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Solvability of Nonlocal Elliptic Problems in Dihedral Angles

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Abstract—In this paper, we consider nonlocal elliptic problems in dihedral and plane angles. Such problems arise in the study of nonlocal problems in bounded domains for the case in which the support of nonlocal terms intersects the boundary. We study the Fredholm and unique solvability of this problem in the corresponding weighted spaces. Results are obtained by means of *a priori* estimates of the solutions and of Green's formula for nonlocal elliptic problems.

Key words: *nonlocal boundary-value problem, nonlocal transmission problem, elliptic problem, Green's formula, Fredholm solvability, unique solvability.*

In the study of elliptic problems with nonlocal conditions, the case in which the support of nonlocal terms intersects the boundary presents the greatest difficulty (see [1–4]). This leads to the appearance of polynomial singularities of solutions near a certain set; therefore, nonlocal elliptic problems are naturally studied in weighted spaces (see [5–7]). In deriving *a priori* estimates of solutions and constructing the right-hand regularizer for nonlocal problems in a bounded domain, we have to deal with model nonlocal boundary-value problems in dihedral angles (see [3, 4]). In the present paper, we propose another approach to studying nonlocal problems based on the use of Green's formula and conjugate nonlocal problems. Such an approach allows us to remove additional constraints (see [3]) on the corresponding "local" model problem and to obtain necessary and sufficient conditions for the Fredholm solvability of nonlocal problems in plane angles and for the unique solvability of such problems in dihedral angles. Simultaneously, certain conjugate problems arise, such as nonlocal transmission problems studied in [8] (in the case of bounded domains with smooth boundary) and in [9] (in the one-dimensional case).

In this paper, for clarity, we restrict ourselves to nonlocal perturbations of the Dirichlet problem for the Laplace operator.

1. STATEMENT OF NONLOCAL ELLIPTIC BOUNDARY-VALUE PROBLEMS

1.1. Let us introduce the dihedral angle

$$\Omega = \{x = (y, z) \colon r > 0,\ b_1 < \varphi < b_2,\ z \in \mathbb{R}^{n-2}\}$$

with the faces

$$\Gamma_j = \{x = (y, z) \colon r > 0,\ \varphi = b_j,\ z \in \mathbb{R}^{n-2}\}, \qquad j = 1, 2,$$

and the edge $M = \{x = (y, z) \colon y = 0,\ z \in \mathbb{R}^{n-2}\}$. Here $x = (y, z) \in \mathbb{R}^n$, $y \in \mathbb{R}^2$, $z \in \mathbb{R}^{n-2}$; and r, φ are polar coordinates of the point y; $0 < b_1 < b_2 < 2\pi$. Consider the following nonlocal

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boundary-value problem in the dihedral angle Ω,

$$\Delta U(x) = f(x), \qquad x \in \Omega, \tag{1.1}$$

$$U(x)|_{\Gamma_j} + a_j U(\mathcal{G}_j y, z)|_{\Gamma_j} = g_j(x), \qquad x \in \Gamma_j. \tag{1.2}$$

Here and further, the index j assumes the values $j = 1, 2$; $a_j \in \mathbb{C}$; $\mathcal{G}_j$ is the operator of rotation by the angle φ_j followed by a dilatation of χ_j times in the plane $\{y\}$; here $b_1 < b_1 + \varphi_1 = b_2 + \varphi_2 = b < b_2$, $0 < \chi_j$.

Let us introduce the space $H_a^l(\Omega)$ as the completion of the set $C_0^\infty(\overline{\Omega} \setminus M)$ in the norm

$$\|w\|_{H_a^l(\Omega)} = \left(\sum_{|\alpha| \le l} \int_\Omega r^{2(a-l+|\alpha|)} |D_x^\alpha w(x)|^2 \, dx \right)^{1/2},$$

where $C_0^\infty(\overline{\Omega} \setminus M)$ is the set of functions infinitely differentiable in $\overline{\Omega}$ with compact supports in $\overline{\Omega} \setminus M$; , $a \in \mathbb{R}$, $l \ge 0$ is an integer. By $H_a^{l-1/2}(\Gamma')$ for $l \ge 1$ we denote the space of traces on $\Gamma' = \{x = (y, z)\colon r > 0,\ \varphi = b',\ z \in \mathbb{R}^{n-2}\}$, $b_1 \le b' \le b_2$, with the norm

$$\|\psi\|_{H_a^{l-1/2}(\Gamma')} = \inf \|w\|_{H_a^l(\Omega)}, \qquad w \in H_a^l(\Omega)\colon w|_{\Gamma'} = \psi.$$

Let us introduce the bounded operator

$$\mathcal{L}_\Omega \colon H_a^2(\Omega) \to H_a^0(\Omega, \Gamma) = H_a^0(\Omega) \times \prod_j H_a^{3/2}(\Gamma_j), \qquad \mathcal{L}_\Omega U = \{\Delta U, U(x)|_{\Gamma_j} + a_j U(\mathcal{G}_j y,\ z)|_{\Gamma_j}\}.$$

By $W^l(Q)$, where $l \ge 0$ is an integer, denote the Sobolev space of generalized functions (distributions) square-integrable together with all the generalized derivatives up to the lth-order inclusive in Q, where $Q \subset \mathbb{R}^n$ is a domain with Lipschitzian boundary. By $W^{l-1/2}(\Upsilon)$, $l \ge 1$, we denote the space of traces on an $(n-1)$-dimensional smooth manifold $\Upsilon \subset \overline{Q}$.

Lemma 1.1. *For all $w \in W^l(Q)$ and $\lambda \in \mathbb{C}$, we have the estimate*

$$|\lambda|^{l-s} \|w\|_{W^s(Q)} \le c(\|w\|_{W^l(Q)} + |\lambda|^l \|w\|_{L_2(Q)}). \tag{1.3}$$

Here $0 < s < l$; $c > 0$ is independent of w, λ.

Lemma 1.2. *For all $w \in W^1(Q)$ and $\lambda \in \mathbb{C}$, we have the estimate*

$$|\lambda|^{1/2} \|w|_\Upsilon\|_{L^2(\Upsilon)} \le c(\|w\|_{W^1(Q)} + |\lambda| \|w\|_{L_2(Q)}). \tag{1.4}$$

Here $c > 0$ is independent of w, λ.

The proof of Lemmas 1.1 and 1.2 is given in [10, Chap. 1]. Using Lemma 1.1 and the properties of weighted spaces, the following result was obtained in [2, Sec. 1].

Lemma 1.3. *For all $w \in H_a^l(\Omega)$ and $\lambda \in \mathbb{C}$, we have*

$$|\lambda|^s \|w\|_{H_{a-s}^{l-s}(\Omega)} \le c(\|w\|_{H_a^l(\Omega)} + |\lambda|^l \|w\|_{H_{a-l}^0(\Omega)}). \tag{1.5}$$

Here $0 < s < l$; $c > 0$ is independent of w, λ.

1.2. Consider the following auxiliary nonlocal boundary-value problem in a plane angle:

$$\Delta u(y) - u(y) = f(y), \qquad y \in K, \tag{1.6}$$

$$u(y)|_{\gamma_j} + a_j u(\mathcal{G}_j y)|_{\gamma_j} = g_j(y), \qquad y \in \gamma_j, \tag{1.7}$$

where $K = \{y \in \mathbb{R}^2 \colon r > 0,\ 0 < b_1 < \varphi < b_2 < 2\pi\}$, $\gamma_j = \{y \in \mathbb{R}^2 \colon r > 0,\ \varphi = b_j\}$.

As above, we introduce the function spaces $H_a^l(K)$ and $H_a^{l-1/2}(\gamma')$, where

$$\gamma' = \{y \in \mathbb{R}^2 \colon r > 0,\ \varphi = b'\}, \qquad b_1 \le b' \le b_2.$$

Let us introduce the space $E_a^l(K)$ as the completion of $C_0^\infty(\overline{K} \setminus \{0\})$ in the norm

$$\|w\|_{E_a^l(K)} = \biggl(\sum_{|\alpha| \le l} \int_K r^{2a}(r^{2(|\alpha|-l)} + 1)|D_y^\alpha w(y)|^2\, dy\biggr)^{1/2}.$$

By $E_a^{l-1/2}(\gamma')$, $l \ge 1$, we denote the space of traces on the ray γ' with the norm

$$\|\psi\|_{E_a^{l-1/2}(\gamma')} = \inf \|w\|_{E_a^l(K)}, \qquad w \in E_a^l(K) \colon w|_{\gamma'} = \psi.$$

For the constructive definitions of the spaces $H_a^{l-1/2}(\Gamma')$, $H_a^{l-1/2}(\gamma')$, and $E_a^{l-1/2}(\gamma')$ equivalent to the ones given above, see [6, Sec. 1]. Now we establish a property of weighted spaces that will be needed later.

Lemma 1.4. *For any $\psi \in E_a^{l-1/2}(\gamma')$, we have the estimate*

$$\biggl(\int_{\gamma'} r^{2(a-(l-1/2))}|\psi|^2\, d\gamma\biggr)^{1/2} \le c\|\psi\|_{E_a^{l-1/2}(\gamma')},$$

where $c > 0$ is independent of ψ.

Proof. It follows from [7, Chap. 6, Sec. 1.3] that the norm $\|u\|_{E_a^l(K)}$ and the norm

$$\biggl(\sum_{k=0}^{l} \int_0^\infty r^{2(a-(l-1/2))} \sum_{j=0}^{l-k} (1+r)^{2(l-k-j)} \|(rD_r)^k u(r,\,\cdot\,)\|_{W^j(b_1,b_2)}^2\, dr\biggr)^{1/2} \tag{1.8}$$

are equivalent; here $u(r,\varphi)$ is the function $u(y)$ written in polar coordinates.

Let us choose a function $u \in E_a^l(K)$ so that $u|_{\gamma'} = \psi$, $\|u\|_{E_a^l(K)} \le 2\|\psi\|_{E_a^{l-1/2}(\gamma')}$. Since $u(r,\varphi)|_{\varphi=b'} = \psi(r)$, by the continuity of the trace operation in Sobolev spaces we have

$$|\psi(r)|^2 \le k_1 \|u(r,\,\cdot\,)\|_{W^l(b_1,b_2)}^2.$$

Combining this with the equivalence of the norm $\|u\|_{E_a^l(K)}$ and of the norms (1.8), we obtain

$$\int_{\gamma'} r^{2(a-(l-1/2))}|\psi|^2\, d\gamma \le k_1 \int_0^\infty r^{2(a-(l-1/2))} \|u(r,\,\cdot\,)\|_{W^l(b_1,b_2)}^2\, dr \le k_2 \|u\|_{E_a^l(K)}^2. \tag{1.9}$$

The assertion of the lemma follows from (1.9) and the inequality $\|u\|_{E_a^l(K)} \le 2\|\psi\|_{E_a^{l-1/2}(\gamma')}$. $\square$

Let us introduce the bounded operator

$$\mathcal{L}_K \colon E_a^2(K) \to E_a^0(K,\gamma) = E_a^0(K) \times \prod_j E_a^{3/2}(\gamma_j), \qquad \mathcal{L}_K u = \{\Delta u - u,\ u(y)|_{\gamma_j} + a_j u(\mathcal{G}_j y)|_{\gamma_j}\}.$$

1.3. Following [3], we consider the model analytic operator function

$$\widetilde{\mathcal{L}}(\lambda)\colon W^2(b_1, b_2) \to W^0[b_1, b_2] = L_2(b_1, b_2) \times \mathbb{C}^2,$$

defined by the formula

$$\widetilde{\mathcal{L}}(\lambda)\widetilde{U} = \left\{ \frac{d^2}{d\varphi^2}\widetilde{U}(\varphi) - \lambda^2 \widetilde{U}(\varphi), \qquad \widetilde{U}(\varphi)|_{\varphi=b_j} + a_j e^{i\lambda \ln \chi_j} \widetilde{U}(\varphi + \varphi_j)|_{\varphi=b_j} \right\}.$$

In the Hilbert spaces $W^2(b_1, b_2)$ and $W^0[b_1, b_2]$, let us introduce the equivalent norms depending on the parameter $\lambda \in \mathbb{C}$ $(|\lambda| \ge 1)$:

$$|||\widetilde{U}|||_{W^2(b_1,b_2)} = (\|\widetilde{U}\|^2_{W^2(b_1,b_2)} + |\lambda|^4 \|\widetilde{U}\|^2_{L_2(b_1,b_2)})^{1/2},$$

$$|||\{\widetilde{F}, \widetilde{G}_j\}|||_{W^0[b_1,b_2]} = \left(\|\widetilde{F}\|^2_{L_2(b_1,b_2)} + \sum_j |\lambda|^3 |\widetilde{G}_j|^2 \right)^{1/2}.$$

Lemma 1.5. *For all* $\lambda \in \mathbb{C}$, *the operator* $\widetilde{\mathcal{L}}(\lambda)$ *is Fredholm,* $\operatorname{ind}\widetilde{\mathcal{L}}(\lambda) = 0$; *for any* $h \in \mathbb{R}$ *there exists a* $q_0 > 1$ *such that for* $\lambda \in J_{h,q_0} = \{\lambda \in \mathbb{C}\colon \operatorname{Im}\lambda = h,\ |\operatorname{Re}\lambda| \ge q_0\}$ *the operator* $\widetilde{\mathcal{L}}(\lambda)$ *has a bounded inverse* $\widetilde{\mathcal{L}}^{-1}(\lambda)\colon W^0[b_1, b_2] \to W^2(b_1, b_2)$ *and*

$$|||\widetilde{\mathcal{L}}^{-1}(\lambda)\widetilde{\Phi}|||_{W^2(b_1,b_2)} \le c|||\widetilde{\Phi}|||_{W^0[b_1,b_2]} \tag{1.10}$$

for all $\widetilde{\Phi} \in W^0[b_1, b_2]$, *where* $c > 0$ *is independent of* λ *and* $\widetilde{\Phi}$; *the operator function*

$$\widetilde{\mathcal{L}}^{-1}(\lambda)\colon W^0[b_1, b_2] \to W^2(b_1, b_2)$$

is finitely meromorphic.

Lemma 1.6. *For any* $0 < \varepsilon < 1/\max|\ln \chi_j|$, *there exists a* $q > 1$ *such that the set*

$$\{\lambda \in \mathbb{C}\colon |\operatorname{Im}\lambda| \le \varepsilon \ln|\operatorname{Re}\lambda|,\ |\operatorname{Re}\lambda| \ge q\}$$

does not contain any poles of the operator function $\widetilde{\mathcal{L}}^{-1}(\lambda)$; *for each pole* λ_0 *of the operator function* $\widetilde{\mathcal{L}}^{-1}(\lambda)$ *there exists a* $\delta > 0$ *such that the set*

$$\{\lambda \in \mathbb{C}\colon 0 < |\operatorname{Im}\lambda - \operatorname{Im}\lambda_0| < \delta\}$$

does not contain any poles of the operator function $\widetilde{\mathcal{L}}^{-1}(\lambda)$.

Lemmas 1.5 and 1.6 were proved in [3, Sec. 2]. In [3, Sec. 3], the following result was also obtained.

Theorem 1.1. *Suppose that on the line* $\operatorname{Im}\lambda = a - 1$ *there are no poles of the operator function* $\widetilde{\mathcal{L}}^{-1}(\lambda)$. *Then for all* $u \in E^2_a(K)$ *we have the estimate*

$$\|u\|_{E^2_a(K)} \le c(\|\mathcal{L}_K u\|_{E^0_a(K,\gamma)} + \|u\|_{L_2(K \cap S)}), \tag{1.11}$$

where $S = \{y \in \mathbb{R}^2\colon 0 < R_1 < r < R_2\}$ *and* $c > 0$ *is independent of* u.

If for all $u \in E^2_a(K)$ *we have the estimate* (1.11), *then on the line* $\operatorname{Im}\lambda = a - 1$ *there are no poles of the operator function* $\widetilde{\mathcal{L}}^{-1}(\lambda)$.

It follows from Theorem 1.1 that the kernel is finite-dimensional and the image of the operator $\mathcal{L}_K$ is closed. To prove that the cokernel of the operator $\mathcal{L}_K$ is finite-dimensional, let us derive Green formulas for nonlocal problems and study problems conjugate to nonlocal boundary-value problems with respect to Green's formula.

2. GREEN FORMULAS FOR NONLOCAL ELLIPTIC PROBLEMS

2.1. Let us introduce the set $\gamma=\{y\in\mathbb{R}^2\colon r>0,\ \varphi=b\}$ (recall that $b=b_j+\varphi_j$). The set γ is the support of nonlocal terms in problem (1.6), (1.7). Let

$$K_1=\{y\in\mathbb{R}^2\colon r>0,\ b_1<\varphi<b\},\qquad K_2=\{y\in\mathbb{R}^2\colon r>0,\ b<\varphi<b_2\}.$$

Suppose n_j is the normal to γ_j directed outside the domain K_j and n is the normal to γ directed outside the domain K_2. By $(\cdot\,,\,\cdot)_{K_j}$, $(\cdot\,,\,\cdot)_{\gamma_j}$, $(\cdot\,,\,\cdot)_{\gamma}$ denote inner products in $L_2(K_j)$, $L_2(\gamma_j)$, $L_2(\gamma)$, respectively.

Theorem 2.1. *For $u\in C_0^\infty(\overline{K}\setminus\{0\})$, $v_j\in C^\infty(\overline{K}_j\setminus\{0\})$, we have Green's formula*

$$\begin{aligned}\sum_j(\Delta u-u,v_j)_{K_j}&+\sum_j\left(u|_{\gamma_j}+a_ju(\mathcal{G}_jy)|_{\gamma_j},\,\frac{\partial v_j}{\partial n_j}\bigg|_{\gamma_j}\right)_{\gamma_j}\\&+\left(u|_\gamma,\,\frac{\partial v_2}{\partial n}\bigg|_\gamma-\frac{\partial v_1}{\partial n}\bigg|_\gamma-\sum_k\bar a_k\chi_k^{-1}\frac{\partial v_k}{\partial n_k}(\mathcal{G}_k^{-1}y)|_\gamma\right)_\gamma\\&=\sum_j(u,\Delta v_j-v_j)_{K_j}+\sum_j\left(\frac{\partial u}{\partial n_j}\bigg|_{\gamma_j},\,v_j|_{\gamma_j}\right)_{\gamma_j}+\left(\frac{\partial u}{\partial n}\bigg|_\gamma,\,v_2|_\gamma-v_1|_\gamma\right)_\gamma,\end{aligned}\tag{2.1}$$

where $\mathcal{G}_k^{-1}$ is the operator of rotation by the angle $-\varphi_k$ followed by a dilatation of $1/\chi_k$ times in the plane $\{y\}$; here and further, the index k assumes the values $k=1,2$.

Proof. Let us multiply $\Delta u-u$ by $\bar v_j$, integrate over K_j, and twice integrate by parts; as a result, we obtain

$$\begin{aligned}&\int_{K_1}(\Delta u-u)\cdot\bar v_1\,dx+\int_{\gamma_1}u|_{\gamma_1}\cdot\frac{\partial\bar v_1}{\partial n_1}\bigg|_{\gamma_1}d\gamma-\int_\gamma u|_\gamma\cdot\frac{\partial\bar v_1}{\partial n}\bigg|_\gamma d\gamma\\&\quad=\int_{K_1}u\cdot(\Delta\bar v_1-\bar v_1)\,dy+\int_{\gamma_1}\frac{\partial u}{\partial n_1}\bigg|_{\gamma_1}\cdot\bar v_1|_{\gamma_1}\,d\gamma-\int_\gamma\frac{\partial u}{\partial n}\bigg|_\gamma\cdot\bar v_1|_\gamma\,d\gamma,\\&\int_{K_2}(\Delta u-u)\cdot\bar v_2\,dx+\int_\gamma u|_\gamma\cdot\frac{\partial\bar v_2}{\partial n}\bigg|_\gamma d\gamma+\int_{\gamma_2}u\bigg|_{\gamma_2}\cdot\frac{\partial\bar v_2}{\partial n_2}\bigg|_{\gamma_2}d\gamma\\&\quad=\int_{K_2}u\cdot(\Delta\bar v_2-\bar v_2)\,dy+\int_\gamma\frac{\partial u}{\partial n}\bigg|_\gamma\cdot\bar v_2|_\gamma\,d\gamma+\int_{\gamma_2}\frac{\partial u}{\partial n_2}\bigg|_{\gamma_2}\cdot\bar v_2|_{\gamma_2}\,d\gamma.\end{aligned}$$

Let us add the last two relations:

$$\begin{aligned}\sum_j\int_{K_j}(\Delta u-u)\cdot\bar v_j\,dx&+\sum_j\int_{\gamma_j}u|_{\gamma_j}\cdot\frac{\partial\bar v_j}{\partial n_j}\bigg|_{\gamma_j}d\gamma+\int_\gamma u|_\gamma\cdot\left(\frac{\partial\bar v_2}{\partial n}\bigg|_\gamma-\frac{\partial\bar v_1}{\partial n}\bigg|_\gamma\right)d\gamma\\&=\sum_j\int_{K_j}u\cdot(\Delta\bar v_j-\bar v_j)\,dx+\sum_j\int_{\gamma_j}\frac{\partial u}{\partial n_j}\bigg|_{\gamma_j}\cdot\bar v_j|_{\gamma_j}\,d\gamma+\int_\gamma\frac{\partial u}{\partial n}\bigg|_\gamma\cdot(\bar v_2|_\gamma-\bar v_1|_\gamma)\,d\gamma.\end{aligned}\tag{2.2}$$

But

$$\begin{aligned}\int_{\gamma_j}u|_{\gamma_j}\cdot\frac{\partial\bar v_j}{\partial n_j}\bigg|_{\gamma_j}d\gamma&=\int_{\gamma_j}(u|_{\gamma_j}+a_ju(\mathcal{G}_jy)|_{\gamma_j})\cdot\frac{\partial\bar v_j}{\partial n_j}\bigg|_{\gamma_j}d\gamma-\int_{\gamma_j}a_ju(\mathcal{G}_jy)|_{\gamma_j}\cdot\frac{\partial\bar v_j}{\partial n_j}\bigg|_{\gamma_j}d\gamma\\&=\int_{\gamma_j}(u|_{\gamma_j}+a_ju(\mathcal{G}_jy)|_{\gamma_j})\cdot\frac{\partial\bar v_j}{\partial n_j}\bigg|_{\gamma_j}d\gamma-\int_\gamma u|_\gamma\cdot a_j\chi_j^{-1}\frac{\partial\bar v_j}{\partial n_j}(\mathcal{G}_j^{-1}y)|_\gamma\,d\gamma,\end{aligned}$$

where $\mathcal{G}_j^{-1}$ is the operator of rotation by the angle $-\varphi_j$ and dilatation of $1/\chi_j$ times in the plane $\{y\}$. Combining this with (2.2), we obtain relation (2.1). $\square$

Remark 2.1. Formula (2.1) can be extended by continuity to the case $u \in E_a^2(K)$, $v_j \in E_{-a+2}^2(K)$. Indeed, $C_0^\infty(\overline{K}\setminus\{0\})$ is dense in $E_a^2(K)$ and $C_0^\infty(\overline{K}_j\setminus\{0\})$ is dense in $E_{-a+2}^2(K_j)$; therefore, there exist sequences $\{u_p\}_{p=1}^\infty \subset C_0^\infty(\overline{K}\setminus\{0\})$ and $\{v_{jq}\}_{q=1}^\infty \subset C_0^\infty(\overline{K}_j\setminus\{0\})$ converging to u and v in $E_a^2(K)$ and $E_{-a+2}^2(K_j)$, respectively. Moreover, for the functions u_p and v_{jq} we have Green's formula (2.1). Passing to the limit as $p, q \to \infty$, we obtain Green's formula for the functions u and v (the passage to the limit is possible by the Cauchy–Bunyakovskii inequality and Lemma 1.4).

2.2. By $(\cdot\,,\,\cdot)_{\beta_1}$, $(\cdot\,,\,\cdot)_{\beta_2}$, $(\cdot\,,\,\cdot)_{\mathbb{C}}$ we denote inner products in $L_2(b_1, b)$, $L_2(b, b_2)$, $\mathbb{C}$, respectively. The proof of the following theorem is similar to that of Theorem 2.1.

Theorem 2.2. *For all $\widetilde{U} \in C^\infty([b_1, b_2])$, $\widetilde{V}_1 \in C^\infty([b_1, b])$, $\widetilde{V}_2 \in C^\infty([b, b_2])$, and $\lambda \in \mathbb{C}$ we have Green's formula with parameter λ:*

$$\begin{aligned}
&\sum_j \left(\frac{d^2}{d\varphi^2}\widetilde{U} - \lambda^2\widetilde{U}, \widetilde{V}_j\right)_{\beta_j} + \sum_j \left(\widetilde{U}|_{\varphi=b_j} + a_j e^{i\lambda \ln\chi_j}\widetilde{U}(\varphi+\varphi_j)|_{\varphi=b_j}, (-1)^j \frac{d\widetilde{V}_j}{d\varphi}\bigg|_{\varphi=b_j}\right)_{\mathbb{C}} \\
&\qquad + \left(\widetilde{U}|_{\varphi=b}, \frac{d\widetilde{V}_1}{d\varphi}\bigg|_{\varphi=b} - \frac{d\widetilde{V}_2}{d\varphi}\bigg|_{\varphi=b} - \sum_k (-1)^k \bar{a}_k e^{-i\bar{\lambda}\ln\chi_k} \frac{d\widetilde{V}_k}{d\varphi}(\varphi-\varphi_k)|_{\varphi=b}\right)_{\mathbb{C}} \\
&= \sum_j \left(\widetilde{U}, \frac{d^2}{d\varphi^2}\widetilde{V}_j - \overline{\lambda}^2\widetilde{V}_j\right)_{\beta_j} + \sum_j \left((-1)^j \frac{d\widetilde{U}}{d\varphi}\bigg|_{\varphi=b_j}, \widetilde{V}_j|_{\varphi=b_j}\right)_{\mathbb{C}} \\
&\qquad + \left(-\frac{d\widetilde{U}}{d\varphi}\bigg|_{\varphi=b}, \widetilde{V}_2|_{\varphi=b} - \widetilde{V}_1|_{\varphi=b}\right)_{\mathbb{C}}.
\end{aligned} \tag{2.3}$$

Remark 2.2. Formula (2.3) can be extended by continuity to the case $\widetilde{U} \in W^2(b_1, b_2)$, $\widetilde{V}_1 \in W^2(b_1, b)$, $\widetilde{V}_2 \in W^2(b, b_2)$ (see Remark 2.2 [11, Chap. 2]).

3. STATEMENT OF NONLOCAL ELLIPTIC TRANSMISSION PROBLEMS

3.1. Formula (2.1) generates the following problem conjugate to problem (1.6), (1.7):

$$\Delta v_j(y) - v_j(y) = f_j(y), \quad y \in K_j, \qquad v_j|_{\gamma_j} = g_j(y), \quad y \in \gamma_j, \tag{3.1}$$

$$v_2|_\gamma - v_1|_\gamma = h_1(y), \qquad \frac{\partial v_2}{\partial n}\bigg|_\gamma - \frac{\partial v_1}{\partial n}\bigg|_\gamma - \sum_k \bar{a}_k \chi_k^{-1} \frac{\partial v_k}{\partial n_k}(\mathcal{G}_k^{-1}y)|_\gamma = h_2(y), \qquad y \in \gamma. \tag{3.2}$$

Problem (3.1), (3.2) is called a *nonlocal transmission problem* in the plane angle K.

Set

$$\mathcal{E}_{-a+2}^0(K, \gamma) = E_{-a+2}^0(K) \times \prod_j E_{-a+2}^{3/2}(\gamma_j) \times \prod_\nu E_{-a+2}^{3/2-\nu}(\gamma);$$

here and further, $\nu = 0, 1$. We also denote

$$\mathcal{E}_{-a+2}^2(K) = \bigoplus_j E_{-a+2}^2(K_j).$$

Consider the bounded operator $\mathcal{M}_K \colon \mathcal{E}_{-a+2}^2(K) \to \mathcal{E}_{-a+2}^0(K, \gamma)$, acting by the formula

$$\mathcal{M}_K v = \left\{ w - v,\ v_j|_{\gamma_j},\ v_2|_\gamma - v_1|_\gamma,\ \frac{\partial v_2}{\partial n}\bigg|_\gamma - \frac{\partial v_1}{\partial n}\bigg|_\gamma - \sum_k \bar{a}_k \chi_k^{-1} \frac{\partial v_k}{\partial n_k}(\mathcal{G}_k^{-1}y)|_\gamma \right\}.$$

Here and further, v_j is the restriction of $v \in \mathcal{E}_{-a+2}^2(K)$ to K_j and $w \equiv \Delta v_j$ for $y \in K_j$. (Note that we cannot assume $w \equiv \Delta v$ for $y \in K$, since the function $v \in \mathcal{E}_{-a+2}^2(K)$ may have a "discontinuity" on γ.)

Lemma 3.1. *For all* $g_j \in E^{3/2}_{-a+2}(\gamma_j)$ *and* $h_\nu \in E^{3/2-\nu}_{-a+2}(\gamma)$, *there exists a function* $v \in \mathcal{E}^2_{-a+2}(K)$ *satisfying conditions* (3.2) *and such that*

$$\|v\|_{\mathcal{E}^2_{-a+2}(K)} \le c\biggl(\sum_j \|g_j\|_{E^{3/2}_{-a+2}(\gamma_j)} + \sum_\nu \|h_\nu\|_{E^{3/2-\nu}_{-a+2}(\gamma)}\biggr),$$

where $c>0$ *is independent of* g_j *and* h_ν.

Proof. By Lemma 3.1′ [6], there exist $w_j \in E^2_{-a+2}(K_j)$ such that

$$w_j|_{\gamma_j} = g_j(y), \qquad y \in \gamma_j, \tag{3.3}$$

$$\|w_j\|_{E^2_{-a+2}(K_j)} \le k_1 \|g_j\|_{E^{3/2}_{-a+2}(\gamma_j)}. \tag{3.4}$$

Repeating the proof of Lemma 3.1′ [6], we construct a $\widehat{w}_2 \in E^2_{-a+2}(K_2)$ such that

$$\widehat{w}_2|_\gamma = h_1(y), \qquad \frac{\partial \widehat{w}_2}{\partial n}\bigg|_\gamma = h_2(y) + \sum_k \bar{a}_k \chi_k^{-1} \frac{\partial w_k}{\partial n_k}(\mathcal{G}_k^{-1}y)|_\gamma, \quad y \in \gamma, \tag{3.5}$$

$$\|\widehat{w}_2\|_{E^2_{-a+2}(K_2)} \le k_2\biggl(\|h_1\|_{E^{3/2}_{-a+2}(\gamma)} + \biggl\|h_2 + \sum_k \bar{a}_k \chi_k^{-1} \frac{\partial v_k}{\partial n_k}(\mathcal{G}_k^{-1}y)|_\gamma\biggr\|_{E^{1/2}_{-a+2}(\gamma)}\biggr). \tag{3.6}$$

Let us introduce the functions $\zeta, \zeta_j \in C_0^\infty(\mathbb{R})$, $\zeta_j(\varphi) = 1$ for $|b_j - \varphi| < \varepsilon/2$, $\zeta_j(\varphi) = 1$ for $|b_j - \varphi| > \varepsilon$ and $\zeta(\varphi) = 1$ for $|b - \varphi| < \varepsilon/2$, $\zeta(\varphi) = 1$ for $|b - \varphi| > \varepsilon$. Here $\varepsilon = \min_j\{|b - b_j|\}/4$. The functions ζ, ζ_j are multipliers in the spaces $E^2_{-a+2}(K_j)$. Combining this with (3.3)–(3.6), we see that the function v satisfies $v = v_1 = \zeta_1 w_1$ for $y \in K_1$ and $v = v_2 = \zeta_2 w_2 + \zeta \widehat{w}_2$ for $y \in K_2$, satisfies the assumptions of the lemma. □

3.2. Let

$$\mathcal{W}^2(b_1, b_2) = W^2(b_1, b) \oplus W^2(b, b_2).$$

As in Sec. 1 for problem (3.1), (3.2), consider the model operator function

$$\widetilde{\mathcal{M}}(\lambda)\colon \mathcal{W}^2(b_1, b_2) \to \mathcal{W}^0[b_1, b_2] = L_2(b_1, b_2) \times \mathbb{C}^2 \times \mathbb{C}^2,$$

defined by the formula

$$\widetilde{\mathcal{M}}(\lambda)\widetilde{V} = \biggl\{\widetilde{W}(\varphi) - \lambda^2 \widetilde{V}(\varphi), \widetilde{V}_j(\varphi)|_{\varphi=b_j}, \widetilde{V}_2(\varphi)|_{\varphi=b} - \widetilde{V}_1(\varphi)|_{\varphi=b}, \frac{d\widetilde{V}_1}{d\varphi}\bigg|_{\varphi=b} - \frac{d\widetilde{V}_2}{d\varphi}\bigg|_{\varphi=b} - \sum_k (-1)^k \bar{a}_k e^{-i\lambda \ln \chi_k} \frac{d\widetilde{V}_k}{d\varphi}(\varphi - \varphi_k)|_{\varphi=b}\biggr\}.$$

Here $\widetilde{V}_j$ is the restriction of $\widetilde{V} \in \mathcal{W}^2(b_1, b_2)$ to K_j and $\widetilde{W}(\varphi) \equiv (d^2/d\varphi^2)\widetilde{V}_1(\varphi)$ for $\varphi \in (b_1, b)$, $\widetilde{W}(\varphi) \equiv (d^2/d\varphi^2)\widetilde{V}_2(\varphi)$ for $\varphi \in (b, b_2)$. Let us establish certain properties of the operator function $\widetilde{\mathcal{M}}(\lambda)$. In the Hilbert spaces $\mathcal{W}^2(b_1, b_2)$ and $\mathcal{W}^0[b_1, b_2]$, we introduce the following equivalent norms depending on the parameter $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$:

$$|||\widetilde{V}|||_{\mathcal{W}^2(b_1, b_2)} = (\|\widetilde{V}\|^2_{\mathcal{W}^2(b_1, b_2)} + |\lambda|^4 \|\widetilde{V}\|^2_{L_2(b_1, b_2)})^{1/2},$$

$$|||\{\widetilde{F}, \widetilde{G}_j, \widetilde{H}_\nu\}|||_{\mathcal{W}^0[b_1, b_2]} = \biggl(\|\widetilde{F}\|^2_{L_2(b_1, b_2)} + \sum_j |\lambda|^3 |\widetilde{G}_j|^2 + \sum_\nu |\lambda|^{3-2\nu} |\widetilde{H}_\nu|^2\biggr)^{1/2}.$$

Lemma 3.2. *For all* $\lambda\in\mathbb{C}$, *the operator* $\widetilde{\mathcal{M}}(\lambda)$ *is Fredholm,* $\operatorname{ind}\widetilde{\mathcal{M}}(\lambda)=0$; *for any* $h\in\mathbb{R}$, *there exists a* $q_0>1$ *such that for* $\lambda\in J_{h,q_0}=\{\lambda\in\mathbb{C}\colon \operatorname{Im}\lambda=h,\ |\operatorname{Re}\lambda|\ge q_0\}$ *the operator* $\widetilde{\mathcal{M}}(\lambda)$ *has a bounded inverse* $\widetilde{\mathcal{M}}^{-1}(\lambda)\colon \mathcal{W}^0[b_1,b_2]\to\mathcal{W}^2(b_1,b_2)$ *and*

$$|||\widetilde{\mathcal{M}}^{-1}(\lambda)\widetilde{\Phi}|||_{\mathcal{W}^2(b_1,b_2)}\le c|||\widetilde{\Phi}|||_{\mathcal{W}^0[b_1,b_2]} \tag{3.7}$$

for all $\widetilde{\Phi}\in\mathcal{W}^0[b_1,b_2]$, *where* $c>0$ *is independent of* λ *and* $\widetilde{\Phi}$; *the operator function*

$$\widetilde{\mathcal{M}}^{-1}(\lambda)\colon \mathcal{W}^0[b_1,b_2]\to\mathcal{W}^2(b_1,b_2)$$

is finitely meromorphic.

Proof. In the case $a_j=0$, i.e., when there are no operators corresponding to nonlocal terms, we denote $\widetilde{\mathcal{M}}(\lambda)$ by $\widetilde{\mathcal{M}}_0(\lambda)$. Following the scheme developed by Agranovich and Vishik in [10], we can show that there exist an $0<\varepsilon_1<\pi/2$ and a $q_1>1$ such that for

$$\lambda\in Q_{\varepsilon_1,q_1}=\{\lambda\colon |\lambda|\ge q_1,\ |\arg\lambda|\le\varepsilon_1\}\cup\{\lambda\colon |\lambda|\ge q_1,\ |\arg\lambda-\pi|\le\varepsilon_1\}$$

there exists an inverse operator $\widetilde{\mathcal{M}}_0^{-1}(\lambda)$ for which we have the estimate

$$|||\widetilde{\mathcal{M}}_0^{-1}(\lambda)\widetilde{\Phi}|||_{\mathcal{W}^2(b_1,b_2)}\le k_1|||\widetilde{\Phi}|||_{\mathcal{W}^0[b_1,b_2]} \tag{3.8}$$

for all $\widetilde{\Phi}\in\mathcal{W}^0[b_1,b_2]$, where $k_1>0$ is independent of λ and $\widetilde{\Phi}$.

Let us introduce the operator $\widetilde{\mathcal{M}}_t(\lambda)=\widetilde{\mathcal{M}}_0(\lambda)+t(\widetilde{\mathcal{M}}(\lambda)-\widetilde{\mathcal{M}}_0(\lambda))$, $0\le t\le 1$. Let us prove that for any $h\in\mathbb{R}$ there exists a $q_0>0$ such that for $\lambda\in J_{h,q_0}$ and $0\le t\le 1$ we have

$$k_2|||\widetilde{\mathcal{M}}_t(\lambda)\widetilde{V}|||_{\mathcal{W}^0[b_1,b_2]}\le|||\widetilde{V}|||_{\mathcal{W}^2(b_1,b_2)}\le k_3|||\widetilde{\mathcal{M}}_t(\lambda)\widetilde{V}|||_{\mathcal{W}^0[b_1,b_2]} \tag{3.9}$$

for all $\widetilde{V}\in\mathcal{W}^2(b_1,b_2)$, where $k_2,k_3>0$ are independent of λ, t, and V.

Let $\widetilde{\mathcal{M}}_t(\lambda)\widetilde{V}=\widetilde{\Phi}$. Then $\widetilde{\mathcal{M}}_0(\lambda)\widetilde{V}=\widetilde{\Phi}+\widetilde{\Psi}$, where

$$\widetilde{\Psi}=\biggl(0,0,0,0,t\sum_k(-1)^k\bar a_k e^{-i\lambda\ln\chi_k}\frac{d\widetilde{V}_k}{d\varphi}(\varphi-\varphi_k)|_{\varphi=b}\biggr).$$

By (3.8), we have

$$|||\widetilde{V}|||_{\mathcal{W}^2(b_1,b_2)}\le k_1|||\widetilde{\Phi}+\widetilde{\Psi}|||_{\mathcal{W}^0[b_1,b_2]}. \tag{3.10}$$

Set $\varepsilon=\min_j\{|b-b_j|\}/4$ and choose a $q_0\ge q_1$ so that $J_{h,q_0}\subset Q_{\varepsilon_1,q_1}$. Then, using inequalities (1.3), (1.4), we obtain

$$\begin{aligned}I_1&=|\lambda|^{1/2}\left|\bar a_1 e^{-i\lambda\ln\chi_1}\frac{d\widetilde{V}_1}{d\varphi}(\varphi-\varphi_1)|_{\varphi=b}\right|\\&\le k_4\left\{\left\|\frac{d\widetilde{V}_1}{d\varphi}\right\|_{W^1(b_1,b_1+\varepsilon/2)}+|\lambda|\left\|\frac{d\widetilde{V}_1}{d\varphi}\right\|_{L^2(b_1,b_1+\varepsilon/2)}\right\}\le k_5|||\widetilde{V}_1|||_{W^2(b_1,b_1+\varepsilon/2)}.\end{aligned} \tag{3.11}$$

Suppose that q_1 is so large that in the domain Q_{ε_1,q_1} Theorem 4.1 [10, Chap. 1] is valid. Then using the Leibniz formula and the interpolation inequality (1.3), from inequality (3.11) and Theorem 4.1 [10, Chap. 1], we obtain

$$\begin{aligned}I_1&\le k_5|||\zeta_1\widetilde{V}_1|||_{W^2(b_1,b_1+\varepsilon/2)}\le k_6\left(\left|\left|\left|\left(\frac{d^2}{d\varphi^2}-\lambda^2\right)(\zeta_1\widetilde{V}_1)\right|\right|\right|_{L_2(b_1,b)}+|\lambda|^{3/2}|\widetilde{V}_1(\varphi)|_{\varphi=b_1}\right)\\&\le k_7\left(\left|\left|\left|\left(\frac{d^2}{d\varphi^2}-\lambda^2\right)\widetilde{V}_1\right|\right|\right|_{L_2(b_1,b)}+|\lambda|^{-1}|||\widetilde{V}_1|||_{W^2(b_1,b)}+|\lambda|^{3/2}|\widetilde{V}_1(\varphi)|_{\varphi=b_1}\right),\end{aligned} \tag{3.12}$$

where ζ_1 the same as in the proof of Lemma 3.1. The estimate of

$$I_2 = |\lambda|^{1/2} \big|\bar{a}_2 e^{-i\lambda \ln \chi_2} (d\widetilde{V}_2/d\varphi)(\varphi - \varphi_2)|_{\varphi=b}\big|$$

is similar to the estimates (3.11), (3.12):

$$I_2 \le k_8 \left(\left|\!\left|\!\left| \left(\frac{d^2}{d\varphi^2} - \lambda^2 \right) \widetilde{V}_2 \right|\!\right|\!\right|_{L_2(b,b_2)} + |\lambda|^{-1} |||\widetilde{V}_2|||_{W^2(b,b_2)} + |\lambda|^{3/2} |\widetilde{V}_2(\varphi)|_{\varphi=b_2} \right). \tag{3.13}$$

Assuming that q_0 is sufficiently large, from (3.10), (3.12), (3.13) we obtain the second one of the inequalities (3.9). The first one of the inequalities (3.9) is obvious. Using the standard method of continuation with respect to the parameter t (see the proof of Theorem 7.1 in [12, Chap. 2, Sec. 7]), inequality (3.9), and the existence of the inverse operator $\widetilde{\mathcal{M}}_0^{-1}(\lambda)$ for $\lambda \in Q_{\varepsilon_1, q_1}$, we see that for $\lambda \in J_{h,q_0}$ the operator $\widetilde{\mathcal{M}}(\lambda)$ also has a bounded inverse satisfying inequality (3.7). Using Theorem 16.4 [13] on the stability of the index of the Fredholm operator with respect to compact perturbations, we can easily verify that the operator $\widetilde{\mathcal{M}}(\lambda)$ is Fredholm for all $\lambda \in \mathbb{C}$ and $\operatorname{ind} \widetilde{\mathcal{M}}(\lambda) = 0$. Hence from the existence of $\widetilde{\mathcal{M}}^{-1}(\lambda)$ for $\lambda \in J_{h,q_0}$ and from Theorem 1 given in [14], we find that the operator function $\widetilde{\mathcal{M}}^{-1}(\lambda)$ is finitely meromorphic. $\square$

Using (3.10)–(3.13), we can carry out the proof of the following lemma, which is similar to that of Lemma 2.2 [3].

Lemma 3.3. *For any $0 < \varepsilon < 1/\max|\ln \chi_j|$, there exists a $q > 1$ such that the set*

$$\{\lambda \in \mathbb{C} \colon |\operatorname{Im} \lambda| \le \varepsilon \ln |\operatorname{Re} \lambda|,\ |\operatorname{Re} \lambda| \ge q\}$$

does not contain any poles of the operator function $\widetilde{\mathcal{M}}^{-1}(\lambda)$; for each pole λ_0 of the operator function $\widetilde{\mathcal{M}}^{-1}(\lambda)$ there exists a $\delta > 0$ such that the set

$$\{\lambda \in \mathbb{C} \colon 0 < |\operatorname{Im} \lambda - \operatorname{Im} \lambda_0| < \delta\}$$

does not contain any poles of the operator function $\widetilde{\mathcal{M}}^{-1}(\lambda)$.

3.3. For the functions v_j, consider the auxiliary system of two equations

$$\Delta v_j(y) = \hat{f}_j(y), \qquad y \in K_j, \tag{3.14}$$

with the boundary conditions and nonlocal transmission conditions (3.2). Let us introduce the space

$$\mathcal{H}^0_{-a+2}(K, \gamma) = H^0_{-a+2}(K) \times \prod_j H^{3/2}_{-a+2}(\gamma_j) \times \prod_\nu H^{3/2-\nu}_{-a+2}(\gamma).$$

We also denote $\mathcal{H}^2_{-a+2}(K) = \bigoplus_j H^2_{-a+2}(K_j)$.

By analogy with Theorem 2.1 from [3], Lemma 3.2 yields the following result.

Lemma 3.4. *Suppose that the line $\operatorname{Im} \lambda = -a + 1$ does not contain any poles of the operator function $\widetilde{\mathcal{M}}^{-1}(\lambda)$. Then the nonlocal transmission problem (3.14), (3.2) has a unique solution $v \in \mathcal{H}^2_{-a+2}(K)$ for any right-hand side of $\{\hat{f}, g_j, h_\nu\} \in \mathcal{H}^0_{-a+2}(K, \gamma)$ and the following estimate is satisfied:*

$$\|v\|_{\mathcal{H}^2_{-a+2}(K)} \le c \|\{\hat{f}, g_j, h_\nu\}\|_{\mathcal{H}^0_{-a+2}(K,\gamma)},$$

where $c > 0$ is independent of $\{\hat{f}, g_j, h_\nu\}$; $v(y) \equiv v_j(y)$, $\hat{f}(y) \equiv \hat{f}_j(y)$ for $y \in K_j$.

Lemma 3.4 will be needed in Sec. 4 to derive *a priori* estimates for the solutions of problem (3.1), (3.2).

4. *A PRIORI* ESTIMATES FOR THE SOLUTIONS OF NONLOCAL ELLIPTIC PROBLEMS

4.1. Let us introduce the set

$$\Gamma = \{x = (y, z)\colon r > 0,\ \varphi = b,\ z \in \mathbb{R}^{n-2}\}.$$

The set Γ is the support of nonlocal terms in problem (1.1), (1.2). We denote

$$\Omega_1 = \{x = (y, z)\colon r > 0,\ b_1 < \varphi < b,\ z \in \mathbb{R}^{n-2}\},$$
$$\Omega_2 = \{x = (y, z)\colon r > 0,\ b < \varphi < b_2,\ z \in \mathbb{R}^{n-2}\}.$$

Suppose that n_j is the normal to Γ_j directed outside the domain Ω_j and n is the normal to Γ directed outside the domain Ω_2. Set $d_1 = \min\{1, \chi_j^{-1}\}/2$, $d_2 = 2\max\{1, \chi_j^{-1}\}$;

$$\Omega_j^p = \Omega_j \cap \{x = (y, z)\colon r_1 d_1^{3-p} < r < r_2 d_2^{3-p},\ |z| < 2^{-p-1}\},$$
$$\Omega^p = \Omega \cap \{x = (y, z)\colon r_1 d_1^{3-p} < r < r_2 d_2^{3-p},\ |z| < 2^{-p-1}\},$$

where $p = 0, \dots, 3$; $0 < r_1 < r_2$.

Denote

$$\mathcal{W}^2(\Omega^p) = \bigoplus_j W^2(\Omega_j^p).$$

Suppose that V_j is the restriction of $V \in \mathcal{W}^2(\Omega^p)$ to Ω_j^p.

Lemma 4.1. *For all $V \in \mathcal{W}^2(\Omega^0)$ and $|\lambda| \ge 1$, the following inequality is valid*:

$$\begin{aligned}\|V\|_{\mathcal{W}^2(\Omega^3)} \le c\biggl(&\sum_j \|\Delta V_j\|_{W^2(\Omega_j^0)} + \sum_j \|V_j|_{\Gamma_j}\|_{W^{3/2}(\Gamma_j\cap\bar\Omega^0)} + \|V_2|_\Gamma - V_1|_\Gamma\|_{W^{3/2}(\Gamma\cap\bar\Omega^0)}\\ &+ \biggl\|\frac{\partial V_2}{\partial n}\biggr|_\Gamma - \frac{\partial V_1}{\partial n}\biggr|_\Gamma - \sum_k \bar a_k \chi_k^{-1} \frac{\partial V_k}{\partial n_k}(\mathcal{G}_k^{-1} y, z)|_\Gamma\biggr\|_{W^{1/2}(\Gamma\cap\bar\Omega^0)}\\ &+ |\lambda|^{-1}\|V\|_{\mathcal{W}^2(\Omega^0)} + |\lambda|\|V\|_{L_2(\Omega^0)}\biggr), \end{aligned} \tag{4.1}$$

where $c > 0$ is independent of λ and V.

Proof. Theorem 1 [15] yields the *a priori* estimate

$$\begin{aligned}\|V\|_{\mathcal{W}^2(\Omega^3)} \le k_1\biggl(&\sum_j \|\Delta V_j\|_{W^2(\Omega_j^2)} + \sum_j \|V_j|_{\Gamma_j}\|_{W^{3/2}(\Gamma_j\cap\bar\Omega^2)} + \|V_2|_\Gamma - V_1|_\Gamma\|_{W^{3/2}(\Gamma\cap\bar\Omega^2)}\\ &+ \biggl\|\frac{\partial V_2}{\partial n}\biggr|_\Gamma - \frac{\partial V_1}{\partial n}\biggr|_\Gamma\biggr\|_{W^{1/2}(\Gamma\cap\bar\Omega^2)} + \|V\|_{L_2(\Omega^2)}\biggr). \end{aligned} \tag{4.2}$$

Let

$$W = \sum_k \bar a_k \chi_k^{-1} \frac{\partial(\zeta_k V_k)}{\partial n_k}(\mathcal{G}_k^{-1} y, z),$$

where ζ_k is the same as in the proof of Lemma 3.1. Obviously,

$$W|_{\Gamma\cap\overline{\Omega}^2} = \sum_k \bar a_k \chi_k^{-1} \frac{\partial V_k}{\partial n_k}(\mathcal{G}_k^{-1} y, z)|_{\Gamma\cap\overline{\Omega}^2}. \tag{4.3}$$

Using Theorem 5.1 [11, Chap. 2], the Leibniz formula, and inequality (1.3), we obtain

$$\begin{aligned}\|W|_{\Gamma\cap\overline{\Omega}^2}\|_{W^{1/2}(\Gamma\cap\overline{\Omega}_j^2)} &\le k_2\sum_k \|\zeta_k V_k\|_{W^2(\Omega_k^1)}\\ &\le k_3\sum_k\{\|\Delta V_k\|_{L_2(\Omega_k^0)} + \|V_k|_{\Gamma_k}\|_{W^{3/2}(\Gamma_k\cap\overline{\Omega}_k^0)} + |\lambda|^{-1}\|V_k\|_{W^2(\Omega_k^0)} + |\lambda|\|V_k\|_{L_2(\Omega_k^0)}\}.\end{aligned} \tag{4.4}$$

From (4.2)–(4.4) we obtain inequality (4.1). □

Lemma 4.2. *Suppose that $v_j \in W^2_{\mathrm{loc}}(\overline{K}_j \setminus \{0\})$ is a solution of the nonlocal transmission problem* (3.14), (3.2) *such that $v \in H^0_{-a}(K)$ and $\{\hat f, g_j, h_\nu\} \in \mathcal{H}^0_{-a+2}(K,\gamma)$. Then $v \in \mathcal{H}^2_{-a+2}(K)$ and*

$$\|v\|_{\mathcal{H}^2_{-a+2}(K)} \le c(\|\{\hat f, g_j, h_\nu\}\|_{\mathcal{H}^0_{-a+2}(K,\gamma)} + \|v\|_{H^0_{-a}(K)}), \tag{4.5}$$

where $c>0$ is independent of v.

Proof. As in the proof of Lemma 3.2 [3], the proof of Lemma 4.2 follows from Lemma 3.1 and the analog of Lemma 4.1 for $n=2$. □

4.2. Suppose that

$$K_j^{ps} = K_j \cap \{r_1 d_1^{3-p}\cdot 2^s < r < r_2 d_2^{3-p}\cdot 2^s\}, \qquad K^{ps} = K \cap \{r_1 d_1^{3-p}\cdot 2^s < r < r_2 d_2^{3-p}\cdot 2^s\},$$

where $0<r_1<r_2$; $s\ge 1$; $p=0,\dots,3$.

Let

$$\mathcal{W}^2(K^{ps}) = \bigoplus_j W^2(K_j^{ps}).$$

Suppose that v_j is the restriction of $v\in\mathcal{W}^2(K^{ps})$ to K_j^{ps}.

Lemma 4.3. *Let $s\ge 1$. Suppose that $v\in\mathcal{W}^2(K^{0s})$,*

$$v_j|_{\gamma_j} = 0, \quad y\in\gamma_j\cap\overline{K}_j^{0s}, \qquad v_2|_\gamma - v_1|_\gamma = 0, \quad y\in\gamma\cap K^{0s},$$

$$\frac{\partial v_2}{\partial n}\Big|_\gamma - \frac{\partial v_1}{\partial n}\Big|_\gamma - \sum_k \bar a_k \chi_k^{-1}\frac{\partial v_k}{\partial n_k}(\mathcal{G}_k^{-1}y)|_\gamma = 0, \quad y\in\gamma\cap K^{0s}.$$

Then for $|\lambda|\ge 1$ we have

$$\begin{aligned}&2^{s(-a+2)}\|v\|_{\mathcal{W}^2(K^{3s})}\\ &\quad\le c\Big(2^{s(-a+2)}\sum_j \|\Delta v_j - v_j\|_{L_2(K_j^{0s})} + |\lambda|^{-1}2^{s(-a+2)}\|v\|_{\mathcal{W}^2(K^{0s})} + |\lambda|2^{s(-a)}\|v\|_{L_2(K^{0s})}\Big),\end{aligned} \tag{4.6}$$

where $c>0$ is independent of v, λ, and s.

Proof. As in the proof of Lemma 3.3 [3], Lemma 4.3 is proved by substituting the function $V(y,z) = e^{i\cdot 2^s(\theta,z)}v(y)$, $\theta\in\mathbb{R}^{n-2}$, $|\theta|=1$, into inequality (4.1), by making the change of variables $y' = 2^s y$, and by multiplying both sides of the resulting inequality by $2^{s(-a)}$. □

Theorem 4.1. *Suppose that $v_j \in W^2_{\mathrm{loc}}(\overline{K}_j\setminus\{0\})$ is a solution of problem* (3.1), (3.2) *such that $v\in E^0_{-a}(K)$ and $\{f, g_j, h_\nu\}\in\mathcal{E}^0_{-a+2}(K,\gamma)$. Then $v\in\mathcal{E}^2_{-a+2}(K)$ and*

$$\|v\|_{\mathcal{E}^2_{-a+2}(K)} \le c(\|\{f, g_j, h_\nu\}\|_{\mathcal{E}^0_{-a+2}(K,\gamma)} + \|v\|_{E^0_{-a}(K)}), \tag{4.7}$$

where $c>0$ is independent of v; $v(y)\equiv v_j(y)$, $f(y)\equiv f_j(y)$ for $y\in K_j$.

Proof. 1) By Lemma 3.1, it suffices to consider the case $\{g_j, h_\nu\} = 0$. Suppose that $r_1 = d_1$, $r_2 = d_2$. Then

$$K_j^{ps} = K_j \cap \{d_1^{4-p} \cdot 2^s < r < d_2^{4-p} \cdot 2^s\}, \qquad K^{ps} = K \cap \{d_1^{4-p} \cdot 2^s < r < d_2^{4-p} \cdot 2^s\},$$

where $s \ge 1$; $p = 0, \dots, 3$. We also denote $K_j^{30} = K_j \cap \{r < d_2\}$. Let us introduce the functions ψ, $\hat\psi \in C^\infty(\mathbb{R})$, $\psi(r) = 1$ for $r < d_2$, $\psi(r) = 0$ for $r > 2d_2$; $\hat\psi(r) = 1$ for $r < 2d_2^2$, $\hat\psi(r) = 0$ for $r > 3d_2^2$.

By Lemma 4.2, we have

$$\|v\|_{\mathcal{E}^2_{-a+2}(K_j^{30})} \le k_1 \|\psi v\|_{\mathcal{H}^2_{-a+2}(K)} \le k_2 \biggl(\sum_j \|\Delta(\psi v_j)\|_{H^0_{-a+2}(K_j)} + \|\psi v\|_{H^0_{-a}(K)} \biggr). \tag{4.8}$$

Let us estimate $\|\Delta(\psi v_j)\|_{H^0_{-a+2}(K_j)}$. Using the Leibniz formula and the constraints on the supports of the functions ψ and $\hat\psi$, we obtain

$$\begin{aligned} \|\Delta(\psi v_j)\|_{H^0_{-a+2}(K_j)} &\le \|\Delta(\psi v_j) - \psi v_j\|_{H^0_{-a+2}(K_j)} + \|\psi v_j\|_{H^0_{-a+2}(K_j)} \\ &\le k_3 (\|\Delta v_j - v_j\|_{E^0_{-a+2}(K_j)} + \|\hat\psi v_j\|_{H^1_{-a+1}(K_j)}). \end{aligned} \tag{4.9}$$

Inequalities (4.8), (4.9) and the interpolation inequality (1.5) yield

$$\|v\|_{\mathcal{E}^2_{-a+2}(K^{30})} \le k_4 \biggl(\sum_j \|f_j\|_{E^0_{-a+2}(K_j)} + |\lambda|^{-1} \|v\|_{\mathcal{E}^2_{-a+2}(K)} + |\lambda| \|v\|_{E^0_{-a}(K)} \biggr). \tag{4.10}$$

2) By Lemma 4.3, for $s \ge 1$ we have

$$\|v\|_{\mathcal{E}^2_{-a+2}(K^{3s})} \le k_5 \biggl(\sum_j \|f_j\|_{E^0_{-a+2}(K_j^{0s})} + |\lambda|^{-1} \|v\|_{\mathcal{E}^2_{-a+2}(K^{0s})} + |\lambda| \|v\|_{E^0_{-a}(K^{0s})} \biggr). \tag{4.11}$$

Adding (4.10) and (4.11) for $s \ge 1$, in the case of a sufficiently large $|\lambda|$ we obtain (4.7). □

By analogy with Theorem 3.1 [3.1], from Lemmas 3.1, 3.4, and 4.3 we obtain the following result.

Theorem 4.2. *Suppose that on the line* $\operatorname{Im}\lambda = -a+1$ *there are no poles of the operator function* $\widetilde{\mathcal{M}}^{-1}(\lambda)$. *Then for* $v \in \mathcal{E}^2_{-a+2}(K)$ *we have the estimate*

$$\|v\|_{\mathcal{E}^2_{-a+2}(K)} \le c(\|\mathcal{M}_K v\|_{\mathcal{E}^0_{-a+2}(K,\gamma)} + \|v\|_{L_2(K \cap S')}), \tag{4.12}$$

where $S' = \{y \in \mathbb{R}^2 \colon 0 < R_1' < r < R_2'\}$; $c > 0$ *is independent of* v.

Conversely, if for all $v \in \mathcal{E}^2_{-a+2}(K)$ *we have the estimate* (4.12), *then on the line* $\operatorname{Im}\lambda = -a+1$ *there are no poles of the operator function* $\widetilde{\mathcal{M}}^{-1}(\lambda)$.

It follows from Theorem 4.2 that $\mathcal{M}_K$ has a finite-dimensional kernel and a closed image. Note that this assertion does not follow from the estimate (4.7) (valid even if there are poles of $\widetilde{\mathcal{M}}^{-1}(\lambda)$ on the line $\operatorname{Im}\lambda = -a+1$), since the embedding $\mathcal{E}^2_{-a+2}(K) \subset E^0_{-a}(K)$ is continuous, but not compact.

In what follows, we shall establish the connection between the kernels of the operators $\mathcal{M}_K$ and $\mathcal{L}_K^*$ ($\mathcal{L}_K^*$ is the operator adjoint to $\mathcal{L}_K$). To study the operator $\mathcal{L}_K^*$, we need an assertion on the *a priori* estimates and smoothness of the solutions of an auxiliary problem. We shall state this assertion in the following section.

4.3. Consider the bounded operator

$$\mathcal{L}\colon W^2(\mathbb{R}^2) \to L_2(\mathbb{R}^2) \times \prod_j W^{3/2}(\mathbb{R}^1), \qquad \mathcal{L}U = (\Delta U, U|_{x_2=0}, U|_{x_2=0}).$$

Note that the problem corresponding to the operator $\mathcal{L}$ is artificial: this is neither a boundary-value problem (since the solution U is sought in the whole space $\mathbb{R}^2$) nor a transmission problem (since on the line $\{x_2 = 0\}$ we are not given the conjugation conditions but rather the trace of the function U, and in fact twice). However, in deriving *a priori* estimates for the solutions of conjugate nonlocal problems, we encounter a problem of exactly such type (in the following section), which can be explained by the specific character of the applied method consisting in the "separation of nonlocal terms."

Let us introduce the bounded operator

$$\mathcal{L}^*\colon L_2(\mathbb{R}^2) \times \prod_j W^{-3/2}(\mathbb{R}^1) \to W^{-2}(\mathbb{R}^2)$$

adjoint to $\mathcal{L}$. The operator $\mathcal{L}^*$ acts on $\{f, g_j\} \in L_2(\mathbb{R}^2) \times \prod_j W^{-3/2}(\mathbb{R}^1)$ by the formula

$$\langle U, \mathcal{L}^*\{f, g_j\}\rangle = (\Delta U, f)_{\mathbb{R}^2} + \sum_j \langle U|_{x_2=0}, g_j\rangle_{\mathbb{R}^1} \qquad \text{for any} \quad U \in W^2(\mathbb{R}^2),$$

where $(\cdot\,,\,\cdot)_{\mathbb{R}^2}$ denotes the inner product in $L_2(\mathbb{R}^2)$, $\langle\cdot\,,\,\cdot\rangle_{\mathbb{R}^1}$ denotes the sesquilinear form on the pair of spaces $W^{3/2}(\mathbb{R}^1)$, $W^{-3/2}(\mathbb{R}^1)$.

Denote $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2\colon x_2 > 0\}$, $\mathbb{R}^2_- = \{x \in \mathbb{R}^2\colon x_2 < 0\}$. For integers $l \ge 0$, let us introduce the space

$$\mathcal{W}^l(\mathbb{R}^2) = W^l(\mathbb{R}^2_+) \oplus W^l(\mathbb{R}^2_-).$$

Lemma 4.4. *For any fixed integer $l \ge 0$, if*

$$\{f, g_j\} \in L_2(\mathbb{R}^2) \times \prod_j W^{-3/2+l}(\mathbb{R}^1), \qquad \mathcal{L}^*\{f, g_j\} \in \begin{cases} W^{-2+l}(\mathbb{R}^2) & \text{for } l < 2, \\ \mathcal{W}^{-2+l}(\mathbb{R}^2) & \text{for } l \ge 2, \end{cases}$$

then $f \in \mathcal{W}^l(\mathbb{R}^2)$ and

$$\|f\|_{\mathcal{W}^l(\mathbb{R}^2)} \le c_l\biggl(\|\mathcal{L}^*\{f, g_j\}\|_{-2+l} + \|f\|_{W^{-1}(\mathbb{R}^2)} + \sum_j \|g_j\|_{W^{-3/2+l}(\mathbb{R}^1)}\biggr), \tag{4.13}$$

where

$$\|\cdot\|_{-2+l} = \begin{cases} \|\cdot\|_{W^{-2+l}(\mathbb{R}^2)} & \text{for } l < 2, \\ \|\cdot\|_{\mathcal{W}^{-2+l}(\mathbb{R}^2)} & \text{for } l \ge 2, \end{cases}$$

and the $c_l > 0$ are independent of $\{f, g_j\}$.

Proof. The proof is carried out according to the scheme [11, Chap. 2] (see Theorems 4.1, 4.3 [11, Chap. 2]).

Note that in contrast to model problems in the whole space (see [11, Chap. 2, Sec. 3]), in our case the operator $\mathcal{L}^*$ contains distributions with supports on the line $\{x_2 = 0\}$. In this connection, note that smoothness of the function f can fail to hold on the line $\{x_2 = 0\}$, even if $\mathcal{L}^*\{f, g_j\}$ is infinitely smooth in $\mathbb{R}^2$. Moreover, Lemma 4.4 means that to increase smoothness of the function f in $\mathbb{R}^2_+$ and $\mathbb{R}^2_-$, it is necessary to require additional smoothness not only for $\mathcal{L}^*\{f, g_j\}$, but also for the distributions g_j. □

4.4. Now let us study the operator adjoint to $\mathcal{L}_K$. For integers $l \ge 0$, by

$$(E_a^l(K))^*, \qquad (E_a^{l+1/2}(\gamma_j))^*, \qquad \text{and} \qquad (E_a^{l+1/2}(\gamma))^*$$

we denote the spaces adjoint to $E_a^l(K)$, $E_a^{l+1/2}(\gamma_j)$, and $E_a^{l+1/2}(\gamma)$ with respect to the inner products in $L_2(K)$, $L_2(\gamma_j)$, and $L_2(\gamma)$, respectively. Obviously, $(E_a^0(K))^* = E_{-a}^0(K)$.

Set $\hat{\gamma}_j = \{y: \varphi = b_j \text{ or } \varphi = b_j + \pi\}$, $\hat{\gamma} = \{y: \varphi = b \text{ or } \varphi = b + \pi\}$. Obviously, $\gamma_j \subset \hat{\gamma}_j$, $\gamma \subset \hat{\gamma}$. For integers $l \ge 0$, by $W_{\overline{K}}^{-l}(\mathbb{R}^2)$, $W^{-l-1/2}(\hat{\gamma}_j)$, and $W^{-l-1/2}(\hat{\gamma})$ we denote the spaces adjoint to $W^l(K)$, $W^{l+1/2}(\hat{\gamma}_j)$, and $W^{l+1/2}(\hat{\gamma})$, respectively.

Let us introduce functions $\psi_p \in C_0^\infty(\mathbb{R}^2)$ such that $\psi_p(y) = 1$ for $r_1 d_1^{3-p} < r < r_2 d_2^{3-p}$, $\psi_p(y) = 0$ for $r < 2r_1 d_1^{3-p}/3$ and $r > 3r_2 d_2^{3-p}/2$. Here $0 < r_1 < r_2$; $p = 0, \dots, 3$.

For $g_j \in (E_a^{l+1/2}(\gamma_j))^*$, by $\psi_p g_j$ we denote the distribution $W^{-l-1/2}(\hat{\gamma}_j)$ defined by the relation $\langle u_{\hat{\gamma}_j}, \psi_p g_j\rangle_{\hat{\gamma}_j} = \langle \psi_p u_{\hat{\gamma}_j}, g_j\rangle_{\gamma_j}$ for all $u_{\hat{\gamma}_j} \in W^{l+1/2}(\hat{\gamma}_j)$. Here $\langle \cdot\,, \cdot\rangle_{\hat{\gamma}_j}$, $\langle \cdot\,, \cdot\rangle_{\gamma_j}$ are sesquilinear forms on the dual pairs $W^{l+1/2}(\hat{\gamma}_j)$, $W^{-l-1/2}(\hat{\gamma}_j)$ and $E^{l+1/2}(\gamma_j)$, $E^{-l-1/2}(\gamma_j)$, respectively. Similarly, for $g \in (E_a^{l+1/2}(\gamma))^*$ we introduce the distribution $\psi_p g_j \in W^{-l-1/2}(\hat{\gamma})$.

For the operator $\mathcal{L}_K: E_a^2(K) \to E_a^0(K, \gamma)$ introduced in Sec. 1, consider the adjoint operator

$$\mathcal{L}_K^*: (E_a^0(K,\gamma))^* \to (E_a^2(K))^*, \qquad \text{where} \quad (E_a^0(K,\gamma))^* = E_{-a}^0(K) \times \prod_j (E_a^{3/2}(\gamma_j))^*.$$

The operator $\mathcal{L}_K^*$ acts on $\{f, g_j\} \in (E_a^0(K,\gamma))^*$ for all $u \in E_a^2(K)$ by the formula

$$\langle u, \mathcal{L}_K^*\{f, g_j\}\rangle = (\Delta u - u, f)_K + \sum_j \langle u|_{\gamma_j} + a_j u(\mathcal{G}_j y)|_{\gamma_j}, g_j\rangle_{\gamma_j}.$$

Here $(\cdot\,, \cdot)_K$ denotes the inner product in $L_2(K)$ and $\langle \cdot\,, \cdot\rangle$ and $\langle \cdot\,, \cdot\rangle_{\gamma_j}$ denote sesquilinear forms on the corresponding dual pairs of spaces.

For integers $l \ge 0$, set $\mathcal{W}^l(K) = \bigoplus_j W^l(K_j)$.

Theorem 4.3. *Suppose that $\{f, g_j\} \in (E_a^0(K,\gamma))^*$, $\mathcal{L}_K^*\{f, g_j\} \in (E_a^2(K))^*$. Then for any fixed integer $l \ge 0$, if*

$$\psi_0 \mathcal{L}_K^*\{f, g_j\} \in \begin{cases} W_{\overline{K}}^{-2+l}(\mathbb{R}^2) & \text{for } l < 2, \\ \mathcal{W}^{-2+l}(K) & \text{for } l \ge 2, \end{cases}$$

then $\psi_3\{f, g_j\} \in \mathcal{W}^l(K) \times \prod_j W^{-3/2+l}(\hat{\gamma}_j)$ and

$$\begin{aligned} &\|\psi_3\{f, g_j\}\|_{\mathcal{W}^l(K)\times\prod_j W^{-3/2+l}(\hat{\gamma}_j)} \\ &\qquad \le c_l(\|\psi_0\mathcal{L}_K^*\{f, g_j\}\|_{-2+l} + \|\psi_0\{f, g_j\}\|_{W_{\overline{K}}^{-1}(\mathbb{R}^2)\times\prod_j W^{-5/2}(\hat{\gamma}_j)}), \end{aligned} \tag{4.14}$$

where

$$\|\cdot\|_{-2+l} = \begin{cases} \|\cdot\|_{W_{\overline{K}}^{-2+l}(\mathbb{R}^2)} & \text{for } l < 2, \\ \|\cdot\|_{\mathcal{W}^{-2+l}(K)} & \text{for } l \ge 2, \end{cases}$$

and the $c_l > 0$ are independent of $\{f, g_j\}$.

Proof. 1) Let us introduce the operator

$$\mathcal{L}_{\mathcal{G}}^*: E_{-a}^0(K) \times \prod_j \{(E_a^{3/2}(\gamma_j))^* \times (E_a^{3/2}(\gamma))^*\} \to (E_a^2(K))^*,$$

acting on $\{f, g_j, g'_j\} \in E^0_{-a}(K) \times \prod_j \{(E^{3/2}_a(\gamma_j))^* \times (E^{3/2}_a(\gamma))^*\}$ for all $u \in E^2_a(K)$ by the formula

$$\langle u, \mathcal{L}^*_{\mathcal{G}}\{f, g_j, g'_j\}\rangle = (\Delta u - u, f)_K + \sum_j \{\langle u|_{\gamma_j}, g_j\rangle_{\gamma_j} + \langle a_j u|_\gamma, g'_j\rangle_\gamma\}.$$

For $g_j \in (E^{3/2}_a(\gamma_j))^*$, we define the distribution $g^{\mathcal{G}}_j \in (E^{3/2}_a(\gamma))^*$ by the relation

$$\langle u_\gamma, g^{\mathcal{G}}_j\rangle_\gamma = \langle u_\gamma(\mathcal{G}_j \cdot), g_j\rangle_{\gamma_j} \qquad \text{for all} \quad u_\gamma \in E^{3/2}_a(\gamma).$$

Note that $\psi_p g^{\mathcal{G}}_j \in W^{-3/2+l}(\hat\gamma)$ if and only if $\psi_p(\mathcal{G}_j \cdot) g_j \in W^{-3/2+l}(\hat\gamma_j)$; moreover, there exist constants $k_1, k_2 > 0$ (depending on l) such that

$$k_1\|\psi_p(\mathcal{G}_j \cdot)g_j\|_{W^{-3/2+l}(\hat\gamma_j)} \le \|\psi_p g^{\mathcal{G}}_j\|_{W^{-3/2+l}(\hat\gamma)} \le k_2\|\psi_p(\mathcal{G}_j \cdot)g_j\|_{W^{-3/2+l}(\hat\gamma_j)}. \tag{4.15}$$

It follows from the definitions of the operators $\mathcal{L}^*_K$ and $\mathcal{L}^*_{\mathcal{G}}$ that

$$\mathcal{L}^*_{\mathcal{G}}\{f, g_j, g^{\mathcal{G}}_j\} = \mathcal{L}^*_K\{f, g_j\}. \tag{4.16}$$

2) Suppose that ε, ζ, ζ_j denote the same as in the proof of Lemma 3.1. Let us introduce functions $\hat\zeta_j, \hat\zeta, \bar\zeta_j, \bar\zeta \in C^\infty_0(\mathbb{R})$ such that $\hat\zeta_j(\varphi) = 1$ for $|b_j - \varphi| < 3\varepsilon/2$, $\hat\zeta_j(\varphi) = 0$ for $|b_j - \varphi| > 2\varepsilon$; $\hat\zeta(\varphi) = 1$ for $|b - \varphi| < 3\varepsilon/2$, $\hat\zeta(\varphi) = 0$ for $|b - \varphi| > 2\varepsilon$; $\bar\zeta_j(\varphi) = 1$ for $|b_j - \varphi| < \varepsilon/8$, $\bar\zeta_j(\varphi) = 0$ for $|b_j - \varphi| > \varepsilon/4$; $\bar\zeta(\varphi) = 1$ for $|b - \varphi| < \varepsilon/8$, $\bar\zeta(\varphi) = 0$ for $|b - \varphi| > \varepsilon/4$.

The support of the function ζ_i does not intersect γ and γ_k for $k \ne i$; hence $\psi_p\zeta_i g_k = 0$, $\psi_p\zeta_i g^{\mathcal{G}}_j = 0$; therefore,

$$\langle u, \mathcal{L}^*_{\mathcal{G}}(\psi_p\zeta_i\{f, g_j, g^{\mathcal{G}}_j\})\rangle = (\psi_p\zeta_i\Delta u - \psi_p\zeta_i u, f)_K + \langle(\psi_p\zeta_i u)|_{\gamma_i}, g_i\rangle_{\gamma_i}.$$

Since the change of variables of rotation type takes the Laplace operator to the Laplace operator and preserves the property of functions to belong to the corresponding Sobolev spaces, we can use Theorem 4.3 [11, Chap. 2].[1] Hence it follows from relations (4.16) and the Leibniz formula that

$$\begin{gathered}\psi_1\zeta_i\{f, g_j, g^{\mathcal{G}}_j\} \in W^l(K) \times \prod_j \{W^{-3/2+l}(\hat\gamma_j) \times W^{-3/2+l}(\hat\gamma)\},\\ \|\psi_1\zeta_i\{f, g_j, g^{\mathcal{G}}_j\}\|_{W^l(K)\times\prod_j\{W^{-3/2+l}(\hat\gamma_j)\times W^{-3/2+l}(\hat\gamma)\}}\\ \le k_3(\|\psi_0\mathcal{L}^*_K\{f, g_j\}\|_{-2+l} + \|\psi_0\hat\zeta_i f\|_{W^{-1}_{\overline{K}}(\mathbb{R}^2)} + \|\psi_0 g_i\|_{W^{-5/2}(\hat\gamma_i)}).\end{gathered} \tag{4.17}$$

Hence, in particular, using (4.15), we find that $\psi_2 g^{\mathcal{G}}_i \in W^{-3/2+l}(\hat\gamma)$ and

$$\|\psi_2 g^{\mathcal{G}}_i\|_{W^{-3/2+l}(\hat\gamma)} \le k_4(\|\psi_0\mathcal{L}^*_K\{f, g_j\}\|_{-2+l} + \|\psi_0\hat\zeta_i f\|_{W^{-1}_{\overline{K}}(\mathbb{R}^2)} + \|\psi_0 g_i\|_{W^{-5/2}(\hat\gamma_i)}). \tag{4.18}$$

3) The support of the function ζ does not intersect γ_j; hence $\psi_p\zeta_i g_j = 0$; therefore,

$$\langle u, \mathcal{L}^*_{\mathcal{G}}(\psi_p\zeta\{f, g_j, g^{\mathcal{G}}_j\})\rangle = (\psi_p\zeta\Delta u - \psi_p\zeta u, f)_K + \sum_k \langle(\psi_p\zeta u)|_\gamma, g^{\mathcal{G}}_k\rangle_\gamma.$$

[1] In Theorem 4.3 [11, Chap. 2], operators with variable coefficients were studied; this led to the imposition of additional constraints on the supports of the functions under consideration. However, it is readily seen that in the case of operators with constant coefficients these constraints can be removed.

Thus, taking into account the properties of the supports of the functions ψ_p and ζ, we can regard the operator $\mathcal{L}^*_{\mathcal{G}}$, acting on $\psi_p\zeta\{f, g_j, g_j^{\mathcal{G}}\}$, as adjoint to the problem

$$\Delta u - u = \hat{f}(y), \quad y \in \mathbb{R}^2, \qquad u|_{\hat\gamma} = \hat g_1(y), \quad u|_{\hat\gamma} = \hat g_2(y), \quad y \in \hat\gamma,$$

which (after the corresponding change of variables of rotation type) coincides up to the lowest term u with the problem discussed in Sec. 4.3. As proved above, $\psi_2 g_k^{\mathcal{G}} \in W^{-3/2+l}(\hat\gamma)$; therefore, we can use Lemma 4.4. Then from relation (4.16) and the Leibniz formula we obtain

$$\psi_3\zeta\{f, g_j, g_j^{\mathcal{G}}\} \in \mathcal{W}^l(K) \times \prod_j \{W^{-3/2+l}(\hat\gamma_j) \times W^{-3/2+l}(\hat\gamma)\},$$

$$\|\psi_3\zeta\{f, g_j, g_j^{\mathcal{G}}\}\|_{\mathcal{W}^l(K)\times\prod_j\{W^{-3/2+l}(\hat\gamma_j)\times W^{-3/2+l}(\hat\gamma)\}} \le k_5\bigg(\|\psi_2\mathcal{L}^*_K\{f, g_j\}\|_{-2+l} + \|\psi_2\hat\zeta f\|_{W^{-1}_{\overline{K}}(\mathbb{R}^2)} + \sum_j \|\psi_2 g_j^{\mathcal{G}}\|_{W^{-3/2+l}(\hat\gamma)}\bigg). \tag{4.19}$$

Note that the space $\mathcal{W}^l(K)$ is now relevant, i.e., the smoothness of the function f fails to hold on the ray γ. This is due to the presence of nonlocal terms in the boundary condition (1.7) and, as a consequence, in the adjoint operator $\mathcal{L}^*_K$.

From inequalities (4.19) and (4.18) we obtain

$$\|\psi_3\zeta\{f, g_j, g_j^{\mathcal{G}}\}\|_{\mathcal{W}^l(K)\times\prod_j\{W^{-3/2+l}(\hat\gamma_j)\times W^{-3/2+l}(\hat\gamma)\}} \le k_6\bigg(\|\psi_0\mathcal{L}^*_K\{f, g_j\}\|_{-2+l} + \|\psi_0\hat\zeta f\|_{W^{-1}_{\overline{K}}(\mathbb{R}^2)} + \sum_j \|\psi_0 g_j\|_{W^{-5/2}(\hat\gamma_j)}\bigg). \tag{4.20}$$

4) The support of the function $\zeta_0 = 1 - \sum_i \zeta_i - \zeta$ does not intersect γ_j and γ; therefore, $\psi_p\zeta_0 g_j = 0$, $\psi_p\zeta_0 g_j^{\mathcal{G}} = 0$; therefore,

$$\langle u, \mathcal{L}^*_{\mathcal{G}}(\psi_p\zeta_0\{f, g_j, g_j^{\mathcal{G}}\})\rangle = (\psi_p\zeta_0\Delta u - \psi_p\zeta_0 u, f)_K.$$

From Theorem 3.1 [11, Chap. 2], relation (4.16), and the Leibniz formula, we obtain

$$\psi_1\zeta_0\{f, g_j, g_j^{\mathcal{G}}\} \in W^l(K) \times \prod_j \{W^{-3/2+l}(\hat\gamma_j) \times W^{-3/2+l}(\hat\gamma)\},$$

$$\|\psi_1\zeta_0\{f, g_j, g_j^{\mathcal{G}}\}\|_{W^l(K)\times\prod_j\{W^{-3/2+l}(\hat\gamma_j)\times W^{-3/2+l}(\hat\gamma)\}} \le k_7(\|\psi_0\mathcal{L}^*_K\{f, g_j\}\|_{-2+l} + \|\psi_0\bar\zeta_0 f\|_{W^{-1}_{\overline{K}}(\mathbb{R}^2)}), \tag{4.21}$$

where $\bar\zeta_0 = 1 - \sum_i \bar\zeta_i - \bar\zeta$.

Now the *a priori* estimate (4.14) follows from inequalities (4.17), (4.20), and (4.21). □

5. SOLVABILITY OF NONLOCAL ELLIPTIC BOUNDARY-VALUE PROBLEMS

In this section, we present the main results concerning the solvability of nonlocal problems in plane and dihedral angles (Theorems 5.1–5.3).

5.1. First, we establish the connection between the kernels of the operators $\mathcal{L}^*_K$ and $\mathcal{M}_K$.

Lemma 5.1. *The kernel* $\ker(\mathcal{L}^*_K)$ *of the operator* $\mathcal{L}^*_K$ *coincides with the set over which the element* $\{v, (\partial v_j/\partial n_j)|_{\gamma_j}\}$ *ranges whenever* $v \in \mathcal{E}^2_{-a+2}(K)$ *and* $v_j \in C^\infty(\overline{K}_j \setminus \{0\})$ *satisfies problem* (3.1), (3.2) *for* $\{f, g_j, h_\nu\} = 0$. *Here* $v(y) \equiv v_j(y)$, $f(y) \equiv f_j(y)$ *for* $y \in K_j$.

Proof. 1) Suppose that $v \in \mathcal{E}^2_{-a+2}(K)$ and $v_j \in C^\infty(\overline{K}_j \setminus \{0\})$ satisfies problem (3.1), (3.2) for $\{f, g_j, h_\nu\} = 0$. Then, by Theorem 2.1, for any $u \in C_0^\infty(\overline{K}_j \setminus \{0\})$ we have

$$\sum_{j=1}(\Delta u - u, v_j)_{K_j} + \sum_j \left(u|_{\gamma_j} + a_j u(\mathcal{G}_j y)|_{\gamma_j}, \frac{\partial v_j}{\partial n_j}\bigg|_{\gamma_j} \right)_{\gamma_j} = 0. \tag{5.1}$$

It follows from the continuity of the operator of embedding of $\mathcal{E}^2_{-a+2}(K)$ in $E^0_{-a}(K)$ that $v \in E^0_{-a}(K)$. In addition, by the Cauchy–Bunyakovskii inequality and Theorem 1.4, we have

$$\begin{aligned}\left| \left(u_{\gamma_j}, \frac{\partial v_j}{\partial n_j}\bigg|_{\gamma_j} \right)_{\gamma_j} \right|^2 &\le k_1 \int_{\gamma_j} r^{2(a-3/2)} |u_{\gamma_j}|^2 \, d\gamma \cdot \int_{\gamma_j} r^{2(-a+3/2)} \left| \frac{\partial v_j}{\partial n_j}\bigg|_{\gamma_j} \right|^2 d\gamma \\ &\le k_2 \|u_{\gamma_j}\|^2_{E_a^{3/2}(\gamma_j)} \cdot \left\| \frac{\partial v_j}{\partial n_j}\bigg|_{\gamma_j} \right\|^2_{E^{1/2}_{-a+2}(\gamma_j)}\end{aligned}$$

for all $u_{\gamma_j} \in E_a^{3/2}(\gamma_j)$. Therefore, $(\partial v_j/\partial n_j)|_{\gamma_j} \in (E_a^{3/2}(\gamma_j))^*$.

Thus

$$\left\{ v, \frac{\partial v_j}{\partial n_j}\bigg|_{\gamma_j} \right\} \in E^0_{-a}(K) \times \prod_j (E_a^{3/2}(\gamma_j))^*$$

and by the definition of the operator $\mathcal{L}_K^*$ and identity (5.1) we have

$$\left\langle u, \mathcal{L}_K^* \left\{ v, \frac{\partial v_j}{\partial n_j}\bigg|_{\gamma_j} \right\} \right\rangle = 0 \qquad \text{for all} \quad u \in C_0^\infty(\overline{K} \setminus \{0\}).$$

But $C_0^\infty(\overline{K} \setminus \{0\})$ is dense in $E^2_a(K)$; therefore, $\{v, (\partial v_j/\partial n_j)|_{\gamma_j}\} \in \ker(\mathcal{L}_K^*)$.

2) Now suppose that, conversely, $\{v, \psi_j\} \in \ker(\mathcal{L}_K^*)$. It follows from Theorem 4.3 that

$$v_j \in C^\infty(\overline{K}_j \setminus \{0\}), \psi_j \in C^\infty(\gamma_j).$$

Then the definition of the operator $\mathcal{L}_K^*$ implies that for any $u \in C_0^\infty(\overline{K} \setminus \{0\})$

$$(\Delta u - u, v)_K = -\sum_j (u|_{\gamma_j} + a_j u(\mathcal{G}_j y)|_{\gamma_j}, \psi_j)_{\gamma_j},$$

which, together with Green's formula (2.1), yields

$$\begin{aligned}\sum_j \left(u|_{\gamma_j} + a_j u(\mathcal{G}_j y)|_{\gamma_j}, \frac{\partial v_j}{\partial n_j}\bigg|_{\gamma_j} - \psi_j \right)_{\gamma_j} + \left(u|_\gamma, \frac{\partial v_2}{\partial n}\bigg|_\gamma - \frac{\partial v_1}{\partial n}\bigg|_\gamma - \sum_k \bar{a}_k \chi_k^{-1} \frac{\partial v_k}{\partial n_k}(\mathcal{G}_k^{-1} y)|_\gamma \right)_\gamma \\ = \sum_j (u, \Delta v_j - v_j)_{K_j} + \sum_j \left(\frac{\partial u}{\partial n_j}\bigg|_{\gamma_j}, v_j|_{\gamma_j} \right)_{\gamma_j} + \left(\frac{\partial u}{\partial n}\bigg|_\gamma, v_2|_\gamma - v_1|_\gamma \right)_\gamma.\end{aligned} \tag{5.2}$$

Setting $\operatorname{supp} u \in K_j$, from (5.2) we find that $\Delta v_j - v_j = 0$.

By Lemma 2.2 [11, Chap. 2], for any system of functions $\{\Theta_{j\mu}\}_{\mu=1}^2$ there exists a function $u \in C_0^\infty(\overline{K} \setminus \{0\})$ in $C_0^\infty(\gamma_j)$ such that

$$u|_{\gamma_j} = \Theta_{j1}, \qquad \frac{\partial u}{\partial n_j}\bigg|_{\gamma_j} = \Theta_{j2}, \qquad u = 0 \quad \text{in the neighborhood of } \gamma.$$

Hence, from (5.2) and the fact that $\Delta v_j - v_j = 0$, we obtain

$$\frac{\partial v_j}{\partial n_j}\Big|_{\gamma_j} - \psi_j = 0 \qquad \text{and} \qquad v_j|_{\gamma_j} = 0.$$

Similarly,

$$v_2|_\gamma - v_1|_\gamma = 0, \qquad \frac{\partial v_2}{\partial n}\Big|_\gamma - \frac{\partial v_1}{\partial n}\Big|_\gamma - \sum_k \bar{a}_k \chi_k^{-1} \frac{\partial v_k}{\partial n_k}(\mathcal{G}_k^{-1} y)|_\gamma = 0.$$

Finally, the fact that $v \in E^0_{-a}(K)$, $v_j \in C^\infty(\overline{K}_j \setminus \{0\})$ and Theorem 4.1 imply that the function v belongs to the space $\mathcal{E}^2_{-a+2}(K)$. $\square$

Theorem 5.1. *The operator $\mathcal{L}_K$ is Fredholm if and only if there are no poles of the operator function $\widetilde{\mathcal{L}}^{-1}(\lambda)$ on the line* $\operatorname{Im}\lambda = a-1$.

Proof. Suppose that the operator $\mathcal{L}_K$ is Fredholm; then, by Theorem 7.1 [13], we have the estimate (1.11) and hence, by Theorem 1.1, there are no poles of the operator function $\widetilde{\mathcal{L}}^{-1}(\lambda)$ on the line $\operatorname{Im}\lambda = a-1$.

Suppose that, conversely, the operator function $\widetilde{\mathcal{L}}^{-1}(\lambda)$ has no poles on the line $\operatorname{Im}\lambda = a-1$. Then, by Theorem 1.1, the operator $\mathcal{L}_K$ has a finite-dimensional kernel and a closed image.

Let us prove that the kernel of the operator $\mathcal{L}_K^*$ is finite-dimensional. Green's formula (2.3), Remark 2.2 and Lemmas 1.5, 3.2 imply that λ_0 is a pole of the operator $\widetilde{\mathcal{L}}^{-1}(\lambda)$ if and only if $\bar{\lambda}_0$ is a pole of the operator $\widetilde{\mathcal{M}}^{-1}(\lambda)$. Hence there are no poles of the operator function $\widetilde{\mathcal{M}}^{-1}(\lambda)$ on the line $\operatorname{Im}\lambda = -a+1$. Therefore, by Theorem 4.2, the operator $\mathcal{M}_K$ has a finite-dimensional kernel whose dimension coincides, by Lemma 5.1, with the dimension of the kernel of the operator $\mathcal{L}_K^*$. $\square$

5.2. Let us proceed with the study of the solvability of the nonlocal boundary-value problem (1.1), (1.2) in a dihedral angle. As in the proof of Lemma 7.3 [6], we can reduce problem (1.1), (1.2) to problem (1.6), (1.7) by using the Fourier transform with respect to z: $U(y,z) \to \widehat{U}(y,\eta)$ and make the change of variables $y' = |\eta| \cdot y$, thus obtaining the following result.

Theorem 5.2. *Suppose there are no poles of the operator function $\widetilde{\mathcal{L}}^{-1}(\lambda)$ on the line* $\operatorname{Im}\lambda = a-1$ *and* $\dim\ker\mathcal{L}_K = \operatorname{codim}\mathcal{R}(\mathcal{L}_K) = 0$. *Then the operator $\mathcal{L}_\Omega$ is an isomorphism.*

Using Green's formula for the nonlocal problem (1.1), (1.2) in the dihedral angle Ω, which is similar to Green's formula (2.1) for problem (1.6), (1.7) in the plane angle K, and repeating the arguments from [6, Sec. 8], we obtain the following necessary condition for the Fredholm property of the operator $\mathcal{L}_\Omega$.

Theorem 5.3. *If the operator $\mathcal{L}_\Omega$ is Fredholm, then the operator $\mathcal{L}_K$ is an isomorphism.*

It follows from Theorems 1.1, 5.2, and 5.3 that if the operator $\mathcal{L}_\Omega$ is Fredholm, then it an isomorphism.

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