpha| \leqslant l+2m-1$, we obtain

$$
\begin{aligned}
\|\zeta_{kq,2}D^\alpha U_k - \zeta_{kq,2}\widehat{D^\alpha U_k}\|_{H^0_{b+|\alpha|-l-2m}(\Omega_k)} &\leqslant k_{10}\varepsilon_0\|D^\alpha U_k\|_{H^1_{b+1+|\alpha|-l-2m}(\Omega_k)} \\
&\leqslant k_{11}\varepsilon_0\|U_k\|_{H_b^{l+2m}(\Omega_k)}.
\end{aligned}
\tag{4.6}
$$

Lemma 3.4 similarly yields that

$$
\|\zeta_{kq,2}\widehat{D^\alpha U_k} - \zeta_{kq,2}D^\alpha\widehat{U}_k\|_{H^0_{b+|\alpha|-l-2m}(\Omega_k)} \leqslant k_{12}\varepsilon_0\|U_k\|_{H_b^{l+2m}(\Omega_k)}.
\tag{4.7}
$$

Now the lemma follows from (4.1)–(4.7) if $\varepsilon_0$ is sufficiently small.

**2.** Repeating the proof of Theorem 2.1 of [11] and using Lemma 5.2 of [12], we deduce the following result from Lemma 4.1 of the present paper and Lemmas 2.4, 2.5 of [11].

**Theorem 4.1.** *Suppose that the hypotheses of Lemma 4.1 hold and $b > l+2m-1$. Then the following estimate holds for all $u \in H_b^{l+2m}(G)$:*

$$\|u\|_{H_b^{l+2m}(G)} \leqslant c\big(\|\mathbf{L}u\|_{\mathcal{H}_b^l(G,\Upsilon)} + \|u\|_{H^0_{b+1-l-2m}(G)}\big), \tag{4.8}$$

*where $c > 0$ is independent of $u$.*

Since the embedding $H_b^{l+2m}(G) \subset H^0_{b+1-l-2m}(G)$ is compact (see [27], Lemma 3.5), Theorem 4.1 implies that the operator $\mathbf{L}$ has finite-dimensional kernel and closed range.

## § 5. Construction of the right regularizer

In this section we construct a right regularizer for $\mathbf{L}$. Along with Theorem 4.1, this enables us to prove the Fredholm solubility of (1.2), (1.3).

**1.** To begin with, we consider the case of functions supported in a neighbourhood of $\mathcal{K}_1$. We shall use the invertibility of the operators $\mathcal{L}_g^{\mathcal{G}}$ $(g \in \mathcal{K}_1)$ with linear transformations as well as some special constructions that "compensate" the nonlinearity of the argument transformations. In subsection 2 we use the results of [11] and Lemma 5.2 of [12] to construct the right regularizer on the whole of $G$.

We first prove the following auxiliary result.

**Lemma 5.1.** *Let $H$, $H_1$, $H_2$ be Hilbert spaces, $\mathcal{A}\colon H \to H_1$ a bounded linear operator, and $\mathcal{T}_0\colon H \to H_2$ a compact linear operator. Suppose that the following inequality holds for some $\varepsilon, c > 0$ and for all $f \in H$:*

$$\|\mathcal{A}f\|_{H_1} \leqslant \varepsilon\|f\|_H + c\|\mathcal{T}_0 f\|_{H_2}. \tag{5.1}$$

*Then there are bounded operators $\mathcal{M}$, $\mathcal{F}\colon H \to H_1$ such that*

$$\mathcal{A} = \mathcal{M} + \mathcal{F},$$

*where $\|\mathcal{M}\| \leqslant 2\varepsilon$ and the operator $\mathcal{F}$ is finite-dimensional.*

*Proof.* It is well known that each compact operator is the limit of a uniformly convergent sequence of finite-dimensional operators (see, for example, [28], Ch. 5, § 85). Hence there are bounded operators $\mathcal{M}_0$, $\mathcal{F}_0\colon H \to H_2$ such that $\mathcal{T}_0 = \mathcal{M}_0 + \mathcal{F}_0$, $\|\mathcal{M}_0\| \leqslant c^{-1}\varepsilon$, and $\mathcal{F}_0$ is finite-dimensional. Using this and (5.1), we see that

$$\|\mathcal{A}f\|_{H_1} \leqslant 2\varepsilon\|f\|_H + c\|\mathcal{F}_0 f\|_{H_2} \quad \text{for all} \quad f \in H. \tag{5.2}$$

We denote by $\ker(\mathcal{F}_0)^\perp$ the orthogonal complement in $H$ to the kernel of $\mathcal{F}_0$. Since the finite-dimensional operator $\mathcal{F}_0$ maps $\ker(\mathcal{F}_0)^\perp$ onto its range in a one-to-one manner, it follows that the subspace $\ker(\mathcal{F}_0)^\perp$ is finite-dimensional. Let $\mathcal{I}$ be the identity operator in $H$, and let $\mathcal{P}_0$ be the orthogonal projection onto $\ker(\mathcal{F}_0)^\perp$. The operator $\mathcal{A}\mathcal{P}_0\colon H \to H_1$ is clearly finite-dimensional. Furthermore, since $\mathcal{I} - \mathcal{P}_0$ is the orthogonal projection onto $\ker(\mathcal{F}_0)$, it follows that $\mathcal{F}_0(\mathcal{I} - \mathcal{P}_0) = 0$. Replacing $f$ by $(\mathcal{I} - \mathcal{P}_0)f$ in (5.2), we get

$$\|\mathcal{A}(\mathcal{I} - \mathcal{P}_0)f\|_{H_1} \leqslant 2\varepsilon\|(\mathcal{I} - \mathcal{P}_0)f\|_H \leqslant 2\varepsilon\|f\|_H \quad \text{for all} \quad f \in H.$$

Putting $\mathcal{M} = \mathcal{A}(\mathcal{I} - \mathcal{P}_0)$ and $\mathcal{F} = \mathcal{A}\mathcal{P}_0$, we prove the lemma.

We now proceed to construct the right regularizer.

**Lemma 5.2.** *Suppose that the hypotheses of Lemma 4.1 hold. Then, for all sufficiently small $\varepsilon$ with $0 < \varepsilon < \operatorname{dist}(\mathcal{K}_1, \mathcal{K}_2 \cup \mathcal{K}_3)/2$, there are bounded operators $\mathbf{R}_1$, $\mathbf{M}_1$ and a compact operator $\mathbf{T}_1$ acting from $\{f \in \mathcal{H}_b^l(G, \Upsilon)\colon \operatorname{supp} f \subset \overline{G} \cap \mathcal{O}_\varepsilon(\mathcal{K}_1)\}$ to $H_b^{l+2m}(G)$, $\mathcal{H}_b^l(G, \Upsilon)$, and $\mathcal{H}_b^l(G, \Upsilon)$ respectively such that*

$$\mathbf{L}\mathbf{R}_1 f = f + \mathbf{M}_1 f + \mathbf{T}_1 f,$$

*where $\|\mathbf{M}_1 f\|_{\mathcal{H}_b^l(G,\Upsilon)} \leqslant c\varepsilon \|f\|_{\mathcal{H}_b^l(G,\Upsilon)}$ and $c > 0$ is independent of $\varepsilon$ and $f$.*

Using partitions of unity, Leibnitz' formula, and Lemma 2.1 of [27], we reduce the proof of Lemma 5.2 to the proof of the following result.

**Lemma 5.3.** *Suppose that the hypotheses of Lemma 4.1 hold. Then, for each $g \in \mathcal{K}_1$ and all sufficiently small $\varepsilon_1 = \varepsilon_1(g) > 0$ there are bounded operators $\mathcal{R}_g$, $\mathcal{M}_g$ and a compact operator $\mathcal{T}_g$ acting from $\{f \in \mathcal{H}_b^{l,N}(\Omega, \Gamma)\colon \operatorname{supp} f \subset \mathcal{V}_{\varepsilon_1}(0)\}$ to $H_b^{l+2m,N}(\Omega)$, $\mathcal{H}_b^{l,N}(\Omega, \Gamma)$, and $\mathcal{H}_b^{l,N}(\Omega, \Gamma)$ respectively such that*

$$\mathcal{L}_g^\omega \mathcal{R}_g f = f + \mathcal{M}_g f + \mathcal{T}_g f, \tag{5.3}$$

*where $\|\mathcal{M}_g f\|_{H_b^l(G,\Gamma)} \leqslant c\varepsilon_1 \|f\|_{H_b^l(G,\Gamma)}$ and $c > 0$ is independent of $\varepsilon_1$ and $f$.*

*Proof.* 1) As above, we put

$$d_1 = \frac{1}{2}\min\{1, \chi_{j\sigma kqs}\}, \qquad d_2 = 2\max\{1, \chi_{j\sigma kqs}\}.$$

We choose $\varepsilon_1 < d_1\varepsilon_0/4$, where $\varepsilon_0$ is defined in Lemma 4.2. We introduce a function $\psi_{\varepsilon_1}(x) = \psi(x/\varepsilon_1)$, where $\psi \in C^\infty(\mathbb{R}^n)$, $\psi(x) = 1$ for $|x| \leqslant 1$, and $\psi(x) = 0$ for $|x| \geqslant 2$. It is obvious that $\psi_{\varepsilon_1} \in C^\infty(\mathbb{R}^n)$, $\psi_{\varepsilon_1}(x) = 1$ for $|x| \leqslant \varepsilon_1$, and $\psi_{\varepsilon_1}(x) = 0$ for $|x| \geqslant 2\varepsilon_1$. Since $|D^\alpha \psi_{\varepsilon_1}| \leqslant c_\alpha r^{-|\alpha|}$, we see from Lemma 2.1 of [27] that

$$\|\psi_{\varepsilon_1} v\|_{H_b^{l+2m}(\Omega_k)} \leqslant c\|v\|_{H_b^{l+2m}(\Omega_k)} \quad \text{for all} \quad v \in H_b^{l+2m}(\Omega_k), \tag{5.4}$$

where $c > 0$ is independent of $\varepsilon_1$. Moreover, we assume that $\psi_{\varepsilon_1}$, being written in the cylindrical coordinates, is independent of $\varphi$.

Put $f_0 = \{f_j\}$, $g = \{g_{j\sigma\mu}\}$, $\{f_0, g\} = \{f_j, g_{j\sigma\mu}\}$.

By assumption, the operator $\mathcal{L}_g^{\mathcal{G}}\colon H_b^{l+2m,N}(\Omega) \to \mathcal{H}_b^{l,N}(\Omega, \Gamma)$ has a bounded inverse $(\mathcal{L}_g^{\mathcal{G}})^{-1}\colon \mathcal{H}_b^{l,N}(\Omega, \Gamma) \to H_b^{l+2m,N}(\Omega)$. Therefore we can introduce the operators

$$\mathcal{R}_1\colon H_b^{l,N}(\Omega) \to H_b^{l+2m,N}(\Omega),$$
$$\mathcal{R}_2\colon \mathcal{H}_b^{l,N}(\Gamma) \to H_b^{l+2m,N}(\Omega)$$

given by

$$\mathcal{R}_1 f_0 = \psi_{\varepsilon_1}(\mathcal{L}_g^{\mathcal{G}})^{-1}\{f_0, 0\},$$
$$\mathcal{R}_2 g = \psi_{\varepsilon_1}(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0, g\},$$

where $\mathcal{H}_b^{l,N}(\Gamma) = \prod_{j,\sigma,\mu} H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})$. Thus the supports of $\mathcal{R}_1 f_0$ and $\mathcal{R}_2 g$ are contained in the ball of radius $2\varepsilon_1$ centered at the origin.

Let us introduce the operators

$$\mathcal{P}\colon H_b^{l+2m,N}(\Omega) \to H_b^{l,N}(\Omega),$$
$$\mathcal{B}^{\mathcal{G}}, \mathcal{B}^{\omega}\colon H_b^{l+2m,N}(\Omega) \to \mathcal{H}_b^{l,N}(\Gamma),$$

given by

$$\mathcal{P}U = \{\mathcal{P}_j U_j\}, \qquad \mathcal{B}^{\mathcal{G}}U = \{\mathcal{B}^{\mathcal{G}}_{j\sigma\mu}U\}, \qquad \mathcal{B}^{\omega}U = \{\mathcal{B}^{\omega}_{j\sigma\mu}U\}.$$

We now establish a relation between the operators $\mathcal{P}$, $\mathcal{B}^{\mathcal{G}}$, $\mathcal{B}^{\omega}$ and $\mathcal{R}_1$, $\mathcal{R}_2$. We use the following well-known property of weighted spaces (see [27], Lemma 3.5):

($*$) *the embedding operator from* $\{v \in H_b^{l+1}(\Omega_j)\colon \operatorname{supp} v \subset \mathcal{V}_d(0),\ d > 0\}$ *to* $H_b^l(\Omega_j)$ *is compact.*

Using Leibnitz' formula, the boundedness of $\operatorname{supp}\psi_{\varepsilon_1}$, and property ($*$), we get

$$\mathcal{P}\mathcal{R}_1 f_0 = \psi_{\varepsilon_1} f_0 + \mathcal{T}_1 f_0, \qquad \mathcal{P}\mathcal{R}_2 g = \mathcal{T}_2 g. \tag{5.5}$$

Here $\mathcal{T}_1\colon H_b^{l,N}(\Omega) \to H_b^{l,N}(\Omega)$ and $\mathcal{T}_2\colon \mathcal{H}_b^{l,N}(\Gamma) \to H_b^{l,N}(\Omega)$ are compact operators. We similarly have

$$\begin{aligned}\mathcal{B}^{\mathcal{G}}\mathcal{R}_2 g = \psi_{\varepsilon_1} g + \Bigg\{ &\sum_{k,q,s} \big(\psi_{\varepsilon_1}(\chi_{j\sigma kqs}x) - \psi_{\varepsilon_1}(x)\big) \\ &\times (B_{j\sigma\mu kqs}[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0,g\}]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} \Bigg\} + \mathcal{T}_3 g,\end{aligned} \tag{5.6}$$

where $\mathcal{T}_3$ is a compact operator in $\mathcal{H}_b^{l,N}(\Gamma)$. Here and in what follows we denote by $[\,\cdot\,]_k$ the $k$-th component of an $N$-dimensional vector, and by $\{\dots\}$ a vector whose components are defined by the indices $j$, $\sigma$, $\mu$.

Let us show that each term of the sum in (5.6) is a compact operator. Let $\zeta_{kq,i}$ be the functions defined by (3.10). We also introduce the functions $\widehat{\psi}_0, \widehat{\psi}_1 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned}&\widehat{\psi}_1(x) = 1 \quad \text{for} \quad 2d_1\varepsilon_1 \leqslant |x| \leqslant d_2\varepsilon_1, && \widehat{\psi}_1(x) = 0 \quad \text{outside} \quad d_1\varepsilon_1 \leqslant |x| \leqslant 2d_2\varepsilon_1,\\ &\widehat{\psi}_0(x) = 1 \quad \text{for} \quad d_1\varepsilon_1 \leqslant |x| \leqslant 2d_2\varepsilon_1, && \widehat{\psi}_0(x) = 0 \quad \text{outside} \quad d_1\varepsilon_1/2 \leqslant |x| \leqslant 4d_2\varepsilon_1.\end{aligned}$$

Since the trace operator is bounded in weighted spaces, we have

$$\begin{aligned}&\big\|\big(\psi_{\varepsilon_1}(\chi_{j\sigma kqs}x) - \psi_{\varepsilon_1}(x)\big) \\ &\qquad \times (B_{j\sigma\mu kqs}[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0,g\}]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}}\big\|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \\ &\quad \leqslant k_2\big\|\zeta_{kq,2}\big(\psi_{\varepsilon_1}(x) - \psi_{\varepsilon_1}(\chi^{-1}_{j\sigma kqs}x)\big)B_{j\sigma\mu kqs}[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0,g\}]_k\big\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)} \\ &\quad \leqslant k_3\big\|\zeta_{kq,1}\widehat{\psi}_1[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0,g\}]_k\big\|_{H_b^{l+2m}(\Omega_k)}.\end{aligned} \tag{5.7}$$

The support of $\widehat{\psi}_1$ is bounded and disjoint from the origin, and $\zeta_{kq,1}$ vanishes near the sides of the angle $\Omega_k$. Hence we can apply Theorem 5.1 of Ch. 2 in [23]. Using the relation $\mathcal{P}_k[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0,g\}]_k = 0$, we see from (5.7) that

$$\begin{aligned}\big\|(\psi_{\varepsilon_1}(\chi_{j\sigma kqs}x) - \psi_{\varepsilon_1}(x)) \\ \times (B_{j\sigma\mu kqs}[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0,g\}]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}}\big\|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \\ \leqslant k_4\big\|\widehat{\psi}_0[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0,g\}]_k\big\|_{H_b^{l+2m-1}(\Omega_k)}.\end{aligned}$$

Since the support of $\widehat{\psi}_0$ is bounded, this inequality and property $(*)$ imply that

$$\Big\{\sum_{k,q,s}(\psi_{\varepsilon_1}(\chi_{j\sigma kqs}x) - \psi_{\varepsilon_1}(x))(B_{j\sigma\mu kqs}[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0,g\}]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}}\Big\}$$

is a compact operator in $\mathcal{H}_b^{l,N}(\Gamma)$. Combining this with (5.6) yields that

$$\mathcal{B}^{\mathcal{G}}\mathcal{R}_2 g = \psi_{\varepsilon_1} g + \mathcal{T}_4 g, \tag{5.8}$$

where $\mathcal{T}_4$ is a compact operator in $\mathcal{H}_b^{l,N}(\Gamma)$.

Finally, we use (5.8) to get the following formula for the composition $\mathcal{B}^\omega\mathcal{R}_2$:

$$\begin{aligned}\mathcal{B}^\omega\mathcal{R}_2 g = \psi_{\varepsilon_1} g + \mathcal{T}_4 g + \Big\{\sum_{k,q,s}\big((B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\omega'_{j\sigma kqs}(y,z),z)|_{\Gamma_{j\sigma}} \\ - (B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}}\big)\Big\}.\end{aligned} \tag{5.9}$$

2) We introduce an operator $\mathcal{R}_g\colon \mathcal{H}_b^{l,N}(\Omega,\Gamma) \to H_b^{l+2m,N}(\Omega)$ by

$$\mathcal{R}_g\{f_0,g\} = \mathcal{R}_1 f_0 - \mathcal{R}'_2\mathcal{B}^\omega\mathcal{R}_1 f_0 + \mathcal{R}_2 g.$$

Here $\mathcal{R}'_2\colon \mathcal{H}_b^{l,N}(\Gamma) \to H_b^{l+2m,N}(\Omega)$ is a bounded operator given by

$$\mathcal{R}'_2 g = \psi_{\varepsilon_1}(d_1x/2)(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0,g\}.$$

Similarly to (5.5) and (5.9), we see that

$$\mathcal{P}\mathcal{R}'_2 g = \mathcal{T}'_2 g, \tag{5.10}$$

$$\begin{aligned}\mathcal{B}^\omega\mathcal{R}'_2 g = \psi_{\varepsilon_1}(d_1x/2) g + \mathcal{T}'_4 g + \Big\{\sum_{k,q,s}\big((B_{j\sigma\mu kqs}[\mathcal{R}'_2 g]_k)(\omega'_{j\sigma kqs}(y,z),z)|_{\Gamma_{j\sigma}} \\ - (B_{j\sigma\mu kqs}[\mathcal{R}'_2 g]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}}\big)\Big\},\end{aligned} \tag{5.11}$$

where $\mathcal{T}'_2$, $\mathcal{T}'_4$ are compact operators acting in the same spaces as $\mathcal{T}_2$, $\mathcal{T}_4$ do.

Let us show that the operator $\mathcal{R}_g$ satisfies (5.3). It follows from (5.5) and (5.10) that

$$\mathcal{P}\mathcal{R}_g\{f_0, g\} = \psi_{\varepsilon_1} f_0 + \mathcal{T}_5\{f_0, g\}, \tag{5.12}$$

where $\mathcal{T}_5\colon \mathcal{H}_b^{l,N}(\Omega,\Gamma) \to H_b^{l,N}(\Omega)$ is a compact operator.

Taking into account that $\psi_{\varepsilon_1}(d_1x/2)\mathcal{B}^\omega\mathcal{R}_1 f_0 \equiv \mathcal{B}^\omega\mathcal{R}_1 f_0$ and using (5.11), we derive that

$$\begin{aligned}
\mathcal{B}^\omega\mathcal{R}_g\{f_0, g\} &= \mathcal{B}^\omega\mathcal{R}_1 f_0 - \mathcal{B}^\omega\mathcal{R}_2'\mathcal{B}^\omega\mathcal{R}_1 f_0 + \mathcal{B}^\omega\mathcal{R}_2 g \\
&= -\mathcal{T}_4'\mathcal{B}^\omega\mathcal{R}_1 f_0 - \Big\{ \sum_{k,q,s} \big((B_{j\sigma\mu kqs}[\mathcal{R}_2'\mathcal{B}^\omega\mathcal{R}_1 f_0]_k)(\omega_{j\sigma kqs}'(y,z),z)|_{\Gamma_{j\sigma}} \\
&\qquad - (B_{j\sigma\mu kqs}[\mathcal{R}_2'\mathcal{B}^\omega\mathcal{R}_1 f_0]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}}\big) \Big\} + \mathcal{B}^\omega\mathcal{R}_2 g.
\end{aligned}$$

Using this and (5.9), we get

$$\begin{aligned}
\mathcal{B}^\omega\mathcal{R}_g g = \psi_{\varepsilon_1} g + \mathcal{T}_6\{f_0, g\} &+ \Big\{ \sum_{k,q,s} \big((B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\omega_{j\sigma kqs}'(y,z),z)|_{\Gamma_{j\sigma}} \\
&- (B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}}\big) \Big\} \\
&- \Big\{ \sum_{k,q,s} \big((B_{j\sigma\mu kqs}[\mathcal{R}_2'\mathcal{B}^\omega\mathcal{R}_1 f_0]_k)(\omega_{j\sigma kqs}'(y,z),z)|_{\Gamma_{j\sigma}} \\
&- (B_{j\sigma\mu kqs}[\mathcal{R}_2'\mathcal{B}^\omega\mathcal{R}_1 f_0]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}}\big) \Big\},
\end{aligned} \tag{5.13}$$

where $\mathcal{T}_6\colon \mathcal{H}_b^{l,N}(\Omega,\Gamma) \to \mathcal{H}_b^{l,N}(\Gamma)$ is a compact operator.

Consider the terms of the first sum in the right-hand side of (5.13). By Lemma 3.5, we have

$$\begin{aligned}
&\big\|(B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\omega_{j\sigma kqs}'(y,z),z)|_{\Gamma_{j\sigma}} \\
&\qquad - (B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}}\big\|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \\
&\quad \leqslant k_5\big(\varepsilon_1\|[\mathcal{R}_2 g]_k\|_{H_b^{l+2m}(\Omega_k)} + \|\zeta_{kq,3}[\mathcal{R}_2 g]_k - \zeta_{kq,3}\widehat{[\mathcal{R}_2 g]}_k\|_{H_b^{l+2m}(\Omega_k)}\big).
\end{aligned} \tag{5.14}$$

Using inequalities (4.2)–(4.7) for the function $U_k = [\mathcal{R}_2 g]_k$, inequality (5.14), and the second formula in (5.5), we get

$$\begin{aligned}
&\big\|(B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\omega_{j\sigma kqs}'(y,z),z)|_{\Gamma_{j\sigma}} \\
&\qquad - (B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}}\big\|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \\
&\quad \leqslant k_6\big(\varepsilon_1\|[\mathcal{R}_2 g]_k\|_{H_b^{l+2m}(\Omega_k)} + \|\mathcal{P}_k[\mathcal{R}_2 g]_k\|_{H_b^{l}(\Omega_k)}\big) \\
&\quad = k_6\big(\varepsilon_1\|\psi_{\varepsilon_1}[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0,g\}]_k\|_{H_b^{l+2m}(\Omega_k)} + \|[\mathcal{T}_2 g]_k\|_{H_b^{l}(\Omega_k)}\big).
\end{aligned}$$

Combining this with inequality (5.4) and using the boundedness of the operator $(\mathcal{L}_g^{\mathcal{G}})^{-1}\colon \mathcal{H}_b^{l,N}(\Omega,\Gamma) \to H_b^{l+2m,N}(\Omega)$, we finally obtain

$$\begin{aligned}\big\|(B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\omega'_{j\sigma kqs}(y,z),z)|_{\Gamma_{j\sigma}} \\ - (B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}}\big\|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \\ \leqslant k_7\big(\varepsilon_1\|g\|_{\mathcal{H}_b^{l,N}(\Gamma)} + \|[\mathcal{T}_2 g]_k\|_{H_b^l(\Omega_k)}\big).\end{aligned} \tag{5.15}$$

Therefore, by Lemma 5.1, we have

$$\begin{aligned}(B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\omega'_{j\sigma kqs}(y,z),z)|_{\Gamma_{j\sigma}} - (B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\mathcal{G}_{j\sigma kqs}y,z)|_{\Gamma_{j\sigma}} \\ = \mathcal{M}_{j\sigma\mu kqs}g + \mathcal{F}_{j\sigma\mu kqs}g\end{aligned}$$

with operators

$$\mathcal{M}_{j\sigma\mu kqs},\ \mathcal{F}_{j\sigma\mu kqs}\colon \mathcal{H}_b^{l,N}(\Gamma) \to H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})$$

such that $\|\mathcal{M}_{j\sigma\mu kqs}\| \leqslant 2k_7\varepsilon_1$ and $\mathcal{F}_{j\sigma\mu kqs}$ is finite-dimensional.

One can similarly prove that each term of the second sum in the right-hand side of (5.13) can be represented as the sum of an operator with small norm and a compact operator. Combining this with (5.13), (5.12) and choosing $\operatorname{supp}\{f_0, g\} \subset \mathcal{V}_{\varepsilon_1}(0)$, we prove the lemma.

**2.** Let us prove that the operator $\mathbf{L}\colon H_b^{l+2m}(G) \to \mathcal{H}_b^l(G,\Upsilon)$ is Fredholm under certain conditions.

**Theorem 5.1.** *Suppose that the hypotheses of Lemma* 4.1 *hold and* $b > l+2m-1$. *Then the operator* $\mathbf{L}\colon H_b^{l+2m}(G) \to \mathcal{H}_b^l(G,\Upsilon)$ *is Fredholm.*

*Proof.* By Theorem 4.1 above and Theorems 7.1, 15.2 of [29], it suffices to construct a right regularizer $\mathbf{R}$ for $\mathbf{L}$.

Repeating the argument of § 3 in [11] and taking Lemma 5.2 of [12] into account, we deduce from Lemma 5.2 of the present paper that there are bounded operators

$$\begin{aligned}\mathbf{R}'\colon \mathcal{H}_b^l(G,\Upsilon) \to H_b^{l+2m}(G), \\ \mathbf{M},\mathbf{T}\colon \mathcal{H}_b^l(G,\Upsilon) \to \mathcal{H}_b^l(G,\Upsilon)\end{aligned}$$

such that

$$\mathbf{L}\mathbf{R}' = \mathbf{I} + \mathbf{M} + \mathbf{T},$$

where $\|\mathbf{M}\| < 1$ and $\mathbf{T}$ is compact. Since $\|\mathbf{M}\| < 1$, it follows that the operator $\mathbf{I}+\mathbf{M}$ has a bounded inverse. Clearly, $\mathbf{R} = \mathbf{R}'(\mathbf{I}+\mathbf{M})^{-1}$ is a right regularizer for $\mathbf{L}$. The theorem is proved.

**3.** Until now, we assumed that $b > l+2m-1$. In this subsection we use the results of [9] to study the case when $b$ is arbitrary but $n = 2$. As mentioned before, we have to consider solutions and right-hand sides of the non-local problem as functions with power singularities not only near the set $\mathcal{K}_1$ but also near $\mathcal{K}_2$ and $\mathcal{K}_3$. This corresponds to the consistency conditions (see § 1).

Thus, let $n = 2$. We introduce the space $\widetilde{H}^l_b(G)$ as the completion of $C_0^\infty(\overline{G}\setminus\mathcal{K})$ with respect to the norm

$$\|u\|_{H^l_b(G)} = \biggl(\sum_{|\alpha|\leqslant l}\int_G \tilde\rho^{2(b-l+|\alpha|)}|D^\alpha u|^2\,dy\biggr)^{1/2},$$

where $\tilde\rho = \tilde\rho(y) = \operatorname{dist}(y,\mathcal{K})$ (compare with § 1). For $l \geqslant 1$, we denote by $\widetilde{H}^{l-1/2}_b(\Upsilon)$ the space of traces on a smooth curve $\Upsilon \subset \overline{G}$ with the norm

$$\|\psi\|_{\widetilde{H}^{l-1/2}_b(\Upsilon)} = \inf \|u\|_{\widetilde{H}^l_b(G)}, \qquad u \in \widetilde{H}^l_b(G)\colon\ u|_\Upsilon = \psi.$$

We assume that the following condition holds.

**Condition 5.1.** *If $g \in \mathcal{K}_3 \cap \omega_{is}(\Upsilon_i) \neq \varnothing$, then $\omega_{is}^{-1}(g) \in \mathcal{K}$.*

This condition guarantees that the set of points where the consistency condition must be imposed is finite. If Condition 5.1 fails, then consecutive shifts of the set $\mathcal{K}_1$ (under the transformations $\omega_{is}$ and their inverses) may form an infinite set, which should be used instead of $\mathcal{K}$ in the definition of weighted spaces.

In this subsection we consider the following bounded operator corresponding to problem[5] (1.2), (1.3):

$$\mathbf{L} = \bigl\{\mathbf{P}(y,D),\mathbf{B}_{i\mu}(y,D)\bigr\}\colon \widetilde{H}^{l+2m}_b(G) \to \widetilde{H}^l_b(G) \times \prod_{i=1}^{N_0}\prod_{\mu=1}^{m} \widetilde{H}^{l+2m-m_{i\mu}-1/2}_b(\Upsilon_i),$$
$$b \in \mathbb{R}.$$

Since solutions and right-hand sides of the non-local problem may now have power singularities near the points of $\mathcal{K}_2$ and $\mathcal{K}_3$, we have to consider the model problems corresponding to these points in weighted spaces but not in the Sobolev spaces.

We fix a point $g \in \mathcal{K}_2 \cup \mathcal{K}_3$. Let $y \mapsto y'(g)$ be a non-degenerate infinitely differentiable argument transformation that maps some neighbourhood $\mathcal{V}(g)$ of the point $g$ onto a neighbourhood $\mathcal{V}_g(0)$ of the origin such that $g$ is mapped to the origin. We denote by $\mathcal{P}(D_y)$, $\mathcal{B}_{i\mu0}(D_y)$ the principal homogeneous parts of the operators $\mathbf{P}(g,D)$, $B_{i\mu0}(g,D)$ written in the new coordinates $y' = y'(g)$ (with $y'$ subsequently redenoted by $y$). Now we write the operators $\mathcal{P}(D_y)$, $\mathcal{B}_{i\mu0}(D_y)$ in the polar coordinates:

$$\mathcal{P}(D_y) = r^{-2m}\widetilde{\mathcal{P}}(\varphi,D_\varphi,rD_r), \qquad \mathcal{B}_{i\mu0}(D_y) = r^{-m_{i\mu}}\widetilde{\mathcal{B}}_{i\mu0}(\varphi,D_\varphi,rD_r).$$

If $g \in \mathcal{K}_2$, then $g \in \Upsilon_i$ for some $i = i(g)$. Since $\Upsilon_i$ is smooth, any sufficiently small neighbourhood $\mathcal{V}(g)$ of $g$ admits a non-degenerate infinitely differentiable argument transformation $y \mapsto y' = y'(g)$ that maps $\mathcal{V}(g)\cap G$ onto the intersection

[5]Notice that equation (1.2) is now considered in $G\setminus\mathcal{K}_3$ but not on the whole of $G$.

of the half-plane $\mathbb{R}^2_+ = \{y\colon |\varphi| < \pi/2\}$ and a neighbourhood of $\mathcal{V}_g(0)$. We introduce a bounded operator

$$\mathcal{L}_g\colon H_b^{l+2m}(K_{\pi/2}) \to H_b^l(K_{\pi/2}) \times \prod_{j=1}^{2}\prod_{\mu=1}^{m} H_b^{l+2m-m_{i\mu}-1/2}(\gamma_j)$$

given by

$$\mathcal{L}_g U = \bigl\{\mathcal{P}(D_y)U,\ \mathcal{B}_{i\mu0}(D_y)U|_{\gamma_j}\bigr\},$$

where $K_{\pi/2} = \{y\colon |\varphi| < \pi/2\}$, $\gamma_j = \{y\colon \varphi = (-1)^j\pi/2\}$, $j = 1,2$. We also introduce a bounded operator

$$\widetilde{\mathcal{L}}_g(\lambda)\colon W_2^{l+2m}(-\pi/2,\pi/2) \to \mathcal{W}_2^l[-\pi/2,\pi/2] = W_2^l(-\pi/2,\pi/2)\times\mathbb{C}^{2m}$$

given by

$$\widetilde{\mathcal{L}}_g(\lambda)\widetilde{U} = \bigl\{\widetilde{\mathcal{P}}(\varphi, D_\varphi,\lambda)\widetilde{U}(\varphi),\ \widetilde{\mathcal{B}}_{i\mu0}(\varphi, D_\varphi,\lambda)\widetilde{U}(\varphi)|_{\varphi=(-1)^j\pi/2}\bigr\}, \qquad j=1,2.$$

If $g \in \mathcal{K}_3$, we introduce the bounded operators

$$\mathcal{L}_g = \mathcal{P}(D_y)\colon H_b^{l+2m}(\mathbb{R}^2) \to H_b^l(\mathbb{R}^2),$$
$$\widetilde{\mathcal{L}}_g(\lambda) = \widetilde{\mathcal{P}}(\varphi, D_\varphi,\lambda)\colon W_{2,2\pi}^{l+2m}(0,2\pi) \to W_{2,2\pi}^l(0,2\pi),$$

where $W_{2,2\pi}^l(0,2\pi)$ is the closure of the set of infinitely differentiable $2\pi$-periodic functions in $W_2^l(0,2\pi)$.

By § 1 of [27] and § 1 of [9], it follows that for each $g \in \mathcal{K}_2\cup\mathcal{K}_3$ there is a finite-meromorphic operator-valued function $\widetilde{\mathcal{L}}_g^{-1}(\lambda)$ with the following properties: (i) its poles, possibly expect finitely many of them, belong to a double angle of opening $< \pi$ containing the imaginary axis, and (ii) $\widetilde{\mathcal{L}}_g^{-1}(\lambda)$ is the bounded inverse to $\widetilde{\mathcal{L}}_g(\lambda)$ for all $\lambda$ which are not poles of $\widetilde{\mathcal{L}}_g^{-1}(\lambda)$.

Using Theorem 1.1 of [27] and the results of § 1 of [9], we see that $\mathcal{L}_g$ is an isomorphism if and only if the line $\operatorname{Im}\lambda = b+1-l-2m$ contains no poles of $\widetilde{\mathcal{L}}_g^{-1}(\lambda)$.

**Theorem 5.2.** *Assume that Conditions* 1.1–1.4, 5.1 *hold. Let* $b\in\mathbb{R}$ *be such that* $\mathcal{L}_g^{\mathcal{G}}$ *is an isomorphism for all* $g\in\mathcal{K}_1$ *and* $\mathcal{L}_g$ *is an isomorphism for all* $g\in\mathcal{K}_2\cup\mathcal{K}_3$. *Then the operator* $\mathbf{L}\colon \widetilde{H}_b^{l+2m}(G)\to\widetilde{\mathcal{H}}_b^l(G,\Upsilon)$ *is Fredholm.*

*Proof.* Note that Lemmas 4.1, 5.2 are true for all $b\in\mathbb{R}$ such that $\mathcal{L}_g^{\mathcal{G}}$ are isomorphisms for all $g\in\mathcal{K}_1$. Hence, using Lemmas 4.1 and 5.2, we can obtain an a priori estimate (4.8) (in spaces $\widetilde{H}_b^l(\,\cdot\,)$) and construct the right regularizer similarly to the proof of Theorem 3.4 in [9].

## § 6. Stability of the index of non-local elliptic problems

In this section we study the influence of the transformations $\omega_{is}$ upon the index of non-local elliptic problems. We show that the index of the problem is determined by the linear part of $\omega_{is}$ in a neighbourhood of $\mathcal{K}_1$. We note that the stability of the index was established in [15] in the case when the support $\bigcup_{i,s}\omega_{is}(\overline{\Upsilon}_i)$ of non-local terms is disjoint from the set $\mathcal{K}_1$ consisting of all points of conjugation of non-local conditions.

**1.** Along with (1.2), (1.3), we consider the following problem:

$$\mathbf{P}(x,D)u = f_0(x), \qquad x \in G, \tag{6.1}$$

$$\widehat{\mathbf{B}}_{i\mu}(x,D)u \equiv \sum_{s=0}^{\widehat{S}_i} (\widehat{B}_{i\mu s}(x,D)u)(\widehat{\omega}_{is}(x))|_{\Upsilon_i} = g_{i\mu}(x), \\ x \in \Upsilon_i, \qquad i=1,\dots,N_0, \qquad \mu=1,\dots,m. \tag{6.2}$$

Here $\mathbf{P}(x,D)$ and $\widehat{B}_{i\mu 0}(x,D) = B_{i\mu 0}(x,D)$ are the same differential operators[6] as in § 1, $\widehat{B}_{i\mu s}(x,D)$, $s=1,\dots,\widehat{S}_i$ are some differential operators of orders $m_{i\mu}$ with complex-valued $C^\infty$-coefficients, and $\widehat{\omega}_{is}$ $(i=1,\dots,N_0,\ \ s=1,\dots,\widehat{S}_i)$ are infinitely differentiable non-degenerate transformations that map some neighbourhood $\mathcal{O}_i$ of the manifold $\Upsilon_i$ onto $\widehat{\omega}_{is}(\mathcal{O}_i)$ such that $\widehat{\omega}_{is}(\Upsilon_i) \subset G$, $\omega_{i0}(x) \equiv x$. We assume that the set

$$\widehat{\mathcal{K}} = \Big\{\bigcup_i (\overline{\Upsilon}_i \setminus \Upsilon_i)\Big\} \cup \Big\{\bigcup_{i,s} \widehat{\omega}_{is}(\overline{\Upsilon}_i \setminus \Upsilon_i)\Big\} \cup \Big\{\bigcup_{j,p}\bigcup_{i,s} \widehat{\omega}_{jp}\big(\widehat{\omega}_{is}(\overline{\Upsilon}_i \setminus \Upsilon_i) \cap \Upsilon_j\big)\Big\}$$

can be represented as $\widehat{\mathcal{K}} = \bigcup_{j=1}^{3} \bigcup_{p=1}^{\widehat{N}_j} \widehat{\mathcal{K}}_{jp}$, where

$$\widehat{\mathcal{K}}_1 = \bigcup_{p=1}^{\widehat{N}_1} \widehat{\mathcal{K}}_{1p} = \partial G \setminus \bigcup_{i=1}^{N_0} \Upsilon_i, \qquad \widehat{\mathcal{K}}_2 = \bigcup_{p=1}^{\widehat{N}_2} \widehat{\mathcal{K}}_{2p} \subset \bigcup_{i=1}^{N_0} \Upsilon_i, \qquad \widehat{\mathcal{K}}_3 = \bigcup_{p=1}^{\widehat{N}_3} \widehat{\mathcal{K}}_{3p} \subset G$$

(compare with (1.1)). Here $\widehat{\mathcal{K}}_{jp}$ are disjoint $(n-2)$-dimensional $C^\infty$-manifolds without boundary (points if $n=2$). Moreover, $\widehat{N}_1 = N_1$ and $\widehat{\mathcal{K}}_{1p} = \mathcal{K}_{1p}$, $p = 1,\dots,N_1$.

Let the transformations $\widehat{\omega}_{is}$ satisfy Conditions 1.3, 1.4. We also assume the operators $\widehat{B}_{i\mu s}(x,D)$ and the transformations $\widehat{\omega}_{is}$ $(s=1,\dots,\widehat{S}_i)$ to be such that, for each point $g \in \widehat{\mathcal{K}}_1 = \mathcal{K}_1$, the operator $\mathcal{L}_g^{\widehat{\omega}}$ (which is defined similarly to $\mathcal{L}_g^{\omega}$ of § 1) is equal to the operator $\mathcal{L}_g^{\mathcal{G}}$ defined in § 1.

Thus $\widehat{\omega}_{is}$ is the linear part of $\omega_{is}$ in a neighbourhood of $\mathcal{K}_1$.

We introduce the bounded operator corresponding to the non-local problem (6.1), (6.2),

$$\widehat{\mathbf{L}} = \big\{\mathbf{P}(x,D),\ \widehat{\mathbf{B}}_{i\mu}(x,D)\big\} \colon H_b^{l+2m}(G) \to \mathcal{H}_b^l(G,\Upsilon).$$

**Theorem 6.1.** *Suppose that the hypotheses of Lemma* 4.1 *hold and* $b > l+2m-1$. *Then the operators* $\mathbf{L}$, $\widehat{\mathbf{L}}\colon H_b^{l+2m}(G) \to \mathcal{H}_b^l(G,\Upsilon)$ *are Fredholm and* $\operatorname{ind}\mathbf{L} = \operatorname{ind}\widehat{\mathbf{L}}$.

*Proof.* We define an operator $\mathbf{L}_t\colon H_b^{l+2m}(G) \to \mathcal{H}_b^l(G,\Upsilon)$ by

$$\mathbf{L}_t u = \big\{\mathbf{P}(x,D)u,\ \mathbf{B}_{i\mu}(x,D) + t(\widehat{\mathbf{B}}_{i\mu}(x,D) - \mathbf{B}_{i\mu}(x,D))\big\}.$$

[6] It suffices to require that the principal homogeneous parts of the operators $\mathbf{P}(x,D)$ and $\widehat{B}_{i\mu 0}(x,D)$ coincide with those in § 1. We assume for simplicity that the non-leading terms coincide as well.

Clearly, $\mathbf{L}_0 = \mathbf{L}$ and $\mathbf{L}_1 = \widehat{\mathbf{L}}$.

In a neighbourhood of $\mathcal{K}_1$, the transformations $\omega_{is}$ and $\widehat{\omega}_{is}$ coincide up to infinitesimals. Therefore the operators $\mathbf{L}_t$ are Fredholm for all $t$ by Theorem 5.1. Furthermore, for all $t_0$ and $t$, we have

$$\|\mathbf{L}_t u - \mathbf{L}_{t_0} u\|_{\mathcal{H}_b^l(G,\Upsilon)} \leqslant k_{t_0}|t-t_0|\,\|u\|_{H_b^{l+2m}(G)},$$

where $k_{t_0} > 0$ is independent of $t \in [0,1]$. Hence Theorem 16.2 of [29] yields that $\operatorname{ind}\mathbf{L}_t = \operatorname{ind}\mathbf{L}_{t_0}$ for all $t$ in some small neighbourhood of $t_0$. These neighbourhoods cover the interval $[0,1]$. Choosing a finite subcovering, we get $\operatorname{ind}\mathbf{L} = \operatorname{ind}\mathbf{L}_0 = \operatorname{ind}\mathbf{L}_1 = \operatorname{ind}\widehat{\mathbf{L}}$. The theorem is proved.

An analogous argument, which uses Theorem 5.2 instead of Theorem 5.1, proves the index stability for non-local problem (1.2), (1.3) in the case when $n = 2$, $b \in \mathbb{R}$.

Let us suppose that $\widehat{N}_j = N_j$, $\widehat{\mathcal{K}}_{jp} = \mathcal{K}_{jp}$, $j = 1,2,3$, $p = 1,\dots,N_j$.

**Theorem 6.2.** *Suppose that the hypotheses of Theorem* 5.2 *hold. Then the operators* $\mathbf{L}$, $\widehat{\mathbf{L}}\colon \widetilde{H}_b^{l+2m}(G) \to \widetilde{\mathcal{H}}_b^l(G,\Upsilon)$ *are Fredholm and* $\operatorname{ind}\mathbf{L} = \operatorname{ind}\widehat{\mathbf{L}}$.

**2.** In this subsection we give another proof of Theorem 6.2, based upon ideas of [15]. (Using Lemma 5.2 of [12], one can similarly prove Theorem 6.1.) Although this proof is more complicated, it makes the situation clear by answering why is the index of the operator completely determined by the linear part of the transformations $\omega_{is}$ in a neighbourhood of $\mathcal{K}_1$. We show that if the operators $\mathbf{L}$ and $\widehat{\mathbf{L}}$ are Fredholm, then the restriction of their difference to the kernel $\ker(\mathbf{P}) \subset \widetilde{H}_b^{l+2m}(G)$ of the operator $\mathbf{P} = \mathbf{P}(y,D)$ (we recall that $x = y$ if $n = 2$) can be "reduced" to the sum of an operator with arbitrarily small norm and an operator whose square is compact. The first operator accounts for the non-linear part of the transformations $\omega_{is}$ near $\mathcal{K}_1$, and the second operator accounts for the transformations that generate the sets $\mathcal{K}_2$ and $\mathcal{K}_3$ (see § 1). This "reduction" does not contradict the example in § 2 because the reduction procedure involves projections onto the subspace $\ker(\mathbf{P})$ of infinite codimension. By the same reason, this argument does not prove that the operator $\widehat{\mathbf{L}}$ is Fredholm whenever $\mathbf{L}$ is Fredholm (or vice versa). It only proves that $\operatorname{ind}\mathbf{L} = \operatorname{ind}\widehat{\mathbf{L}}$ provided that both operators are Fredholm.

1) We introduce the operators

$$\mathbf{B}, \widehat{\mathbf{B}}\colon \widetilde{H}_b^{l+2m}(G) \to \widetilde{\mathcal{H}}_b^l(\partial G) = \prod_{i=1}^{N_0}\prod_{\mu=1}^{m} \widetilde{H}_b^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$$

given by $\mathbf{B} = \{\mathbf{B}_{i\mu}(y,D)\}$, $\widehat{\mathbf{B}} = \{\widehat{\mathbf{B}}_{i\mu}(y,D)\}$. We denote by $\mathbf{C}$, $\widehat{\mathbf{C}}$ the restrictions of $\mathbf{B}$, $\widehat{\mathbf{B}}$ to the subspace $\ker(\mathbf{P}) \subset \widetilde{H}_b^{l+2m}(G)$. The operators $\mathbf{L}$, $\widehat{\mathbf{L}}$ are Fredholm by Theorem 5.1. Hence Lemma 1.1 of [15] implies that $\mathbf{C}$, $\widehat{\mathbf{C}}$ are also Fredholm. Now, to prove Theorem 6.2, it suffices to show that $\operatorname{ind}\mathbf{C} = \operatorname{ind}\widehat{\mathbf{C}}$.

2) We denote by $\mathbf{C}^1$, $\widehat{\mathbf{C}}^1$ the restrictions of $\mathbf{C}$, $\widehat{\mathbf{C}}$ to the subspace $\ker(\mathbf{C})^\perp \subset \ker(\mathbf{P})$. It is obvious that $\mathbf{C}^1 = \mathbf{C}\mathbf{I}_0$ and $\widehat{\mathbf{C}}^1 = \widehat{\mathbf{C}}\mathbf{I}_0$, where $\mathbf{I}_0\colon \ker(\mathbf{C})^\perp \to \ker(\mathbf{P})$

is the embedding of $\ker(\mathbf{C})^\perp$ to $\ker(\mathbf{P})$. Clearly, $\dim\ker(\mathbf{I}_0)=0$ and $\operatorname{codim}\mathcal{R}(\mathbf{I}_0)=\dim\ker(\mathbf{C})=m_0<\infty$. Therefore Theorem 12.2 of [29] yields that

$$\begin{aligned}\operatorname{ind}\mathbf{C}^1&=\operatorname{ind}\mathbf{C}+\operatorname{ind}\mathbf{I}_0=\operatorname{ind}\mathbf{C}-m_0,\\ \operatorname{ind}\widehat{\mathbf{C}}^1&=\operatorname{ind}\widehat{\mathbf{C}}+\operatorname{ind}\mathbf{I}_0=\operatorname{ind}\widehat{\mathbf{C}}-m_0.\end{aligned}$$

Thus it suffices to prove that $\operatorname{ind}\mathbf{C}^1=\operatorname{ind}\widehat{\mathbf{C}}^1$.

3) We denote by $\mathbf{P}_\perp$ the operator that orthogonally projects $\widetilde{\mathcal{H}}_b^l(\partial G)$ onto $\mathcal{R}(\mathbf{C}^1)^\perp$. Since $\operatorname{codim}\mathcal{R}(\mathbf{C}^1)<\infty$, it follows that $\mathbf{P}_\perp$ is finite-dimensional. Hence,

$$\operatorname{ind}\widehat{\mathbf{C}}^1=\operatorname{ind}\big(\mathbf{C}^1+(\mathbf{I}-\mathbf{P}_\perp)(\widehat{\mathbf{C}}^1-\mathbf{C}^1)\big).$$

Therefore it suffices to prove that

$$\operatorname{ind}\mathbf{C}^1=\operatorname{ind}\big(\mathbf{C}^1+(\mathbf{I}-\mathbf{P}_\perp)(\widehat{\mathbf{C}}^1-\mathbf{C}^1)\big).$$

Since $\mathbf{C}^1u$, $\mathbf{C}^1u+(\mathbf{I}-\mathbf{P}_\perp)(\widehat{\mathbf{C}}^1-\mathbf{C}^1)u\in\mathcal{R}(\mathbf{C}^1)$ for $u\in\ker(\mathbf{C})^\perp$, we may regard $\mathbf{C}^1$, $\mathbf{C}^1+(\mathbf{I}-\mathbf{P}_\perp)(\widehat{\mathbf{C}}^1-\mathbf{C}^1)$ as operators from $\ker(\mathbf{C})^\perp$ to $\mathcal{R}(\mathbf{C}^1)$. This increases the indices of these operators by the same number $m_1=\operatorname{codim}\mathcal{R}(\mathbf{C}^1)$.

It is clear that the operator $\mathbf{C}^1\colon\ker(\mathbf{C})^\perp\to\mathcal{R}(\mathbf{C}^1)$ has a bounded inverse $\mathbf{R}_1=(\mathbf{C}^1)^{-1}\colon\mathcal{R}(\mathbf{C}^1)\to\ker(\mathbf{C})^\perp$ and $\operatorname{ind}\mathbf{C}^1=0$. By Theorem 12.2 of [29], we have

$$\operatorname{ind}\big(\mathbf{C}^1+(\mathbf{I}-\mathbf{P}_\perp)(\widehat{\mathbf{C}}^1-\mathbf{C}^1)\big)=\operatorname{ind}\big(\mathbf{I}+\mathbf{R}_1(\mathbf{I}-\mathbf{P}_\perp)(\widehat{\mathbf{C}}^1-\mathbf{C}^1)\big).$$

It remains to show that $\operatorname{ind}\big(\mathbf{I}+\mathbf{R}_1(\mathbf{I}-\mathbf{P}_\perp)(\widehat{\mathbf{C}}^1-\mathbf{C}^1)\big)=0$.

4) We introduce a function $\psi_\varepsilon\in C_0^\infty(\mathbb{R}^2)$ such that $\psi_\varepsilon(y)=1$ for $y\in\mathcal{O}_{\varepsilon/2}(\mathcal{K})$, $\psi_\varepsilon(y)=0$ for $y\notin\mathcal{O}_\varepsilon(\mathcal{K})$, and

$$|D^\alpha\psi_\varepsilon(y)|\leqslant k_\alpha(\tilde\rho(y))^{-|\alpha|},\qquad y\in\mathcal{O}_\varepsilon(\mathcal{K}),\tag{6.3}$$

where $k_\alpha>0$ is independent of $\varepsilon$. We consider the operators $\mathbf{A}_1$, $\mathbf{A}_2\colon\ker(\mathbf{C})^\perp\to\ker(\mathbf{C})^\perp$ given by

$$\begin{aligned}\mathbf{A}_1u&=\mathbf{R}_1(\mathbf{I}-\mathbf{P}_\perp)(\widehat{\mathbf{B}}-\mathbf{B})\psi_\varepsilon u,\\ \mathbf{A}_2u&=\mathbf{R}_1(\mathbf{I}-\mathbf{P}_\perp)(\widehat{\mathbf{B}}-\mathbf{B})(1-\psi_\varepsilon)u.\end{aligned}$$

Clearly, $\mathbf{I}+\mathbf{A}_1+\mathbf{A}_2=\mathbf{I}+\mathbf{R}_1(\mathbf{I}-\mathbf{P}_\perp)(\widehat{\mathbf{C}}^1-\mathbf{C}^1)$. Since the support of $(1-\psi_\varepsilon)u$ is disjoint from $\mathcal{K}_1$, the proof of Theorem 3.1 in [15] shows that $(\mathbf{A}_2)^2$ is compact.

Let us study $\mathbf{A}_1$. Since the operator $\mathbf{R}_1(\mathbf{I}-\mathbf{P}_\perp)$ is bounded, we have

$$\|\mathbf{A}_1u\|_{\widetilde{H}_b^{l+2m}(G)}\leqslant c\|(\widehat{\mathbf{B}}-\mathbf{B})\psi_\varepsilon u\|_{\widetilde{\mathcal{H}}_b^l(\partial G)}.$$

Using partitions of unity and estimates (4.2)–(4.7) and (6.3), we obtain

$$\begin{aligned}\|\mathbf{A}_1u\|_{\widetilde{H}_b^{l+2m}(G)}&\leqslant c_1\big(\varepsilon\|\psi_\varepsilon u\|_{\widetilde{H}_b^{l+2m}(G)}\\ &\qquad+\|\mathbf{P}\psi_\varepsilon u\|_{\widetilde{H}_b^{l}(G)}\big)+k_1(\varepsilon)\|u\|_{\widetilde{H}_b^{l+2m-1}(G)}\\ &\leqslant c_2\big(\varepsilon\|u\|_{\widetilde{H}_b^{l+2m}(G)}\\ &\qquad+\|\mathbf{P}\psi_\varepsilon u\|_{\widetilde{H}_b^{l}(G)}\big)+k_1(\varepsilon)\|u\|_{\widetilde{H}_b^{l+2m-1}(G)}.\end{aligned}\tag{6.4}$$

Since $u \in \ker(\mathbf{P})$, we see from (6.4) and Leibnitz' formula that

$$\|\mathbf{A}_1 u\|_{\widetilde{H}_b^{l+2m}(G)} \leqslant c_2\varepsilon\|u\|_{\widetilde{H}_b^{l+2m}(G)} + k_2(\varepsilon)\|u\|_{\widetilde{H}_b^{l+2m-1}(G)}, \tag{6.5}$$

where $c_2$ is independent of $\varepsilon$. Using (6.5), the compactness of the embedding $\widetilde{H}_b^{l+2m}(G) \subset \widetilde{H}_b^{l+2m-1}(G)$, and Lemma 5.1, we conclude that $\mathbf{A}_1 = \mathbf{M}_1 + \mathbf{F}_1$, where $\|\mathbf{M}_1\| \leqslant 2c_2\varepsilon$ and $\mathbf{F}_1$ is finite-dimensional.

Thus, we have $\mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\widehat{\mathbf{C}}^1 - \mathbf{C}^1) = \mathbf{M}_1 + \mathbf{F}_1 + \mathbf{A}_2$. Choosing $\varepsilon$ to be sufficiently small, we see from Theorems 15.4 and 16.2 of [29] that

$$\operatorname{ind}\bigl(\mathbf{I} + \mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\widehat{\mathbf{C}}^1 - \mathbf{C}^1)\bigr) = 0.$$

Theorem 6.2 is proved.

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