Generalized Solutions of Nonlocal Elliptic Problems

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Abstract

An elliptic equation of order $2m$ with general nonlocal boundary-value conditions, in a plane bounded domain G with piecewise smooth boundary, is considered. Generalized solutions belonging to the Sobolev space $W_2^m(G)$ are studied. The Fredholm property of the unbounded operator corresponding to the elliptic equation, acting on $L_2(G)$, and defined for functions from the space $W_2^m(G)$ that satisfy homogeneous nonlocal conditions is proved.

Introduction

In the one-dimensional case, nonlocal problems were studied by A. Sommerfeld [1], J. D. Tamarkin [2], M. Picone [3]. T. Carleman [4] considered the problem of finding a function harmonic on a two-dimensional bounded domain and subjected to a nonlocal condition connecting the values of this function at different points of the boundary. A. V. Bitsadze and A. A. Smarskii [5] suggested another setting of a nonlocal problem arising in plasma theory: to find a function harmonic on a two-dimensional bounded domain and satisfying nonlocal conditions on shifts of the boundary that can take points of the boundary inside the domain. Different generalizations of the above nonlocal problems were investigated by many authors [6, 7, 8, 9, 10, 11, 12].

It turns out that the most difficult situation occurs if the support of nonlocal terms intersects the boundary. In that case, solutions of nonlocal problems can have power-law singularities near some points even if the boundary and the right-hand sides are infinitely smooth [13, 14]. For this reason, such problems are naturally studied in weighted spaces (introduced by V. A. Kondrat'ev for boundary-value problems in nonsmooth domains [15]). The most complete theory of nonlocal problems in weighted spaces is developed by A. L. Skubachevskii [13, 16, 17, 18, 19].

Note that the investigation of nonlocal problems is motivated both by significant theoretical progress in that direction and important applications arising in biophysics, theory of diffusion processes [20], plasma theory [21], and so on.

In the present paper, we study *generalized solutions* of an elliptic equation of order $2m$ in a two-dimensional bounded domain G, satisfying nonlocal boundary-value conditions that are set on parts Γ_j of the boundary $\partial G = \bigcup_j \overline{\Gamma_j}$. By generalized solutions, we mean functions from the Sobolev space $W^m(G) = W_2^m(G)$. We prove that an unbounded operator acting on $L_2(G)$ and corresponding to the above nonlocal problem has the Fredholm property.

Note that solutions of nonlocal problems can be sought on the space of "smooth" functions, namely, on the Sobolev space $W^{2m}(G)$ (see [22, 23]) or on weighted spaces $H_a^{2m}(G)$, where

$$\|u\|_{H_a^k(G)} = \left(\sum_{|\alpha|\le k} \int_G \rho^{2(a-k+|\alpha|)} |D^\alpha u|^2 \right)^{1/2},$$

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$k \ge 0$ is an integer, $a \in \mathbb{R}$, $\rho = \rho(y) = \operatorname{dist}(y, \mathcal{K})$, and $\mathcal{K} = \bigcup_j \overline{\Gamma_j} \setminus \Gamma_j$ is the set formed by finitely many points of conjugation of nonlocal conditions (see [13, 17]). In both cases, a *bounded* operator corresponds to the nonlocal problem. Whether or not this operator has the Fredholm property depends on spectral properties of some auxiliary problems with a parameter. In turn, these spectral properties are affected by the values of the coefficients in nonlocal conditions and by a geometrical structure of the support of nonlocal terms and the boundary near the set $\mathcal{K}$. However, if we consider generalized solutions (i.e., functions from $W^m(G)$), then the corresponding *unbounded* operator turns out to have the Fredholm property irrespective of the above factors.

Earlier the Fredholm property of an unbounded nonlocal operator on $L_2(G)$ was studied either for the case in which nonlocal conditions were set on shifts of the boundary [19] or in the case of a nonlocal perturbation of the Dirichlet problem for a second-order elliptic equation [11, 12]. Elliptic equations of order $2m$ with general nonlocal conditions are investigated for the first time.

1 Setting of Nonlocal Problems in Bounded Domains

1.1 Setting of Problem

Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary ∂G. Consider a set $\mathcal{K} \subset \partial G$ consisting of finitely many points. Let $\partial G \setminus \mathcal{K} = \bigcup\limits_{i=1}^{N} \Gamma_i$, where Γ_i are open (in the topology of ∂G) C^∞-curves. We assume that, in a neighborhood of each point $g \in \mathcal{K}$, the domain G is a plane angle.

Denote by $\mathbf{P}(y, D_y) = \mathbf{P}(y, D_{y_1}, D_{y_2})$ and $B_{i\mu s}(y, D_y) = B_{i\mu s}(y, D_{y_1}, D_{y_2})$ differential operators of order $2m$ and $m_{i\mu}$ ($m_{i\mu} \le m-1$), respectively, with complex-valued C^∞ coefficients, and let $\mathbf{P}^0(y, D_y)$ and $B^0_{i\mu s}(y, D_y)$ denote their principal homogeneous parts ($i = 1, \dots, N$; $\mu = 1, \dots, m$; $s = 0, \dots, S_i$). Here $D_y = (D_{y_1}, D_{y_2})$, $D_{y_j} = -i\partial/\partial y_j$.

Now we formulate conditions on the operators $\mathbf{P}(y, D_y)$ and $B_{i\mu 0}(y, D_y)$ (these operators will correspond to a "local" elliptic problem). We assume that the operator $\mathbf{P}(y, D_y)$ is *properly elliptic* on $\overline{G}$; in particular, the following estimate holds for all $\theta \in \mathbb{R}^2$ and $y \in \overline{G}$:

$$A^{-1}|\theta|^{2m} \le |\mathbf{P}^0(y, \theta)| \le A|\theta|^{2m}, \quad A > 0. \tag{1.1}$$

Further, let $y \in \overline{\Gamma_i}$. One may assume with no loss of generality that the curve $\overline{\Gamma_i}$ is defined by the equation $y_2 = 0$ near the point y. We suppose that the *system* $\{B_{i\mu 0}(y, D_y)\}_{\mu=1}^m$ *satisfies the Lopatinsky condition with respect to the operator* $\mathbf{P}(y, D_y)$ *for all* $i = 1, \dots, N$. In other words, let the polynomial

$$B'_{i\mu 0}(y, \tau) \equiv \sum_{\nu=1}^{m} b_{i\mu\nu}(y)\tau^{\nu-1} \equiv B^0_{i\mu 0}(y, 1, \tau) \ \big(\operatorname{mod} \mathbf{M}^+(y, \tau)\big)$$

be the residue of dividing $B^0_{i\mu 0}(y, 1, \tau)$ by $\mathbf{M}^+(y, \tau)$, where

$$\mathbf{M}^+(y, \tau) = \prod_{\nu=1}^{m} (\tau - \tau_\nu^+(y)),$$

while $\tau_1^+(y), \dots, \tau_m^+(y)$ are the roots of the polynomial $\mathbf{P}^0(y, 1, \tau)$ with positive imaginary parts (note that $\mathbf{P}^0(y, 1, \tau)$, $B^0_{i\mu 0}(y, 1, \tau)$, and $\mathbf{M}^+(y, \tau)$ are considered as polynomials in τ). In this case, the validity of the Lopatinsky condition means that

$$d_i(y) = \det \|b_{i\mu\nu}(y)\|_{\mu,\nu=1}^m \ne 0.$$

Since each of the curves $\overline{\Gamma_i}$, $i = 1, \dots, N$, is a compact, it follows that

$$D = \min_{i=1,\dots,N} \inf_{y \in \overline{\Gamma_i}} |d_i(y)| > 0. \tag{1.2}$$

We emphasize that the operators $B_{i\mu 0}(y, D_y)$ are not necessarily normal on $\overline{\Gamma_i}$.

For an integer $k \ge 0$, denote by $W^k(G) = W_2^k(G)$ the Sobolev space with the norm

$$\|u\|_{W^k(G)} = \left(\sum_{|\alpha| \le k} \int_G |D^\alpha u|^2 \, dy \right)^{1/2}$$

(we set $W^0(G) = L_2(G)$ for $k = 0$). For an integer $k \ge 1$, we introduce the space $W^{k-1/2}(\Gamma)$ of traces on a smooth curve $\Gamma \subset \overline{G}$ with the norm

$$\|\psi\|_{W^{k-1/2}(\Gamma)} = \inf \|u\|_{W^k(G)} \quad (u \in W^k(G) : u|_\Gamma = \psi). \tag{1.3}$$

Denote $\mathbf{B}^0_{i\mu} u = B_{i\mu 0}(y, D_y)u(y)|_{\Gamma_i}$. As we have mentioned above, the operators $\mathbf{P}(y, D_y)$ and $\mathbf{B}^0_{i\mu}$ will *correspond to a "local" boundary-value problem.*

Now we define operators corresponding to nonlocal conditions near the set $\mathcal{K}$. Let Ω_{is} ($i = 1, \dots, N$; $s = 1, \dots, S_i$) be C^∞-diffeomorphisms taking some neighborhood $\mathcal{O}_i$ of the curve $\overline{\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})}$ onto the set $\Omega_{is}(\mathcal{O}_i)$ in such a way that

$$\Omega_{is}(\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G,$$

$$\Omega_{is}(g) \in \mathcal{K} \qquad \text{for} \qquad g \in \overline{\Gamma_i} \cap \mathcal{K}. \tag{1.4}$$

Here $\varepsilon > 0$, $\mathcal{O}_\varepsilon(\mathcal{K}) = \{y \in \mathbb{R}^2 : \operatorname{dist}(y, \mathcal{K}) < \varepsilon\}$ is the ε-neighborhood of the set $\mathcal{K}$. Thus, under the transformations Ω_{is}, the curves $\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ are mapped strictly inside the domain G, whereas the set of end points $\overline{\Gamma_i} \cap \mathcal{K}$ is mapped to itself.

Let us specify the structure of the transformations Ω_{is} near the set $\mathcal{K}$. Denote by the symbol Ω^{+1}_{is} the transformation $\Omega_{is} : \mathcal{O}_i \to \Omega_{is}(\mathcal{O}_i)$ and by Ω^{-1}_{is} the transformation $\Omega^{-1}_{is} : \Omega_{is}(\mathcal{O}_i) \to \mathcal{O}_i$ inverse to Ω_{is}. The set of all points

$$\Omega^{\pm 1}_{i_q s_q}(\dots \Omega^{\pm 1}_{i_1 s_1}(g)) \in \mathcal{K} \qquad (1 \le s_j \le S_{i_j},\ j = 1, \dots, q),$$

i.e., the set of all points that can be obtained by consecutively applying the transformations $\Omega^{+1}_{i_j s_j}$ or $\Omega^{-1}_{i_j s_j}$ (taking the points of $\mathcal{K}$ to $\mathcal{K}$) to the point $g \in \mathcal{K}$ is called an *orbit* of g and is denoted by $\operatorname{Orb}(g)$.

Clearly, for any $g, g' \in \mathcal{K}$ either $\operatorname{Orb}(g) = \operatorname{Orb}(g')$ or $\operatorname{Orb}(g) \cap \operatorname{Orb}(g') = \varnothing$. In what follows, we suppose that the set $\mathcal{K}$ consists of a unique orbit. (All results can be directly generalized for the case in which $\mathcal{K}$ consists of finitely many mutually disjoint orbits.) The set (orbit) $\mathcal{K}$ consists of N points, which we denote by g_j, $j = 1, \dots, N$.

Take a small number ε (see Remark 1.3 below) such that there exist neighborhoods $\mathcal{O}_{\varepsilon_1}(g_j)$ of the points $g_j \in \mathcal{K}$ satisfying the following conditions:

1. $\mathcal{O}_{\varepsilon_1}(g_j) \supset \mathcal{O}_\varepsilon(g_j)$;

2. in the neighborhood $\mathcal{O}_{\varepsilon_1}(g_j)$, the boundary ∂G is a plane angle;

3. $\overline{\mathcal{O}_{\varepsilon_1}(g_j)} \cap \overline{\mathcal{O}_{\varepsilon_1}(g_k)} = \varnothing$ for any $g_j, g_k \in \mathcal{K}$, $k \ne j$;

4. if $g_j \in \overline{\Gamma_i}$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_\varepsilon(g_j) \subset \mathcal{O}_i$ and $\Omega_{is}\big(\mathcal{O}_\varepsilon(g_j)\big) \subset \mathcal{O}_{\varepsilon_1}(g_k)$.

For each point $g_j \in \overline{\Gamma_i} \cap \mathcal{K}$, we fix a transformation $y \mapsto y'(g_j)$ of the argument; this transformation is the composition of the shift by the vector $-\overrightarrow{Og_j}$ and a rotation by some angle such that the set $\mathcal{O}_{\varepsilon_1}(g_j)$ is mapped onto the neighborhood $\mathcal{O}_{\varepsilon_1}(0)$ of the origin, while the sets

$$G \cap \mathcal{O}_{\varepsilon_1}(g_j) \qquad \text{and} \qquad \Gamma_i \cap \mathcal{O}_{\varepsilon_1}(g_j)$$

are mapped onto the intersection of a plane angle

$$K_j = \{y \in \mathbb{R}^2 :\ r > 0,\ |\omega| < \omega_j\}$$

with the neighborhood $\mathcal{O}_{\varepsilon_1}(0)$ and the intersection of the side

$$\gamma_{j\sigma} = \{y \in \mathbb{R}^2 :\ \omega = (-1)^\sigma \omega_j\}$$

($\sigma = 1$ or $\sigma = 2$) of the angle K_j with the neighborhood $\mathcal{O}_{\varepsilon_1}(0)$, respectively. Here (ω, r) are the polar coordinates of the point y and $0 < \omega_j < \pi$.

Condition 1.1. *The above change of variables $y \mapsto y'(g_j)$ for $y \in \mathcal{O}_\varepsilon(g_j)$, $g_j \in \overline{\Gamma_i} \cap \mathcal{K}$, reduces the transformation $\Omega_{is}(y)$ $(i = 1, \dots, N,\ s = 1, \dots, S_i)$ to the composition of a rotation and a homothety in the new variables y'.*

Remark 1.1. In particular, Condition 1.1 combined with the assumption $\Omega_{is}(\Gamma_i) \subset G$ means that, if $g \in \Omega_{is}(\overline{\Gamma_i} \cap \mathcal{K}) \cap \overline{\Gamma_j} \cap \mathcal{K} \neq \varnothing$, then the curves $\Omega_{is}(\overline{\Gamma_i})$ and $\overline{\Gamma_j}$ are not tangent to each other at the point g.

Consider a number ε_0, $0 < \varepsilon_0 \le \varepsilon$, satisfying the following condition: if $g_j \in \overline{\Gamma_i}$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_{\varepsilon_0}(g_k) \subset \Omega_{is}\big(\mathcal{O}_\varepsilon(g_j)\big)$. Introduce a function $\zeta \in C^\infty(\mathbb{R}^2)$ such that

$$\zeta(y) = 1 \quad \text{for} \quad y \in \mathcal{O}_{\varepsilon_0/2}(\mathcal{K}), \qquad \operatorname{supp}\zeta \subset \mathcal{O}_{\varepsilon_0}(\mathcal{K}). \tag{1.5}$$

Now we define nonlocal operators $\mathbf{B}^1_{i\mu}$ by the formula

$$\mathbf{B}^1_{i\mu} u = \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)(\zeta u)\big)\big(\Omega_{is}(y)\big) \qquad \text{for} \qquad y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K}),$$

$$\mathbf{B}^1_{i\mu} u = 0 \qquad \text{for} \qquad y \in \Gamma_i \setminus (\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})),$$

where $\big(B_{i\mu s}(y, D_y)u\big)\big(\Omega_{is}(y)\big) = B_{i\mu s}(x, D_x)u(x)|_{x=\Omega_{is}(y)}$. Since $\mathbf{B}^1_{i\mu} u = 0$ whenever $\operatorname{supp} u \subset \overline{G} \setminus \overline{\mathcal{O}_{\varepsilon_0}(\mathcal{K})}$, we say that the operators $\mathbf{B}^1_{i\mu}$ *correspond to nonlocal terms supported near the set $\mathcal{K}$.*

For any $\rho > 0$, we denote $G_\rho = \{y \in G : \operatorname{dist}(y, \partial G) > \rho\}$. Consider operators $\mathbf{B}^2_{i\mu}$ satisfying the following condition (cf. [13, 18, 22]).

Condition 1.2. *There exist numbers $\varkappa_1 > \varkappa_2 > 0$ and $\rho > 0$ such that the inequalities*

$$\|\mathbf{B}^2_{i\mu} u\|_{W^{2m-m_{i\mu}-1/2}(\Gamma_i)} \le c_1 \|u\|_{W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})}, \tag{1.6}$$

$$\|\mathbf{B}^2_{i\mu} u\|_{W^{2m-m_{i\mu}-1/2}(\Gamma_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})})} \le c_2 \|u\|_{W^{2m}(G_\rho)} \tag{1.7}$$

hold for any

$$u \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})}) \cap W^{2m}(G_\rho),$$

where $i = 1, \dots, N$, $\mu = 1, \dots, m$, and $c_1, c_2 > 0$.

It follows from (1.6) that $\mathbf{B}^2_{i\mu} u = 0$ whenever $\operatorname{supp} u \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$. For this reason, we say that the operators $\mathbf{B}^2_{i\mu}$ *correspond to nonlocal terms supported outside the set $\mathcal{K}$.*

We will suppose throughout that Conditions 1.1 and 1.2 hold.

We study the following nonlocal elliptic problem:

$$\mathbf{P}(y, D_y)u = f_0(y) \quad (y \in G), \tag{1.8}$$

$$\mathbf{B}_{i\mu} u \equiv \mathbf{B}^0_{i\mu} u + \mathbf{B}^1_{i\mu} u + \mathbf{B}^2_{i\mu} u = 0 \quad (y \in \Gamma_i;\ i = 1, \dots, N;\ \mu = 1, \dots, m), \tag{1.9}$$

where $f_0 \in L_2(G)$. Introduce the space $W^m_B(G)$ consisting of functions $u \in W^m(G)$ that satisfy homogeneous nonlocal conditions (1.9): $\mathbf{B}_{i\mu} u = 0$.

Consider the unbounded operator $\mathbf{P} : \operatorname{Dom}(\mathbf{P}) \subset L_2(G) \to L_2(G)$ given by

$$\mathbf{P}u = \mathbf{P}(y, D_y)u, \qquad u \in \operatorname{Dom}(\mathbf{P}) = \{u \in W^m_B(G) : \mathbf{P}(y, D_y)u \in L_2(G)\}.$$

Definition 1.1. A function u is called a *generalized solution* of problem (1.8), (1.9) with right-hand side $f_0 \in L_2(G)$ if $u \in \mathrm{Dom}\,(\mathbf{P})$ and $\mathbf{P}u = f_0$.

One can give another (equivalent) definition for a generalized solution. To do so, we write the operator $\mathbf{P}(y, D_y)$ in the divergent form,

$$\mathbf{P}(y, D_y) = \sum_{0\le|\xi|,|\beta|\le m} D^\beta p_{\xi\beta}(y) D^\xi,$$

where $p_{\xi\beta}$ are infinitely differentiable functions.

For any set $X \in \mathbb{R}^2$ having a nonempty interior, denote by $C_0^\infty(X)$ the set of functions infinitely differentiable on $\overline{X}$ and compactly supported on X.

Definition 1.2. A function u is called a *generalized solution* of problem (1.8), (1.9) with right-hand side $f_0 \in L_2(G)$ if $u \in W_B^m(G)$ and the integral identity

$$\sum_{0\le|\xi|,\,|\beta|\le m} \int\limits_G p_{\xi\beta}(y) D^\xi u \overline{D^\beta v}\, dy = \int\limits_G f_0 \overline{v}\, dy$$

holds for any $v \in C_0^\infty(G)$.

Remark 1.2. Generalized solutions a priori belong to the space $W^m(G)$, whereas Condition 1.2 is formulated for functions belonging to the space W^{2m} inside the domain and near the smooth part of the boundary. Such a formulation can be justified by the fact that any generalized solution belongs to W^{2m} outside an arbitrarily small neighborhood of the set $\mathcal{K}$ (see Lemma 2.1 below).

Remark 1.3. We have supposed above that the number ε is small (whereas $\varkappa_1, \varkappa_2, \rho$ can be arbitrary). Let us show that this leads to no loss of generality. Let us have a number $\hat\varepsilon$, $0 < \hat\varepsilon < \varepsilon$. Take a number $\hat\varepsilon_0$, $0 < \hat\varepsilon_0 \le \hat\varepsilon$, satisfying the following condition: if $g_j \in \overline{\Gamma_i}$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_{\hat\varepsilon_0}(g_k) \subset \Omega_{is}\big(\mathcal{O}_{\hat\varepsilon}(g_j)\big)$. Consider a function $\hat\zeta \in C^\infty(\mathbb{R}^2)$ such that $\hat\zeta(y) = 1$ for $y \in \mathcal{O}_{\hat\varepsilon_0/2}(\mathcal{K})$ and $\operatorname{supp}\hat\zeta \subset \mathcal{O}_{\hat\varepsilon_0}(\mathcal{K})$. Introduce the operators $\mathbf{B}^1_{i\mu}$ as follows:

$$\mathbf{B}^1_{i\mu}u = \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)(\zeta u)\big)\big(\Omega_{is}(y)\big) \qquad \text{for} \qquad y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K}),$$

$$\mathbf{B}^1_{i\mu}u = 0 \qquad \text{for} \qquad y \in \Gamma_i \setminus (\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})).$$

Clearly,

$$\mathbf{B}^0_{i\mu} + \mathbf{B}^1_{i\mu} + \mathbf{B}^2_{i\mu} = \mathbf{B}^0_{i\mu} + \hat{\mathbf{B}}^1_{i\mu} + \hat{\mathbf{B}}^2_{i\mu},$$

where $\hat{\mathbf{B}}^2_{i\mu} = \mathbf{B}^1_{i\mu} - \hat{\mathbf{B}}^1_{i\mu} + \mathbf{B}^2_{i\mu}$. Since $\mathbf{B}^1_{i\mu}u - \hat{\mathbf{B}}^1_{i\mu}u = 0$ near the set $\mathcal{K}$, it follows that the operator $\mathbf{B}^1_{i\mu} - \hat{\mathbf{B}}^1_{i\mu}$ satisfies Condition 1.2 for some suitable $\varkappa_1, \varkappa_2, \rho$ (see [22, § 1] for more details). Thus, we can always choose ε to be as small as necessary (possibly at the expense of a modification of the operator $\mathbf{B}^2_{i\mu}$ and the values of $\varkappa_1, \varkappa_2, \rho$).

1.2 Example of Nonlocal Problem

One can consider the following example as a model one.

Example 1.1. Let $\mathbf{P}(y, D_y)$ and $B_{i\mu s}(y, D_y)$ be the same operators as above. Let Ω_{is} ($i = 1, \dots, N$; $s = 1, \dots, S_i$) be C^∞-diffeomorphisms taking some neighborhood $\mathcal{O}_i$ of the (*whole*) curve Γ_i to the set $\Omega_{is}(\mathcal{O}_i)$ in such a way that $\Omega_{is}(\Gamma_i) \subset G$. Consider the following nonlocal problem:

$$\mathbf{P}(y, D_y)u = f_0(y) \quad (y \in G), \tag{1.10}$$

$$
B_{i\mu 0}(y, D_y)u(y)|_{\Gamma_i} + \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)u\big)\big(\Omega_{is}(y)\big)\big|_{\Gamma_i} = 0 \qquad (y \in \Gamma_i;\ i = 1, \dots, N;\ \mu = 1, \dots, m). \tag{1.11}
$$

We emphasize that *a priori the transformations* Ω_{is} *are not supposed to satisfy condition* (1.4); *however, we further represent the nonlocal operators as the sum of the operators* $\mathbf{B}^0_{i\mu}$, $\mathbf{B}^1_{i\mu}$, *and* $\mathbf{B}^2_{i\mu}$, *and the transformations occurring in the definition of the operators* $\mathbf{B}^1_{i\mu}$ *will satisfy condition* (1.4). To obtain this representation, we take a small ε such that, for any point $g \in \mathcal{K}$, the set $\overline{\mathcal{O}_\varepsilon(g)}$ intersects the curve $\overline{\Omega_{is}(\Gamma_i)}$ only if $g \in \overline{\Omega_{is}(\Gamma_i)}$. If $g \in \overline{\Gamma_i} \cap \mathcal{K}$ and $\Omega_{is}(g) \in \mathcal{K}$, then we assume that the transformation $\Omega_{is}(y)$ satisfies Condition 1.1 for $y \in \mathcal{O}_\varepsilon(g)$.

Remark 1.4. By Remark 1.1, Condition 1.1 is a restriction on the geometrical structure of the support of nonlocal terms near the set $\mathcal{K}$. However, if $\Omega_{is}(\overline{\Gamma_i} \cap \mathcal{K}) \subset \partial G \setminus \mathcal{K}$, then we impose no restrictions on the geometrical structure of the curve $\Omega_{is}(\overline{\Gamma_i})$ near the boundary ∂G (cf. [13, 17]).

Let $\zeta \in C^\infty(\mathbb{R}^2)$ be a function satisfying relations (1.5). Introduce the operators

$$
\mathbf{B}^0_{i\mu} u = B_{i\mu 0}(y, D_y)u(y)|_{\Gamma_i},
$$

$$
\mathbf{B}^1_{i\mu} u = \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)(\zeta u)\big)\big(\Omega_{is}(y)\big)\big|_{\Gamma_i},
$$

$$
\mathbf{B}^2_{i\mu} u = \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)((1-\zeta) u)\big)\big(\Omega_{is}(y)\big)\big|_{\Gamma_i}
$$

(see figures 1.1 and 1.2). Since the support of the function ζ is concentrated near the set $\mathcal{K}$, one may assume that the transformations Ω_{is} occurring in the definition of the operator $\mathbf{B}^1_{i\mu}$ are defined on some neighborhood of the set $\mathcal{K}$ and satisfy condition (1.4). Moreover, it follows from [22, Sec. 1.2] that the operator $\mathbf{B}^2_{i\mu}$ satisfies Condition 1.2. Therefore, problem (1.10), (1.11) can be represented in the form (1.8), (1.9).

1.3 Nonlocal Problems near the Set $\mathcal{K}$

When studying problem (1.8), (1.9), one must pay special attention to the behavior of solutions near the set $\mathcal{K}$ of conjugation points. Now we consider the corresponding model problems.

Denote by $u_j(y)$ the function $u(y)$ for $y \in \mathcal{O}_{\varepsilon_1}(g_j)$. If $g_j \in \overline{\Gamma_i}$, $y \in \mathcal{O}_\varepsilon(g_j)$, and $\Omega_{is}(y) \in \mathcal{O}_{\varepsilon_1}(g_k)$, then we denote the function $u(\Omega_{is}(y))$ by $u_k(\Omega_{is}(y))$. In this notation, nonlocal problem (1.8), (1.9) acquires the following form in the ε-neighborhood of the set (orbit) $\mathcal{K}$:

$$
\mathbf{P}(y, D_y)u_j = f_0(y) \quad (y \in \mathcal{O}_\varepsilon(g_j) \cap G),
$$

$$
B_{i\mu 0}(y, D_y)u_j(y)|_{\mathcal{O}_\varepsilon(g_j)\cap\Gamma_i} + \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)(\zeta u_k)\big)\big(\Omega_{is}(y)\big)\big|_{\mathcal{O}_\varepsilon(g_j)\cap\Gamma_i} = f_{i\mu}(y)
$$

$$
\big(y \in \mathcal{O}_\varepsilon(g_j) \cap \Gamma_i;\ i \in \{1 \le i \le N : g_j \in \overline{\Gamma_i}\};\ j = 1, \dots, N;\ \mu = 1, \dots, m\big),
$$

where $f_{i\mu} = -\mathbf{B}^2_{i\mu} u$.

Let $y \mapsto y'(g_j)$ be the change of variables described in Sec. 1.1. Denote $K^\varepsilon_j = K_j \cap \mathcal{O}_\varepsilon(0)$ and $\gamma^\varepsilon_{j\sigma} = \gamma_{j\sigma} \cap \mathcal{O}_\varepsilon(0)$. Introduce the functions

$$
U_j(y') = u_j(y(y')), \qquad f_j(y') = f_0(y(y')), \qquad y' \in K^\varepsilon_j,
$$

$$
f_{j\sigma\mu}(y') = f_{i\mu}(y(y')), \qquad y' \in \gamma^\varepsilon_{j\sigma},
$$

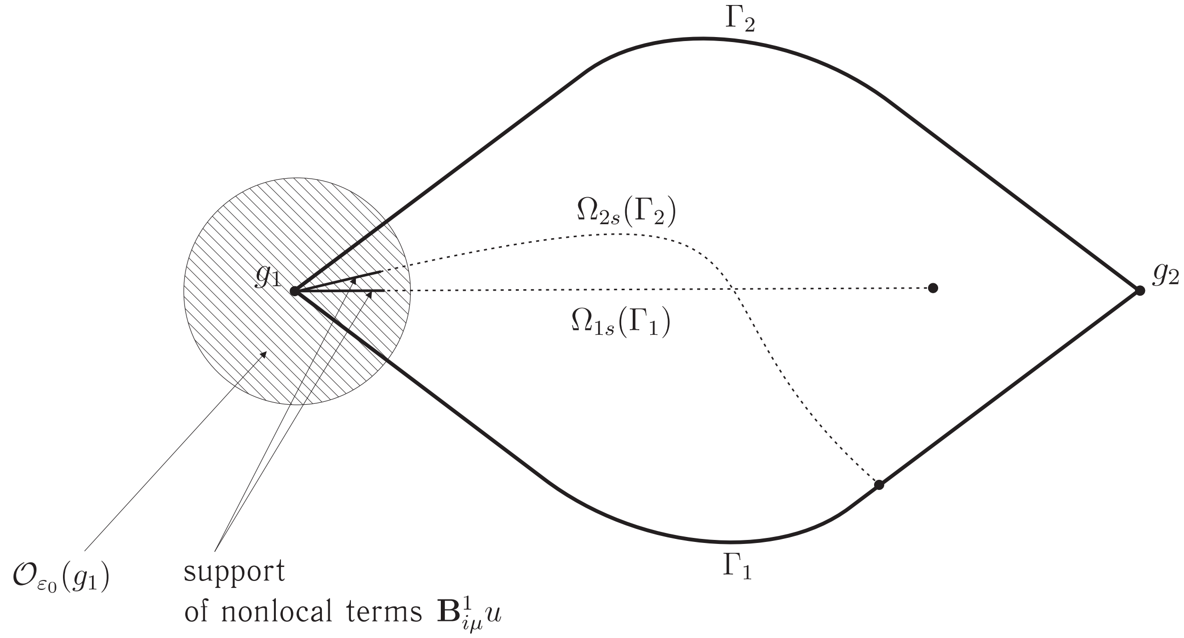

Figure 1.1: To problem (1.10), (1.11)

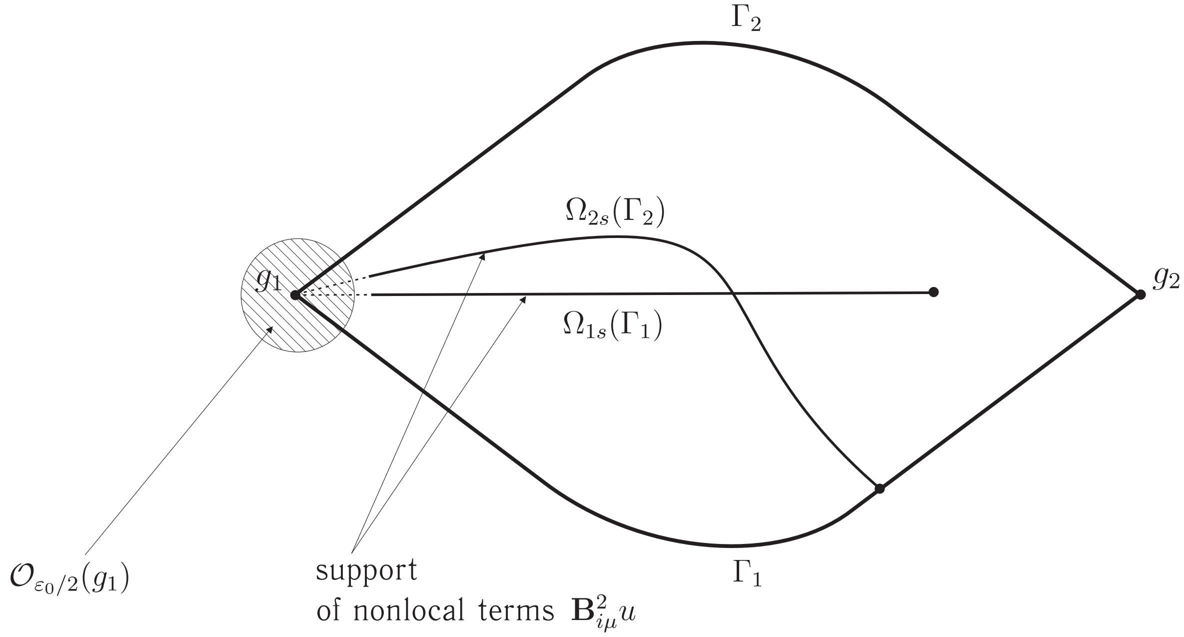

Figure 1.2: To problem (1.10), (1.11)

where $\sigma = 1$ ($\sigma = 2$) if, under the transformation $y \mapsto y'(g_j)$, the curve Γ_i is mapped to the side γ_{j1} (γ_{j2}) of the angle K_j. Denote y' by y again. Then, by virtue of Condition 1.1, problem (1.8), (1.9) acquires the form

$$\mathbf{P}_j(y, D_y)U_j = f_j(y) \quad (y \in K_j^\varepsilon), \tag{1.12}$$

$$\sum_{k,s}(B_{j\sigma\mu ks}(y, D_y)U_k)(\mathcal{G}_{j\sigma ks}y) = f_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma}^\varepsilon). \tag{1.13}$$

Here (and below unless otherwise stated) $j, k = 1, \dots, N$; $\sigma = 1, 2$; $\mu = 1, \dots, m$; $s = 0, \dots, S_{j\sigma k}$; $\mathbf{P}_j(y, D_y)$ and $B_{j\sigma\mu ks}(y, D_y)$ are differential operators of order $2m$ and $m_{j\sigma\mu}$ ($m_{j\sigma\mu} \le m-1$), respectively, with C^∞ complex-valued coefficients, i.e.,

$$\mathbf{P}_j(y, D_y) = \sum_{|\alpha|\le 2m} p_{j\alpha}(y)D_y^\alpha, \qquad B_{j\sigma\mu ks}(y, D_y) = \sum_{|\alpha|\le m_{j\sigma\mu}} b_{j\sigma\mu ks\alpha}(y)D_y^\alpha;$$

$\mathcal{G}_{j\sigma ks}$ is the operator of rotation by an angle $\omega_{j\sigma ks}$ and of the homothety with a coefficient $\chi_{j\sigma ks}$ ($\chi_{j\sigma ks} > 0$) in the y-plane. Moreover,

$$|(-1)^\sigma b_j + \omega_{j\sigma ks}| < b_k \qquad \text{for} \qquad (k, s) \ne (j, 0)$$

(cf. Remark 1.1) and

$$\omega_{j\sigma j0} = 0, \qquad \chi_{j\sigma j0} = 1$$

(i.e., $\mathcal{G}_{j\sigma j0}y \equiv y$).

2 The Fredholm Property of Nonlocal Problems

In this section, we prove the following result.

Theorem 2.1. *Let the operator* $\mathbf{P}(y, D_y)$ *be properly elliptic on* $\overline{G}$, *and let the system* $\{B_{i\mu0}(y, D_y)\}_{\mu=1}^m$ *satisfy the Lopatinsky condition on the curve* $\overline{\Gamma_i}$ *with respect to* $\mathbf{P}(y, D_y)$ *for all* $i = 1, \dots, N$. *Assume that Conditions* 1.1 *and* 1.2 *are fulfilled. Then the operator* $\mathbf{P}$ *has the Fredholm property.*

Remark 2.1. One can assign a bounded operator (acting from $W^{2m}(G)$ to $L_2(G)$) to problem (1.8), (1.9). Such an operator is studied in [22, 23]; it is proved that, unlike the case treated in the present paper, whether or not the bounded operator has the Fredholm property depends both on spectral properties of auxiliary nonlocal problems with a parameter and on the validity of some algebraic relations between the operators $\mathbf{P}(y, D_y)$, $\mathbf{B}_{i\mu}^0$, and $\mathbf{B}_{i\mu}^1$ at the points of the set $\mathcal{K}$.

2.1 Finite Dimensionality of the Kernel

In this subsection, we prove that the kernel of the operator $\mathbf{P}$ is of finite dimension. To do this, we preliminarily study the smoothness of generalized solution of problem (1.8), (1.9). We first study the smoothness outside a neighborhood of the set $\mathcal{K}$ and then near $\mathcal{K}$. The following lemma generalizes part 1 of Theorem 5 in [24].

Lemma 2.1. *Let Condition* 1.2 *hold, and let* $u \in W^m(G)$ *be a generalized solution of problem* (1.8), (1.9) *with right-hand side* $f_0 \in L_2(G)$. *Then*

$$u \in W^{2m}\big(G \setminus \overline{\mathcal{O}_\delta(\mathcal{K})}\big) \qquad \textit{for any} \qquad \delta > 0. \tag{2.1}$$

Proof. 1) Denote by $W^k_{\mathrm{loc}}(G)$ the set of distributions v on G such that $\psi v \in W^k(G)$ for all $\psi \in C_0^\infty(G)$. It follows from Theorem 3.2 in [25, Chap. 2] that

$$u \in W^{2m}_{\mathrm{loc}}(G). \tag{2.2}$$

This relation and estimate (1.7) imply that

$$\mathbf{B}^2_{i\mu}u \in W^{2m-m_{i\mu}-1/2}(\Gamma_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})}). \tag{2.3}$$

Fix an arbitrary point $g \in \Gamma_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})}$. Take a number $\delta > 0$ such that

$$\overline{\mathcal{O}_\delta(g) \cap \Gamma_i} \subset \Gamma_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})}. \tag{2.4}$$

Then the function u is a solution of the following "local" problem in the neighborhood $\mathcal{O}_\delta(g)$:

$$\mathbf{P}(y, D_y)u = f_0(y) \quad (y \in \mathcal{O}_\delta(g) \cap G), \tag{2.5}$$
$$B_{i\mu 0}(y, D_y)u = f'_{i\mu}(y) \quad (y \in \mathcal{O}_\delta(g) \cap \Gamma_i;\ \mu = 1, \dots, m), \tag{2.6}$$

where $f'_{i\mu}(y) = -\mathbf{B}^1_{i\mu}u(y) - \mathbf{B}^2_{i\mu}u(y)$ for $y \in \mathcal{O}_\delta(g)\cap\Gamma_i$. It follows from relations (2.2), (2.3), and (2.4) and from the definition of the operator $\mathbf{B}^1_{i\mu}$ that $f'_{i\mu} \in W^{2m-m_{i\mu}-1/2}(\mathcal{O}_\delta(g) \cap \Gamma_i)$.

Applying Theorem 8.2 in [25, Chap. 2][1] to problem (2.5), (2.6), we obtain

$$u \in W^{2m}(\mathcal{O}_{\delta/2}(g) \cap G). \tag{2.7}$$

By using a partition of unity, we infer from (2.2) and (2.7) that

$$u \in W^{2m}\big(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})}\big). \tag{2.8}$$

2) It follows from the belonging (2.8) and from inequality (1.6) that

$$\mathbf{B}^2_{i\mu}u \in W^{2m-m_{i\mu}-1/2}(\Gamma_i). \tag{2.9}$$

Taking into account (2.9), we can repeat the arguments of part 1) of this proof for arbitrary $g \in \Gamma_i$ and $\delta > 0$ such that

$$\overline{\mathcal{O}_\delta(g) \cap \Gamma_i} \subset \Gamma_i.$$

As a result, we obtain the belonging (2.7) valid for an arbitrary point $g \in \Gamma_i$. Combining this fact with relation (2.2) and using a partition of unity, we deduce (2.1). □

Now we study the smoothness of solutions of problem (1.8), (1.9) in a neighborhood of the set $\mathcal{K}$. Since generalized solutions can have power-law singularities near the set $\mathcal{K}$ (see [13]), it is natural to consider these solutions in weighted spaces. Let us introduce these spaces.

Assume that either $Q = \{y \in \mathbb{R}^2 :\ r > 0,\ |\omega| < b\}$ or $Q = \{y \in \mathbb{R}^2 :\ 0 < r < d,\ |\omega| < b\}$, $0 < b < \pi$, $d > 0$, or $Q = G$. In the first and second cases, we set $\mathcal{M} = \{0\}$, while in the third case we set $\mathcal{M} = \mathcal{K}$. Introduce the space $H^k_a(Q)$ as the completion of the set $C_0^\infty(\overline{Q} \setminus \mathcal{M})$ with respect to the norm

$$\|w\|_{H^k_a(Q)} = \left(\sum_{|\alpha|\le k} \int\limits_Q \rho^{2(a-k+|\alpha|)} |D^\alpha_y w|^2 dy \right)^{1/2},$$

[1]It is additionally supposed in Theorem 8.2 in [25, Chap. 2] that the operators $B_{i\mu 0}(y, D_y)$ are normal on Γ_i, while their orders are not equal to one another. However, it is easy to check that the theorem mentioned remains valid without these assumptions (see [25, Chap. 2, Sec. 8.3]).

where $a \in \mathbb{R}$, $k \geq 0$ is an integer, and $\rho = \rho(y) = \operatorname{dist}(y, \mathcal{M})$. For integer $k \geq 1$, denote by $H_a^{k-1/2}(\gamma)$ the space of traces on a smooth curve $\gamma \subset \overline{Q}$ with the norm

$$\|\psi\|_{H_a^{k-1/2}(\gamma)} = \inf \|w\|_{H_a^k(Q)} \quad (w \in H_a^k(Q) : w|_\gamma = \psi). \tag{2.10}$$

Let u be a generalized solution of problem (1.8), (1.9), and let $U_j(y') = u_j(y(y'))$, $j = 1, \dots, N$, be the functions corresponding to the set (orbit) $\mathcal{K}$ and satisfying problem (1.12), (1.13) with right-hand side $\{f_j, f_{j\sigma\mu}\}$ (see Sec. 1.3).

Set

$$d_1 = \min\{\chi_{j\sigma ks}, 1\}/2, \qquad d_2 = 2\max\{\chi_{j\sigma ks}, 1\}.$$

Take a sufficiently small ε such that $d_2\varepsilon < \varepsilon_1$. It follows from Lemma 2.1 that

$$U_j \in W^{2m}(K_j^{d_2\varepsilon} \cap \{|y| > \delta\}) \qquad \text{for any} \qquad \delta > 0. \tag{2.11}$$

Further, it follows from the belonging $U_j \in W^m(K_j^{d_2\varepsilon})$ and from Lemma 5.2 in [26] that

$$U_j \in H_{a-m}^m(K_j^{d_2\varepsilon}) \subset H_{a-2m}^0(K_j^{d_2\varepsilon}), \quad a > 2m-1. \tag{2.12}$$

Finally, $f_j \in L_2(K_j^\varepsilon)$ and, by virtue of Lemma 2.1 and estimate (1.6), $f_{j\sigma\mu} \in W^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon)$. Therefore, by Lemma 5.2 in [26],

$$f_j \in H_a^0(K_j^\varepsilon), \quad f_{j\sigma\mu} \in H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon), \quad a > 2m-1. \tag{2.13}$$

The following two lemmas enable us to prove that $U_j \in H_a^{2m}(K_j^{\varepsilon/d_2^3})$ whenever relations (2.11)–(2.13) hold.

Set

$$K_{jq} = K_j \cap \{\varepsilon d_2^{-3} d_1^{4-q}/2 < |y| < \varepsilon d_2^{-3} d_2^{4-q}\}, \qquad q = 0, \dots, 4.$$

Lemma 2.2. *Let Condition* 1.1 *hold. Then the estimate*

$$\sum_j \|U_j\|_{W^{2m}(K_{j4})} \leq c \sum_j \big\{ \|\mathbf{P}_j(y, D_y)U_j\|_{L_2(K_{j1})} + \sum_{\sigma,\mu} \|\mathbf{B}_{j\sigma\mu}(y, D_y)U|_{\gamma_{j\sigma}\cap\overline{K_{j1}}}\|_{W^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}\cap\overline{K_{j1}})} + \|U_j\|_{L_2(K_{j1})} \big\} \tag{2.14}$$

holds for any $U \in \prod_j W^{2m}(K_{j0})$, where $c > 0$ does not depend on U.

Proof. It follows from the general theory of elliptic problems that

$$\|U_j\|_{W^{2m}(K_{j4})} \leq k_1 \big(\|\mathbf{P}_j(y, D_y)U_j\|_{L_2(K_{j3})} + \sum_{\sigma,\mu} \|B_{j\sigma\mu j0}(y, D_y)U_j|_{\gamma_{j\sigma}\cap\overline{K_{j3}}}\|_{W^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}\cap\overline{K_{j3}})} + \|U_j\|_{L_2(K_{j3})} \big). \tag{2.15}$$

Let $(k, s) \neq (j, 0)$; then the set $\mathcal{G}_{j\sigma ks}(\gamma_{j\sigma}) \cap \overline{K_{k2}}$ lies strictly inside the domain K_{k1}. Therefore, using the boundedness of the trace operator on the corresponding Sobolev spaces, we obtain (similarly to (2.15))

$$\begin{aligned} &\|B_{j\sigma\mu ks}(y, D_y)U_k(\mathcal{G}_{j\sigma ks}y)|_{\gamma_{j\sigma}\cap\overline{K_{j3}}}\|_{W^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}\cap\overline{K_{j3}})} \\ &\leq k_2 \|B_{j\sigma\mu ks}(y, D_y)U_k|_{\mathcal{G}_{j\sigma ks}(\gamma_{j\sigma})\cap\overline{K_{k2}}}\|_{W^{2m-m_{j\sigma\mu}-1/2}(\mathcal{G}_{j\sigma ks}(\gamma_{j\sigma})\cap\overline{K_{k2}})} \\ &\leq k_3 (\|\mathbf{P}_j(y, D_y)U_k\|_{L_2(K_{k1})} + \|U_k\|_{L_2(K_{k1})}). \end{aligned} \tag{2.16}$$

Estimates (2.15) and (2.16) imply (2.14). □

Remark 2.2. Assume that the norm (in $C^0(\overline{K_{j1}})$) of the coefficients $p_{j\alpha}$ of the operators $\mathbf{P}_j(y, D_y)$ and the norms (in $C^{2m-m_{j\sigma\mu}}(\overline{K_{j0}})$) of the coefficients $b_{j\sigma\mu ks\alpha}$ of the operators $B_{j\sigma\mu ks}(y, D_y)$ do not exceed some constant C. Let the norms (in $C^1(\overline{K_{j1}})$) of the coefficients $p_{j\alpha}$, $|\alpha| = 2m$, at senior terms of the operators $\mathbf{P}_j(y, D_y)$ not exceed the same constant C. In that case, the constant c occurring in inequality (2.14) depends only on C, on the constant A in (1.1), and on the constant D in (1.2).

Lemma 2.3. *Let Condition* 1.1 *hold. Assume that a function U satisfies relations* (2.11) *and* (2.12) *and is a solution of problem* (1.12), (1.13) *with right-hand side $\{f_j, f_{j\sigma\mu}\}$ satisfying relations* (2.13). *Then $U \in \prod\limits_j H_a^{2m}(K_j^{\varepsilon/d_2^3})$ and*

$$\sum_j \|U_j\|_{H_a^{2m}(K_j^{\varepsilon/d_2^3})} \le c \sum_j \{\|f_j\|_{H_a^0(K_j^\varepsilon)} + \sum_{\sigma,\mu} \|f_{j\sigma\mu}\|_{H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon)} + \|U_j\|_{H_{a-2m}^0(K_j^\varepsilon)}\}, \tag{2.17}$$

where $c > 0$ does not depend on U.

Proof. Set

$$K_{jq}^s = K_j \cap \{\varepsilon d_2^{-3} d_1^{4-q} 2^{-s-1} < |y| < \varepsilon d_2^{-3} d_2^{4-q} 2^{-s}\}, \qquad s = 0, 1, 2, \dots .$$

Clearly,

$$\bigcup_{s=0}^{\infty} K_{j1}^s = K_j^\varepsilon, \qquad \bigcup_{s=0}^{\infty} K_{j4}^s = K_j^{\varepsilon/d_2^3}. \tag{2.18}$$

Set $U_j^s(y') = U_j(2^{-s}y')$ and make the change of variables $y = 2^{-s}y'$ in the equation

$$\mathbf{P}_j(y, D_y)U_j \equiv \sum_{|\alpha|\le 2m} p_{j\alpha}(y) D_y^\alpha U_j(y) = f_j(y) \quad (y \in K_{j1}^s)$$

and in the nonlocal conditions

$$\sum_{k,s} \sum_{|\alpha|\le m_{j\sigma\mu}} b_{j\sigma\mu ks\alpha}(x) D_x^\alpha U_j(x)|_{x=\mathcal{G}_{j\sigma ks}y} = f_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma} \cap \overline{K_{j1}^s});$$

multiplying the first equation obtained by $2^{-s\cdot 2m}$ and the second one by $2^{-s\cdot m_{j\sigma\mu}}$, we have

$$\sum_{|\alpha|\le 2m} p_{j\alpha}^s(y') 2^{s(|\alpha|-2m)} D_{y'}^\alpha U_j^s(y') = 2^{-s\cdot 2m} f_j^s(y') \quad (y' \in K_{j1}^0), \tag{2.19}$$

$$\sum_{k,s} \sum_{|\alpha|\le m_{j\sigma\mu}} b_{j\sigma\mu ks\alpha}^s(x') 2^{s(|\alpha|-m_{j\sigma\mu})} D_{x'}^\alpha U_j^s(x')|_{x'=\mathcal{G}_{j\sigma ks}y'} = 2^{-s\cdot m_{j\sigma\mu}} f_{j\sigma\mu}^s(y') \quad (y' \in \gamma_{j\sigma} \cap \overline{K_{j1}^0}), \tag{2.20}$$

where

$$p_{j\alpha}^s(y') = p_{j\alpha}(2^{-s}y'), \qquad b_{j\sigma\mu ks\alpha}^s(x') = b_{j\sigma\mu ks\alpha}(2^{-s}x'),$$
$$f_j^s(y') = f_j(2^{-s}y'), \qquad f_{j\sigma\mu}^s(y') = f_{j\sigma\mu}(2^{-s}y').$$

Applying Lemma 2.2 to problem (2.19), (2.20), we obtain

$$\sum_j \|U_j^s\|_{W^{2m}(K_{j4}^0)} \le k_1 \sum_j \{\|2^{-s\cdot 2m} f_j^s\|_{L_2(K_{j1}^0)} + \sum_{\sigma,\mu} \|2^{-s\cdot m_{j\sigma\mu}} f_{j\sigma\mu}^s\|_{W^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}\cap\overline{K_{j1}^0})} + \|U_j^s\|_{L_2(K_{j1}^0)}\}, \tag{2.21}$$

where $k_1 > 0$ does not depend on s due to Remark 2.2.

Consider a function $\Phi_{j\sigma\mu} \in H_a^{2m-m_{j\sigma\mu}}(K_j)$ satisfying the following conditions: $\Phi_{j\sigma\mu}|_{\gamma_{j\sigma}^\varepsilon} = f_{j\sigma\mu}$ and

$$\|\Phi_{j\sigma\mu}\|_{H_a^{2m-m_{j\sigma\mu}}(K_j^\varepsilon)} \le 2\|f_{j\sigma\mu}\|_{H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon)} \tag{2.22}$$

(the existence of such a function follows from (2.10)). Then $\Phi_{j\sigma\mu}^s|_{\gamma_{j\sigma}\cap\overline{K_{j1}^0}} = f_{j\sigma\mu}^s$, where $\Phi_{j\sigma\mu}^s(y') = \Phi_{j\sigma\mu}(2^{-s}y')$. Therefore, relations (2.21) and (1.3) imply

$$\sum_j \|U_j^s\|_{W^{2m}(K_{j4}^0)} \le k_1 \sum_j \big\{\|2^{-s\cdot 2m} f_j^s\|_{L_2(K_{j1}^0)} + \sum_{\sigma,\mu} \|2^{-s\cdot m_{j\sigma\mu}}\Phi_{j\sigma\mu}^s\|_{W^{2m-m_{j\sigma\mu}}(K_{j1}^0)} + \|U_j^s\|_{L_2(K_{j1}^0)}\big\}. \tag{2.23}$$

Making the inverse change of variables $y' = 2^s y$ in inequality (2.23), we obtain

$$\sum_j \sum_{|\alpha|\le 2m} \|2^{-s|\alpha|} D_y^\alpha U_j\|_{L_2(K_{j4}^s)} \le k_1 \sum_j \big\{\|2^{-s\cdot 2m} f_j\|_{L_2(K_{j1}^s)} + \sum_{\sigma,\mu} \sum_{|\alpha|\le 2m-m_{j\sigma\mu}} \|2^{-s(|\alpha|+m_{j\sigma\mu})}\Phi_{j\sigma\mu}\|_{L_2(K_{j1}^s)} + \|U_j\|_{L_2(K_{j1}^s)}\big\}. \tag{2.24}$$

Multiplying inequality (2.24) by $2^{-s(a-2m)}$, summing with respect to s, and taking into account (2.22) and (2.18), we deduce (2.17). □

Combining Lemma 2.3 with Lemma 2.1 yields $u \in H_a^{2m}(G)$, $a > 2m-1$, where u is an arbitrary generalized solution of problem (1.8), (1.9) with the right-hand side $f_0 \in L_2(G)$.

It follows from Lemma 2.1 in [16] and from Theorem 3.2 in [17] that the set of solutions from $H_a^{2m}(G)$ of problem (1.8), (1.9) with right-hand side $f_0 = 0$ is of finite dimension for almost all $a > 2m-1$. Thus, we have proved the following result.

Lemma 2.4. *Let Conditions* 1.1 *and* 1.2 *hold. Then the kernel of the operator* $\mathbf{P}$ *is of finite dimension.*

2.2 Closedness of the Operator and its Image. Finite Dimensionality of the Cokernel

To prove that the operator $\mathbf{P}$ has the Fredholm property, we need to consider problem (1.8), (1.9) on weighted spaces with weight a such that $0 < a \le m$. Now the difficulty is that the belonging $u \in H_a^{2m}(G)$ does not imply that $\mathbf{B}_{i\mu}^2 u \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$; therefore, the sum

$$\mathbf{B}_{i\mu}u = \mathbf{B}_{i\mu}^0 u + \mathbf{B}_{i\mu}^1 u + \mathbf{B}_{i\mu}^2 u$$

does not necessarily belong to $H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$. One can only guarantee that $\mathbf{B}_{i\mu}u \in H_{a'}^{2m-m_{i\mu}-1/2}(\Gamma_i)$, where $a' > 2m-1$ (which follows from the fact that $\mathbf{B}_{i\mu}u \in W^{2m-m_{i\mu}-1/2}(\Gamma_i)$ and from Lemma 5.2 in [26]). However, it is proved in [23, Sec. 6] that

$$\{\mathbf{P}(y, D_y)u,\ \mathbf{B}_{i\mu}u\} \in \mathcal{H}_a^0(G,\Gamma) \dotplus \mathcal{R}_a^0(G,\Gamma) \qquad \text{for all} \qquad u \in H_a^{2m}(G),\ a > 0,$$

where $\mathcal{H}_a^0(G,\Gamma) = H_a^0(G) \times \prod_{i,\mu} H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$ and $\mathcal{R}_a^0(G,\Gamma)$ is some *finite-dimensional* space naturally embedded in $\{0\} \times \prod_{i,\mu} H_{a'}^{2m-m_{i\mu}-1/2}(\Gamma_i)$ for any $a' > 2m-1$. In particular, this means that the space $\mathcal{R}_a^0(G,\Gamma)$ contains only functions of the form $\{0, f_{i\mu}\}$, where $f_{i\mu} \in H_{a'}^{2m-m_{i\mu}-1/2}(\Gamma_i)$ and $f_{i\mu} \notin H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$. Fix some $a' > 2m-1$. Then any function

$$\{f_0, f_{i\mu}\} \in \mathcal{H}_a^0(G,\Gamma) \dotplus \mathcal{R}_a^0(G,\Gamma)$$

can be represented as follows:

$$\{f_0, f_{i\mu}\} = \{f_0, f_{i\mu}^1\} + \{0, f_{i\mu}^2\},$$

where $\{f_0, f_{i\mu}^1\} \in \mathcal{H}_a^0(G,\Gamma)$ and $\{0, f_{i\mu}^2\} \in \mathcal{R}_a^0(G,\Gamma)$, and its norm is given by

$$\|\{f_0, f_{i\mu}\}\|_{\mathcal{H}_a^0(G,\Gamma)\dot{+}\mathcal{R}_a^0(G,\Gamma)} = \Big(\|\{f_0, f_{i\mu}^1\}\|^2_{\mathcal{H}_a^0(G,\Gamma)} + \sum_{i,\mu} \|f_{i\mu}^2\|^2_{H_{a'}^{2m-m_{i\mu}-1/2}(\Gamma_i)}\Big)^{1/2}.$$

Furthermore, it follows from Theorem 6.1 in [23] that the operator

$$\mathbf{L}_a = \{\mathbf{P}(y, D_y),\ \mathbf{B}_{i\mu}\} : H_a^{2m}(G) \to \mathcal{H}_a^0(G,\Gamma) \dot{+} \mathcal{R}_a^0(G,\Gamma), \quad a > 0,$$

has the Fredholm property for almost all $a > 0$. In other words, if $u \in\ H_a^{2m}(G)$, then $\mathbf{L}_a u$ "belongs" to the space $\mathcal{H}_a^0(G,\Gamma)$ up to a function of the form $\{0, f_{i\mu}\}$ from the *finite-dimensional* space $\mathcal{R}_a^0(G,\Gamma)$.

Using the Fredholm property of the operator $\mathbf{L}_a$, we prove the following result.

Lemma 2.5. *Let Conditions* 1.1 *and* 1.2 *hold. Then the operator* $\mathbf{P}$ *is closed, its image* $\mathcal{R}(\mathbf{P})$ *is closed, and* $\operatorname{codim} \mathcal{R}(\mathbf{P}) < \infty$.

Proof. 1) Let $0 < a \le m$. We consider the auxiliary unbounded operator

$$\mathbf{P}_a : \mathrm{Dom}\,(\mathbf{P}_a) \subset L_2(G) \to L_2(G)$$

given by

$$\mathbf{P}_a u = \mathbf{P}(y, D_y)u, \quad u \in \mathrm{Dom}\,(\mathbf{P}_a) = \{u \in H_a^{2m}(G) :\ \mathbf{B}_{i\mu}u = 0,\ \mathbf{P}(y, D_y)u \in L_2(G)\}.$$

Fix a number a, $0 < a \le m$, such that the operator $\mathbf{L}_a$ has the Fredholm property. Let us show that the operator $\mathbf{P}_a$ also has the Fredholm property.

Since $\mathbf{L}_a$ has the Fredholm property, it follows from the compactness of the embedding $H_a^{2m}(G) \subset H_a^0(G)$ (see Lemma 3.5 in [15]) and from Theorem 7.1 in [27] that

$$\|u\|_{H_a^{2m}(G)} \le k_1(\|\mathbf{L}_a u\|_{\mathcal{H}_a^0(G,\Gamma)\dot{+}\mathcal{R}_a^0(G,\Gamma)} + \|u\|_{H_a^0(G)}) \tag{2.25}$$

for all $u \in H_a^{2m}(G)$.

Now we take a function $u \in \mathrm{Dom}\,(\mathbf{P}_a)$. Then $\mathbf{L}_a u = \{\mathbf{P}(y, D_y)u, 0\}$, $\mathbf{P}(y, D_y)u \in L_2(G) \subset H_a^0(G)$, and hence

$$\|\mathbf{L}_a u\|_{\mathcal{H}_a^0(G,\Gamma)\dot{+}\mathcal{R}_a^0(G,\Gamma)} = \|\mathbf{P}(y, D_y)u\|_{H_a^0(G)}.$$

Combining this relation with (2.25) and taking into account the boundedness of the embedding $L_2(G) \subset H_a^0(G)$ for $a > 0$, we obtain

$$\|u\|_{H_a^{2m}(G)} \le k_2(\|\mathbf{P}(y, D_y)u\|_{H_a^0(G)} + \|u\|_{H_a^0(G)}) \le k_3(\|\mathbf{P}(y, D_y)u\|_{L_2(G)} + \|u\|_{L_2(G)}), \tag{2.26}$$

where $u \in \mathrm{Dom}\,(\mathbf{P}_a)$. It follows from inequality (2.26) that the operator $\mathbf{P}_a$ is closed. Therefore, using (2.26) and applying Lemma 7.1 in [27] again, we obtain that $\dim\ker \mathbf{P}_a < \infty$ (clearly, $\ker \mathbf{P}_a = \ker \mathbf{L}_a$) and the image $\mathcal{R}(\mathbf{P}_a)$ is closed.

Consider an arbitrary function $f_0 \in L_2(G)$. Clearly, $f_0 \in H_a^0(G)$. By Corollary 6.1 in [23], there exist functionals $F_1, \dots, F_{q_0}$ from the adjoint space $\mathcal{H}_a^0(G,\Gamma)^*$ such that problem (1.8), (1.9) admits a solution $u \in H_a^{2m}(G)$ whenever

$$\langle \{f_0, 0\},\ F_q\rangle = 0, \qquad q = 1, \dots, q_0.$$

Since

$$|\langle \{f_0, 0\},\ F_q\rangle| \le k_4\|f_0\|_{H_a^0(G)} \le k_5\|f_0\|_{L_2(G)},$$

it follows from Riesz' theorem on the general form of a continuous linear functional on a Hilbert space that there exist functions $f_1, \dots, f_{q_0} \in L_2(G)$ such that

$$\langle \{f_0, 0\},\ F_q\rangle = (f_0,\ f_q)_{L_2(G)}, \qquad q = 1, \dots, q_0.$$

Therefore, $\operatorname{codim} \mathcal{R}(\mathbf{P}_a) \le q_0$.

Thus, we have proved that the operator $\mathbf{P}_a$ has the Fredholm property.

2) Since $H_a^{2m}(G) \subset H_{a-m}^m(G) \subset W^m(G)$ for $a \le m$, it follows that

$$\mathbf{P}_a \subset \mathbf{P}. \tag{2.27}$$

In particular, relation (2.27) implies that the image $\mathcal{R}(\mathbf{P})$ is closed and

$$\operatorname{codim} \mathcal{R}(\mathbf{P}) \le \operatorname{codim} \mathcal{R}(\mathbf{P}_a) \le q_0.$$

It remains to prove that the operator $\mathbf{P}$ is closed.[2] Denote by $h_1, \dots, h_k$ some basis of the space

$$\mathcal{R}(\mathbf{P}_a)^\perp = \mathcal{R}(\mathbf{P}) \ominus \mathcal{R}(\mathbf{P}_a).$$

Then there exist functions $v_1, \dots, v_k \in \operatorname{Dom}(\mathbf{P})$ such that $\mathbf{P}v_j = h_j$, $j = 1, \dots, k$. Since $h_j \notin \mathcal{R}(\mathbf{P}_a)$, it follows that $v_j \notin \operatorname{Dom}(\mathbf{P}_a)$. It is also clear that the functions $v_1, \dots, v_k$ are linearly independent because the functions $h_1, \dots, h_k$ have this property.

Consider the finite-dimensional space

$$\mathcal{N} = \operatorname{span}(v_1, \dots, v_k, \ker \mathbf{P}) \ominus \ker \mathbf{P}_a.$$

It is easy to see that $\mathcal{N} \cap \operatorname{Dom} \mathbf{P}_a = \{0\}$. Indeed, if $u \in \mathcal{N} \cap \operatorname{Dom} \mathbf{P}_a$, then

$$u = \sum_{i=1}^{k} \alpha_i v_i + v,$$

where α_i are some constants and $v \in \ker \mathbf{P}$. Therefore, taking into account (2.27), we have

$$\sum_{i=1}^{k} \alpha_i h_i = \mathbf{P}u = \mathbf{P}_a u \in \mathcal{R}(\mathbf{P}_a).$$

Hence, $\alpha_i = 0$, $i = 1, \dots, k$, which implies that $u = v$. Using (2.27) again, we see that $u = v \in \ker \mathbf{P}_a$. Combining this fact with the definition of the space $\mathcal{N}$ yields $u = 0$.

Let $\operatorname{Gr} \mathbf{P}$ ($\operatorname{Gr} \mathbf{P}_a$) denote the graph of the operator $\mathbf{P}$ ($\mathbf{P}_a$). As is known, the operator $\mathbf{P}$ ($\mathbf{P}_a$) is closed if and only if its graph $\operatorname{Gr} \mathbf{P}$ ($\operatorname{Gr} \mathbf{P}_a$) is closed in $L_2(G) \times L_2(G)$.

Note that $\operatorname{Gr} \mathbf{P}_a$ is closed (as the graph of the closed operator) and $\operatorname{Gr} \mathbf{P}_a \subset \operatorname{Gr} \mathbf{P}$, while the spaces $\mathcal{N}$ and $\mathcal{R}(\mathbf{P}_a)^\perp$ are of finite dimension. Therefore, to prove that the operator $\mathbf{P}$ is closed, it suffices to show that

$$\operatorname{Gr} \mathbf{P} \subset \operatorname{Gr} \mathbf{P}_a \dotplus (\mathcal{N} \times \mathcal{R}(\mathbf{P}_a)^\perp). \tag{2.28}$$

Clearly, the sum in (2.28) is direct. Indeed, if

$$(u, f) \in \operatorname{Gr} \mathbf{P}_a \cap (\mathcal{N} \times \mathcal{R}(\mathbf{P}_a)^\perp),$$

then $u \in \operatorname{Dom} \mathbf{P}_a \cap \mathcal{N} = \{0\}$, and hence $(u, f) = (u, \mathbf{P}_a u) = (0, 0)$.

[2] Note that the closedness of the image of some operator $\mathbf{P}$ on a Hilbert space and the finite dimensionality of its kernel and cokernel do not imply the closedness of $\mathbf{P}$ itself; this can be shown by using arguments close to that in [28, Chap. 2, Sec. 18]. However, if we additionally suppose that the operator $\mathbf{P}$ is an extension of a Fredholm operator, then we prove that $\mathbf{P}$ is closed.

Further, let $(u, f) \in \operatorname{Gr} \mathbf{P}$, i.e., $u \in \operatorname{Dom} \mathbf{P}$ and $f = \mathbf{P}u$. We represent the function f as follows:

$$f = f_1 + f_2,$$

where $f_1 \in \mathcal{R}(\mathbf{P}_a)$ and $f_2 \in \mathcal{R}(\mathbf{P}_a)^\perp$. Take an element $u_1 \in \operatorname{Dom}(\mathbf{P}_a)$ such that $\mathbf{P}_a u_1 = f_1$. Then $u_2 = u - u_1 \in \operatorname{Dom}(\mathbf{P})$ and $\mathbf{P}u_2 = f_2$. Without loss of generality, one can assume that

$$u_2 \perp \ker \mathbf{P}_a; \tag{2.29}$$

if this relation fails, one must take the projection u_{2a} of the element u_2 to $\ker \mathbf{P}_a$ and replace u_1 by $u_1 + u_{2a}$ and u_2 by $u_2 - u_{2a}$. Clearly, $(u_1, f_1) \in \operatorname{Gr} \mathbf{P}_a$ and, due to (2.29), $(u_2, f_2) \in \mathcal{N} \times \mathcal{R}(\mathbf{P}_a)^\perp$.

Thus, we have proved relation (2.28), and the lemma is true. □

Lemmas 2.4 and 2.5 imply Theorem 2.1.

Remark 2.3. Using results in [29], one can prove that Theorem 2.1 remains valid if the transformations Ω_{is} are *nonlinear* near the points of the set $\mathcal{K}$, while the linear parts of Ω_{is} satisfy Condition 1.1 at the points of $\mathcal{K}$.

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