Parabolic problems with the Preisach hysteresis operator in boundary conditions

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Abstract

Parabolic initial boundary-value problems coupled (via the boundary condition) with ordinary differential equations whose right-hand side contains the Preisach hysteresis operator are considered. In particular, these problems model thermocontrol processes in chemical reactors, climate-control systems, biological cells, etc. For the Preisach operator with and without time delay, solvability, periodicity of solutions, and global B-attractors are studied.

Key words: Thermocontrol problem, hysteresis, Preisach operator, periodicity, global attractor
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1 Introduction

We consider a parabolic initial boundary-value problem coupled with an ordinary differential equation whose right-hand side contains the so-called Preisach hysteresis operator. In particular, this problem models thermocontrol processes in chemical reactors, climate-control systems, biological cells, etc. In

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these processes, the temperature inside a domain at a moment t is controlled by a "thermostat" acting on the boundary. The feedback is based on temperature measurements performed by thermal sensors inside the domain at the moment $t - \tau$ ($\tau \geq 0$). The presence of the Preisach operator corresponds to the fact that the power of the thermostat changes continuously, while the ordinary differential equation means that the temperature of the thermostat also changes continuously. We investigate existence, uniqueness, and periodicity of solutions as well as their large-time behavior.

A thermocontrol model similar to ours (with $\tau = 0$ and somewhat different switching law) was originally proposed in [7,8]. By reducing the problem to an equivalent set-valued integro-differential equation, the existence of a solution was proved. Some questions related to optimal control for heat conduction problems with hysteresis were considered, e.g., in [4].

The question whether *periodic* solutions exist turns out to be much more difficult. In [6], a one-dimensional thermocontrol problem with the hysteresis operator on the boundary described by the rectangular hysteresis loop is considered under the assumption that the temperature of the thermostat changes by jump. Thus, there is no coupling with an ordinary differential equation in that case. The existence of a periodic solution is proved. Its uniqueness in a class of the so-called "two-phase" periodic solutions is established.

Periodicity of solutions of a one-dimensional[2] problem, with the same hysteresis functional as in the previous references, in the case where a thermostat changes its temperature continuously, was considered in [19]. The existence of a periodic solution was proved. The periodicity of solutions for a one-dimensional Stefan problem with hysteresis-type boundary conditions was investigated in [9].

Switching systems described by ordinary differential equations with hysteresis were considered by many authors (see e.g., [1, 3, 14, 17, 20]).

In the multidimensional case, the periodicity of solutions for parabolic equations involving continuous hysteresis operators was studied in [12, 25] (see also [24] and references therein).

The first investigation of periodicity of solutions for a multidimensional initial boundary-value problem involving a hysteresis-type control on the boundary and coupled with an ordinary differential equation was carried out in [10].

In the present work, we prove the existence of periodic solutions, provided that the hysteresis phenomenon is modelled by the continuous Preisach operator.

[2] When saying "one-" or "multidimensional" we mean that the space variable in the parabolic equation is one- or multidimensional, respectively.

The time delay τ between the temperature measurement moment and the thermostat reaction is either positive ($\tau > 0$) or absent ($\tau = 0$). It is proved that the solution for $\tau = 0$ can be approximated by the solutions for $\tau > 0$ as $\tau \to 0$. In particular, this allows one to show that there is a periodic solution in the case $\tau = 0$ which is a limit of periodic solutions for $\tau > 0$ as $\tau \to 0$.

Another important question concerns the behavior of solutions as $t \to +\infty$. Until now, this question was studied in the case where the hysteresis operator enters a parabolic equation itself [4, 11, 12, 24]. In our work, we prove the existence of the so-called *minimal global B-attractor*, i.e., a minimal closed set which attracts any bounded set of initial data (see Definition 8.3). We show that this attractor is a compact connected set.

The paper is organized as follows. In Sec. 2, we formulate auxiliary results concerning initial boundary-value problems for parabolic equations. In Sec. 3, we define the Preisach operator, which is a continuous model of the hysteresis phenomenon (see [13, 23]). The setting of the thermocontrol problem with time delay $\tau > 0$ is given in Sec. 4. In the same section, we prove the existence and uniqueness of the solution. In Sec. 5, using properties of the Preisach operator and the Schauder fixed-point theorem, we show that there exists a T-periodic solution of the thermocontrol problem, provided that $T > \tau$ and the right-hand side of the parabolic equation is T-periodic in time. In Sec. 6, we prove the existence and uniqueness of the solution of the thermocontrol problem without time delay ($\tau = 0$). In Sec. 7, we obtain a periodic solution in the case $\tau = 0$ as a limit of periodic solutions for $\tau > 0$ as $\tau \to 0$. In the case where the right-hand side does not explicitly depend on t, we prove the existence of a stationary solution. Sufficient conditions under which the stationary solution is unique are given. In Sec. 8, using the technique developed in [15] (see also [22]), we study the large-time behavior of solutions for the problem in question. Namely, we prove the existence of a compact connected minimal global B-attractor. Some open questions are formulated in Sec. 9.

2 Strong and Mild Solutions of Parabolic Problems

In this section, we recall some facts about solvability of linear parabolic problems and regularity of their solutions.

2.1 Setting of the problem

Let $Q \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded domain with boundary Γ of class C^∞.

We introduce the differential expression

$$Pu(x) = \Delta\psi(x) - p(x)\psi(x) \quad (x \in Q),$$

where $p \in C^\infty(\mathbb{R}^n)$, $p(x) \ge 0$.

Let $T > 0$, $Q_T = Q \times (0, T)$, and $\Gamma_T = \Gamma \times (0, T)$. In this section, we consider the following parabolic initial boundary-value problem:

$$v_t(x,t) = Pv(x,t) + f(x,t) \quad ((x,t) \in Q_T), \tag{2.1}$$

$$v(x,0) = \psi(x) \quad (x \in Q), \tag{2.2}$$

$$\gamma\frac{\partial v}{\partial \nu} + \sigma(x)v(x,t) = 0 \quad ((x,t) \in \Gamma_T), \tag{2.3}$$

where $f \in L_2(Q_T)$, ν is the outward normal to Γ_T at the point (x,t), $\gamma \ge 0$, $\sigma \in C^\infty(\mathbb{R}^n)$ is a real-valued function, $\sigma(x) \ge 0$. We also assume that $\sigma(x) \ge \sigma_0 > 0$ if $\gamma = 0$ and $p(x) \not\equiv 0$ if $\sigma(x) \equiv 0$.

Denote by $W_2^k(Q)$ ($k \in \mathbb{N}$) the Sobolev space with the norm

$$\|\psi\|_{W_2^k(Q)} = \left(\sum_{|\alpha|\le k}\int_Q |D^\alpha\psi(x)|^2\,dx\right)^{1/2}.$$

By $\mathring{W}_2^k(Q)$ we denote the closure in $W_2^k(Q)$ of the set $C_0^\infty(Q)$ consisting of infinitely differentiable functions supported in Q.

We will throughout use the following equivalent norm in $W_2^1(Q)$ (which we denote by the same symbol $\|\cdot\|_{W_2^1(Q)}$ as the standard norm):

$$\|\psi\|_{W_2^1(Q)} = \begin{cases} \left(\int_Q(|\nabla\psi|^2 + \tilde{p}(x)|\psi|^2)\,dx\right)^{1/2} & \text{if } \gamma = 0, \\ \left(\int_Q(|\nabla\psi|^2 + \tilde{p}(x)|\psi|^2)\,dx + \int_\Gamma \gamma^{-1}\sigma(x)|\psi|^2\,d\Gamma\right)^{1/2} & \text{if } \gamma > 0, \end{cases} \tag{2.4}$$

where $\tilde{p}(x) = p(x) - \inf\limits_{y\in Q} p(y) + 1 \ge 1$ in Q and γ is the same as in the boundary condition (2.3)

Denote by $W_2^{2,1}(Q \times (a,b))$ ($a < b$) the anisotropic Sobolev space with the norm

$$\|v\|_{W_2^{2,1}(Q\times(a,b))} = \left(\int_a^b \|v(\cdot,t)\|^2_{W_2^2(Q)}\,dt + \int_a^b \|v_t(\cdot,t)\|^2_{L_2(Q)}\,dt\right)^{1/2}$$

and, for any Banach space B, by $C([a,b],B)$ ($a < b$) the space of B-valued functions continuous on the segment $[a,b]$ with the norm

$$\|v\|_{C([a,b],B)} = \max_{t\in[a,b]} \|v(\cdot,t)\|_B.$$

If $B = \mathbb{C}$ or $\mathbb{R}$, we will write $C[a,b]$.

Definition 2.1 *A function $v \in W_2^{2,1}(Q_T) \cap C([0,T], W_2^1(Q))$ is called a* strong solution *of problem* (2.1)–(2.3) *in Q_T if v satisfies Eq.* (2.1) *a.e. in Q_T and conditions* (2.2), (2.3) *in the sense of traces.*

In what follows, we omit the term "strong" whenever it leads to no confusion.

2.2 Solvability and a priori estimates

We introduce the unbounded linear operator $\mathbf{P} : \mathrm{D}(\mathbf{P}) \subset L_2(Q) \to L_2(Q)$ given by

$$\mathbf{P}\psi = P\psi, \qquad \mathrm{D}(\mathbf{P}) = \left\{ \psi \in W_2^2(Q) : \gamma \frac{\partial \psi(x)}{\partial \nu} + \sigma(x)\psi(x) = 0 \ (x \in \Gamma) \right\}.$$

It is well known that the operator $\mathbf{P}$ is a generator of an analytic semigroup of contractions $\mathbf{S}_t : L_2(Q) \to L_2(Q)$, $t \geq 0$. The following lemma yields the representation of a solution of problem (2.1)–(2.3) by means of the semigroup $\mathbf{S}_t$.

Lemma 2.1 *For any*

$$f \in L_2(Q_T), \qquad \psi \in \begin{cases} \mathring{W}_2^1(Q) & \text{if } \gamma = 0, \\ W_2^1(Q) & \text{if } \gamma > 0, \end{cases}$$

there exists a unique solution v of problem (2.1)–(2.3) *in Q_T. This solution is represented as*

$$v(\cdot,t) = \mathbf{S}_t\psi(\cdot) + \int_0^t \mathbf{S}_{t-s} f(\cdot,s)\, ds \quad (t \in [0,T]), \tag{2.5}$$

where the integral converges in the $L_2(Q)$ norm, and the following estimate holds:

$$\|v\|_{W_2^{2,1}(Q_T)} + \|v\|_{C([0,T],W_2^1(Q))} \leq c_1(\|f\|_{L_2(Q_T)} + \|\psi\|_{W_2^1(Q)}), \tag{2.6}$$

where $c_1 = c_1(T) > 0$ does not depend on f and ψ and is bounded on any segment $[T_1,T_2]$ $(0 < T_1 < T_2)$.

PROOF. The assertion of the lemma follows from Theorem 3.7 in [2, Chap. 1], inequality (3.9) in [2, Chap. 1], Theorem 1.14.5 in [21], and Theorem 4.3.3 in [21].

Lemma 2.2 *For $f = 0$ and any*

$$\psi \in \begin{cases} \mathring{W}_2^1(Q) & \text{if } \gamma = 0, \\ W_2^1(Q) & \text{if } \gamma > 0, \end{cases}$$

there exists a unique solution v of problem (2.1)–(2.3) in Q_T and

$$\|v(\cdot, T)\|_{L_2(Q)} \le e^{-\omega T}\|\psi\|_{L_2(Q)}, \qquad \|v(\cdot, T)\|_{W_2^1(Q)} \le e^{-\omega T}\|\psi\|_{W_2^1(Q)}, \tag{2.7}$$

where the norm $\|\cdot\|_{W_2^1(Q)}$ is given by (2.4) and $\omega > 0$ does not depend on ψ and T.

PROOF. The existence and uniqueness of the solution v follows from Lemma 2.1. It remains to prove the inequalities in (2.7).

Let $\{\lambda_k\}_{k=1}^{\infty}$ and $\{e_k\}_{k=1}^{\infty}$ denote the sequence of eigenvalues and the corresponding system of real-valued eigenfunctions (orthonormal in $L_2(Q)$) of the operator $\mathbf{P}$.

It is well known that $0 < \lambda_1 < \lambda_2 \le \dots \le \lambda_k \le \dots$ and the system of eigenfunctions $\{e_k\}_{k=1}^{\infty}$ forms an orthonormal basis for $L_2(Q)$.

Furthermore, the functions $e_k/\sqrt{\lambda_k - p_0 + 1}$, where $p_0 = \inf_{x\in Q} p(x)$, form an orthonormal basis for $\mathring{W}_2^1(Q)$ if $\gamma = 0$ and for $W_2^1(Q)$ if $\gamma > 0$ with respect to the norm $\|\cdot\|_{W_2^1(Q)}$ given by (2.4).

The function ψ can be expanded into the Fourier series

$$\psi(x) = \sum_{k=1}^{\infty} \psi_k e_k(x),$$

where $\psi_k = \int_Q \psi(x)e_k(x)\,dx$, which converges in $W_2^1(Q)$.

Further, the function $v(x, T)$ is of the form

$$v(x, T) = \sum_{k=1}^{\infty} e^{-\lambda_k T}\psi_k e_k(x), \tag{2.8}$$

where the series in (2.8) converges in $W_2^1(Q)$. Using (2.8), we obtain

$$\|v(\cdot, T)\|_{L_2(Q)}^2 \le e^{-2\lambda_1 T}\sum_{k=1}^{\infty}\psi_k^2 = e^{-2\lambda_1 T}\|\psi\|_{L_2(Q)}^2,$$

$$\|v(\cdot, T)\|_{W_2^1(Q)}^2 \le e^{-2\lambda_1 T}\sum_{k=1}^{\infty}(\lambda_k - p_0 + 1)\psi_k^2 = e^{-2\lambda_1 T}\|\psi\|_{W_2^1(Q)}^2.$$

Setting $\omega = \lambda_1$, we complete the proof.

Remark 2.1 *The solution $v(x,t)$ from Lemma 2.2 is given by $v(\cdot,t) = \mathbf{S}_t\psi(\cdot)$ (cf. (2.5)). On the other hand, the spaces $W_2^1(Q)$ and $\mathring{W}_2^1(Q)$ are both dense in $L_2(Q)$. Therefore, due to the first inequality in (2.7), we have*

$$\|\mathbf{S}_t\psi\|_{L_2(Q)} \le e^{-\omega t}\|\psi\|_{L_2(Q)} \quad \forall t \ge 0,\ \psi \in L_2(Q). \tag{2.9}$$

Lemmas 2.1 and 2.2 imply the following result.

Corollary 2.1 *For any*

$$f \in L_2(Q_T), \qquad \psi \in \begin{cases} \mathring{W}_2^1(Q) & \text{if } \gamma = 0, \\ W_2^1(Q) & \text{if } \gamma > 0, \end{cases}$$

there exists a unique solution v of problem (2.1)–(2.3) in Q_T and

$$\|v(\cdot,T)\|_{W_2^1(Q)} \le c_1\|f\|_{L_2(Q_T)} + e^{-\omega T}\|\psi\|_{W_2^1(Q)}, \tag{2.10}$$

where $c_1 > 0$ is the same as in Lemma 2.1 and $\omega > 0$ is the same as in Lemma 2.2.

The next lemma allows one to estimate the norm $\|v(\cdot,T)\|_{W_2^2(Q)}$ of the solution of problem (2.1)–(2.3) in Q_T, provided that the right-hand side f is Hölder continuous in t in a neighborhood of $t = T$.

Lemma 2.3 *Let*

$$f \in L_2(Q_T), \qquad \psi \in \begin{cases} \mathring{W}_2^1(Q) & \text{if } \gamma = 0, \\ W_2^1(Q) & \text{if } \gamma > 0. \end{cases}$$

Suppose there are numbers $T_0 \in [0,T)$, $L > 0$, and $\sigma \in (0,1]$ such that

$$\begin{aligned} \|f(\cdot,t)\|_{L_2(Q)} &\le L, & \forall t \in [T_0,T], \\ \|f(\cdot,t_2) - f(\cdot,t_1)\|_{L_2(Q)} &\le L|t_2 - t_1|^\sigma & \forall t_1,t_2 \in [T_0,T]. \end{aligned}$$

Then the solution v of problem (2.1)–(2.3) in Q_T satisfies the inequality

$$\|v(\cdot,T)\|_{W_2^2(Q)} \le c_2(\|f\|_{L_2(Q_{T_0})} + \|\psi\|_{L_2(Q)} + L),$$

where $c_2 = c_2(T,T_0) > 0$ does not depend on f, ψ, and L and is bounded on the set $\{T \in [T_1,T_2],\ T_0 \in [0,T-\varepsilon]\}$ for any $0 < \varepsilon < T_1 < T_2$.

PROOF. It follows from the uniqueness of the solution (see Lemma 2.1) that the function $v(x,t+T_0)$ is a solution of problem (2.1)–(2.3) in Q_{T-T_0} with $f(x,t)$ and $\psi(x)$ replaced by $f(x,t+T_0)$ and $v(x,T_0)$, respectively. Therefore, using Theorem 3.2 in [18, Chap. 4], we obtain

$$\|\mathbf{P}v(\cdot,T)\|_{L_2(Q)} \le k_1(L + \|v(\cdot,T_0)\|_{L_2(Q)}),$$

where $k_1, k_2, \ldots > 0$ do not depend on f, ψ, v.

On the other hand, it is well known that the operator $\mathbf{P}$ has a bounded inverse $\mathbf{P}^{-1}$, which is also bounded as the operator acting from $L_2(Q)$ into $W_2^2(Q)$. Hence,

$$\|v(\cdot,T)\|_{W_2^2(Q)} \le k_2\|\mathbf{P}v(\cdot,T)\|_{L_2(Q)} \le k_3(L + \|v(\cdot,T_0)\|_{L_2(Q)}). \tag{2.11}$$

Finally, using representation (2.5) and estimate (2.9), we have

$$\|v(\cdot,T_0)\|_{L_2(Q)} \le k_4(\|\psi\|_{L_2(Q)} + \|f\|_{L_2(Q_{T_0})}). \tag{2.12}$$

Combining (2.11) and (2.12), we complete the proof.

For any functions $\psi(x)$ and $v(x,t)$, we denote

$$\psi_m = \int_Q m(x)\psi(x)\,dx, \qquad v_m(t) = \int_Q m(x)v(x,t)\,dx \quad (t \ge 0),$$

where $m \in L_\infty(Q)$ is a given function.

Lemma 2.4 *Let*

$$f \in L_2(Q_T), \qquad \psi \in \begin{cases} \mathring{W}_2^1(Q) & \text{if } \gamma = 0, \\ W_2^1(Q) & \text{if } \gamma > 0, \end{cases}.$$

Let v be a solution of problem (2.1)–(2.3) *in Q_T. Then*

$$\|v(\cdot,t_2) - v(\cdot,t_1)\|_{L_2(Q)} \le c_1(\|f\|_{L_2(Q_T)} + \|\psi\|_{W_2^1(Q)})(t_2 - t_1)^{1/2}, \tag{2.13}$$

$$|v_m(t_2) - v_m(t_1)| \le c_3(\|f\|_{L_2(Q_T)} + \|\psi\|_{W_2^1(Q)})(t_2 - t_1)^{1/2} \tag{2.14}$$

for all $0 \le t_1 < t_2 \le T$, where $c_1 = c_1(T) > 0$ is the constant occurring in Lemma 2.1 *and $c_3 = c_3(T) > 0$ does not depend on f, ψ, t_1, t_2 and is bounded on any segment $[T_1, T_2]$ $(0 < T_1 < T_2)$.*

PROOF. Using the Schwartz inequality and Lemma 2.1, we obtain (2.13):

$$\begin{aligned}\|v(\cdot,t_2) - v(\cdot,t_1)\|_{L_2(Q)}^2 &= \int_Q dx \left|\int_{t_1}^{t_2} v_t(x,t)\,dt\right|^2 \le \|v_t\|_{L_2(Q_T)}^2(t_2 - t_1) \\ &\le \|v\|_{W_2^{2,1}(Q_T)}^2(t_2 - t_1) \le c_1^2(\|f\|_{L_2(Q_T)} + \|\psi\|_{W_2^1(Q)})^2(t_2 - t_1).\end{aligned}$$

Inequality (2.14) follows by applying the Schwartz inequality.

3 The Preisach Hysteresis Operator

3.1 Preisach operator for continuous functions

In this section, we introduce the Preisach hysteresis operator having been thoroughly investigated by Preisach, Brokate, Krasnoselskii, Pokrovskii, Visintin, and others (see, e.g., [13, 23] and references therein).

We denote by $BV(t_0, t_1)$, $t_0 < t_1$, the Banach space of real-valued functions having finite total variation on the segment $[t_0, t_1]$ and by $C_r[t_0, t_1)$ the linear space of functions which are continuous on the right in $[t_0, t_1)$. For any couple $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ such that $\rho_1 < \rho_2$, we introduce the *delayed relay operator*

$$h_\rho : C[t_0, t_1] \times \{0, 1\} \to BV(t_0, t_1) \cap C_r[t_0, t_1)$$

by the following rule. For any $r \in C[t_0, t_1]$ and $\chi = 0$ or 1, the function $z = h_\rho(r, \chi, t_0) : [t_0, t_1] \to \{0, 1\}$ is defined as follows. Let $X_t = \{t' \in (t_0, t] : r(t') = \rho_1 \text{ or } \rho_2\}$. Then we set

$$z(t_0) = \begin{cases} 1 & \text{if } r(t_0) \le \rho_1, \\ \chi & \text{if } \rho_1 < r(t_0) < \rho_2, \\ 0 & \text{if } r(t_0) \ge \rho_2 \end{cases}$$

and for $t \in (t_0, t_1]$

$$z(t) = \begin{cases} z(t_0) & \text{if } X_t = \varnothing, \\ 1 & \text{if } X_t \ne \varnothing \text{ and } r(\max X_t) = \rho_1, \\ 0 & \text{if } X_t \ne \varnothing \text{ and } r(\max X_t) = \rho_2. \end{cases}$$

We will say that χ is the *initial configuration of the delayed relay operator* h_ρ.

Thus, the function $h_\rho(r, \chi, t_0)(t)$ equals 1 if $r(t) \le \rho_1$, equals 0 if $r(t) \ge \rho_2$, and equals either 1 or 0 if $r(t) \in (\rho_1, \rho_2)$, depending on the value of r at the "previous" moment (Fig. 3.1).

The following properties of the delayed relay operator are stated in Proposition 1.1 in [23, Chap. 4].

Lemma 3.1 *For any couple* $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ *such that* $\rho_1 < \rho_2$*, the following assertions hold*:

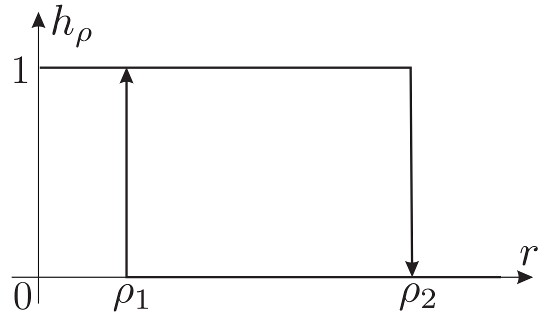


Fig. 3.1. The delayed relay operator h_ρ, $\rho = (\rho_1, \rho_2)$

(1) *Semigroup property: if* $t_0 \le \tau_0 < \tau_1 \le t_1$, *then*

$$[h_\rho(r, \chi, t_0)](\tau_1) = [h_\rho(r, z(\tau_0), \tau_0)](\tau_1),$$

where $z(\tau_0) = [h_\rho(r, \chi, t_0)](\tau_0)$.

(2) *Monotonicity with respect to* r: *if* $r_1(t) \ge r_2(t)$ *for* $t \in [t_0, t_1]$, *then*

$$[h_\rho(r_1, \chi, t_0)](t) \le [h_\rho(r_2, \chi, t_0)](t).$$

(3) *The function* $[h_\rho(r, \chi, t_0)](t)$ *is Borel measurable with respect to* ρ.

Set

$$\mathcal{P} = \{\rho = (\rho_1, \rho_2) \in \mathbb{R}^2 : \rho_1 < \rho_2\}.$$

Denote by $\mathcal{R}$ the set of Borel measurable functions $\mathcal{P} \to \{0, 1\}$ and by ξ_ρ (or simply ξ) a generic element of $\mathcal{R}$, which is called a *relay configuration*. Let μ be a fixed *finite nonnegative Borel measure* over $\mathcal{P}$. Without loss of generality, we assume that

$$\mu(\mathcal{P}) = 1.$$

We will consider $\mathcal{R}$ as a metric space with the distance

$$d(\xi_1, \xi_2) = \int_{\mathcal{P}} |\xi_{1\rho} - \xi_{2\rho}| \, d\mu(\rho) \qquad \forall \xi_1, \xi_2 \in \mathcal{R}.$$

We introduce the *Preisach operator*

$$\mathcal{H} : C[t_0, t_1] \times \mathcal{R} \to L_\infty(t_0, t_1) \cap C_r[t_0, t_1),$$
$$[\mathcal{H}(r, \xi, t_0)](t) = \int_{\mathcal{P}} [h_\rho(r, \xi_\rho, t_0)](t) d\mu(\rho), \qquad t \in [t_0, t_1].$$

In this context, we will say that ξ is the *initial configuration of the Preisach operator* $\mathcal{H}$.

If the measure μ is supported at finitely many points $\rho^{(i)} = (\rho_1^{(i)}, \rho_2^{(i)})$, then the operator $\mathcal{H}(r, \xi, t_0)$ is a linear combination of finitely many discontinuous delayed relay operators $h_{\rho^{(i)}}(r, \xi_{\rho^{(i)}}, t_0)$.

For the operator $\mathcal{H}$ to be continuous, we have to take a "linear combination" of infinitely many delayed relay operators (see Condition 3.1 and Lemma 3.3 below).

The physical interpretation of the continuous Preisach operator in terms of thermocontrol processes is as follows. The value of $\mathcal{H}(r,\xi,t_0)(t)$ corresponds to the power of the heating (cooling) elements on the boundary of a domain, depending on some averaged temperature $r(t)$ of the domain. The value 1 corresponds to the most powerful heating regime and the value 0 to the most powerful cooling regime.

Fix some numbers $\rho_1^* < \rho_2^*$ and $\delta > 0$ such that

$$\rho_1^* + \delta < \rho_2^* - \delta.$$

Let the support of the measure μ be a subset of the set

$$\mathcal{P}^* = \mathcal{P} \cap [\rho_1^* - \delta, \rho_1^* + \delta] \times [\rho_2^* - \delta, \rho_2^* + \delta].$$

If the value of r is less than $\rho_1^* - \delta$, then the value of $\mathcal{H}$ equals 1 (maximum heating), if the value of r is greater than $\rho_2^* + \delta$, then the value of $\mathcal{H}$ equals 0 (maximum cooling). If r is between $\rho_1^* - \delta$ and $\rho_1^* + \delta$ and decreases then $\mathcal{H}$ increases (*gradual* increase of heater's power). If r is between $\rho_1^* - \delta$ and $\rho_1^* + \delta$ and increases then $\mathcal{H}$ does not change. Similarly for r between $\rho_2^* - \delta$ and $\rho_2^* + \delta$ (one should swap "increase" and "decrease"). The dependence of $\mathcal{H}$ on r is schematically depicted in Fig. 3.2.

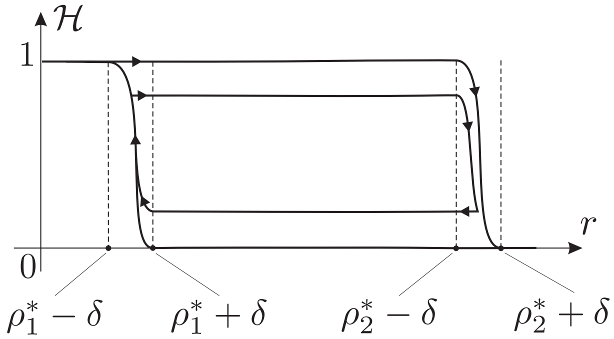


Fig. 3.2. The continuous Preisach operator $\mathcal{H}$

The following properties of the Preisach operator result from the analogous properties of the delayed relay operator h_ρ (see Lemma 3.1).

Lemma 3.2 *(1) Semigroup property: if $t_0 \le \tau_0 < \tau_1 \le t_1$, then*

$$[\mathcal{H}(r,\xi,t_0)](\tau_1) = [\mathcal{H}(r,z_\rho(\tau_0),\tau_0)](\tau_1),$$

where $z_\rho(\tau_0) = [h_\rho(r,\xi_\rho,t_0)](\tau_0)$.

(2) Monotonicity with respect to r: if $r_1(t) \ge r_2(t)$ for $t \in [t_0, t_1]$, then

$$[\mathcal{H}(r_1, \xi, t_0)](t) \le [\mathcal{H}(r_2, \xi, t_0)](t).$$

In what follows, we need continuity properties of the Preisach operator. To ensure their validity, we impose a restriction on the measure μ. Denote by Ψ the class of functions $\psi(\sigma)$ $(\sigma \ge 0)$ that are Lipschitz continuous with Lipschitz constant 1.

Condition 3.1 *For any $\psi \in \Psi$, the μ-measure of the curve $\rho_1 + \rho_2 = \psi(\rho_2 - \rho_1)$, $\rho \in \mathcal{P}$, equals zero.*

Lemma 3.3 (see Sec. 38.6 in [13]) *Condition 3.1 holds if and only if the operator $\mathcal{H} : C[t_0, t_1] \times \mathcal{R} \to C[t_0, t_1]$ is uniformly continuous, i.e., there is a nonnegative function $c(\varepsilon)$ (which does not depend on t_0 and t_1) such that $c(\varepsilon) \to 0$ as $\varepsilon \to 0$ and*

$$\|\mathcal{H}(r_1, \xi_1, t_0) - \mathcal{H}(r_2, \xi_2, t_0)\|_{C[t_0,t_1]} \le c(\varepsilon)$$

whenever

$$\|r_1 - r_2\|_{C[t_0,t_1]} + \int_{\mathcal{P}} |\xi_{1\rho} - \xi_{2\rho}|\, d\mu(\rho) \le \varepsilon.$$

Denote

$$R_i(\lambda_1, \lambda_2) = \{\rho \in \mathcal{P} : \lambda_1 \le \rho_i \le \lambda_2\}, \qquad \lambda_1 \le \lambda_2,\ i = 1, 2,$$

$$k(\lambda) = \sup_{\substack{0 \le \lambda_2 - \lambda_1 \le \lambda, \\ i=1,2}} 2\mu(R_i(\lambda_1, \lambda_2)), \qquad \lambda > 0.$$

Condition 3.2 *There is a constant $C > 0$ such that $k(\lambda) \le C\lambda$ for $\lambda > 0$.*

Lemma 3.4 (see Theorem 3.9 in Chap. 3 in [23]) *Let Conditions 3.1 and 3.2 hold. Suppose that $\xi \in \mathcal{R}$, $r \in C[t_0, t_1]$, and*

$$\|r\|_{1/2} = \sup_{s_0, s_1 \in [t_0,t_1],\, s_0 \ne s_1} \frac{|r(s_1) - r(s_0)|}{|s_1 - s_0|^{1/2}} < \infty.$$

Then

$$|\mathcal{H}(r, \xi, t_0)(s_1) - \mathcal{H}(r, \xi, t_0)(s_0)| \le C\|r\|_{1/2}(s_1 - s_0)^{1/2},$$

where C is the same as in Condition 3.2 and $t_0 \le s_0 \le s_1 \le t_1$.

Now we define the Preisach operator for piecewise continuous functions. Fix some points

$$t_0 < t_1 < t_2.$$

Denote by $\tilde{C}[t_0, t_2]$ the Banach space of functions $r(t)$ continuous on the right and such that their restrictions on the intervals (t_{j-1}, t_j), $j = 1, 2$ (which we denote by $r_j(t)$), belong to $C[t_{j-1}, t_j]$. The norm in $\tilde{C}[t_0, t_2]$ is given by

$$\|r\|_{\tilde{C}[t_0,t_2]} = \max_{j=1,2} \|r_j\|_{C[t_{j-1},t_j]}.$$

We define the operators

$$h_\rho : \tilde{C}[t_0, t_2] \times \{0, 1\} \to BV(t_0, t_2) \cap C_r[t_0, t_2),$$

$$\mathcal{H} : \tilde{C}[t_0, t_2] \times \mathcal{R} \to L_\infty(t_0, t_1) \cap C_r[t_0, t_2)$$

by using the semigroup property:

$$[h_\rho(r, \xi_\rho, t_0)](t) = \begin{cases} [h_\rho(r_1, \xi_\rho, t_0)](t), & t \in [t_0, t_1), \\ [h_\rho(r_2, z_\rho(t_1), t_1)](t), & t \in [t_1, t_2]; \end{cases}$$

$$[\mathcal{H}(r, \xi, t_0)](t) = \begin{cases} [\mathcal{H}(r_1, \xi, t_0)](t), & t \in [t_0, t_1), \\ [\mathcal{H}(r_2, z(t_1), t_1)](t), & t \in [t_1, t_2], \end{cases}$$

where $z_\rho(t_1) = [h_\rho(r_1, \xi_\rho, t_0)](t_1)$.

Lemma 3.5 *Let Condition* 3.1 *hold. Then the operator $\mathcal{H}$ maps $\tilde{C}[t_0, t_2] \times \mathcal{R}$ to $\tilde{C}[t_0, t_2]$. Moreover, for any fixed $\xi \in \mathcal{R}$, the operator $\mathcal{H} : \tilde{C}[t_0, t_2] \to \tilde{C}[t_0, t_2]$ is continuous* (*uniformly with respect to t_0, t_1, and t_2*).

PROOF. We fix $\xi \in \mathcal{R}$. Due to Lemma 3.3, $\mathcal{H}(r, \xi, t_0) \in C[t_{j-1}, t_j]$, $j = 1, 2$. This implies that $\mathcal{H}(r, \xi, t_0) \in \tilde{C}[t_0, t_2]$.

Suppose that $r, q \in \tilde{C}[t_0, t_2]$ and $\|r - q\|_{\tilde{C}[t_0,t_1]} \le \varepsilon$, where $\varepsilon > 0$.

By Lemma 3.3, we have

$$|H(r, \xi, t_0)(t) - H(q, \xi, t_0)(t)| \le c(\varepsilon), \quad t \in [t_0, t_1). \tag{3.1}$$

Denote

$$z_\rho^r(t_1) = [h_\rho(r_1, \xi_\rho, t_0)](t_1), \quad z_\rho^q(t_1) = [h_\rho(q_1, \xi_\rho, t_0)](t_1).$$

Using the monotonicity property of the operator h_ρ (see Lemma 3.1) and Lemma 3.3, we obtain

$$
\begin{aligned}
&\int_{\mathcal{P}} |z_\rho^r(t_1) - z_\rho^q(t_1)|\, d\mu(\rho) \\
&\qquad \le \int_{\mathcal{P}} [h_\rho(r_1 - \varepsilon, \xi_\rho, t_0)](t_1)\, d\mu(\rho) - \int_{\mathcal{P}} [h_\rho(r_1 + \varepsilon, \xi_\rho, t_0)](t_1)\, d\mu(\rho) \\
&\qquad = H(r_1 - \varepsilon, \xi, t_0)(t_1) - H(r_1 + \varepsilon, \xi, t_0)(t_1) \le c(2\varepsilon).
\end{aligned}
\tag{3.2}
$$

It follows from inequalities (3.2) and Lemma 3.3 that

$$
\begin{aligned}
&|\mathcal{H}(r, \xi, t_0)(t) - \mathcal{H}(q, \xi, t_0)(t)| \\
&\qquad = |\mathcal{H}(r_2, z_\rho^r(t_1), t_1)(t) - \mathcal{H}(q_2, z_\rho^q(t_1), t_1)(t)| \le c(\varepsilon + c(2\varepsilon)), \quad t \in [t_1, t_2].
\end{aligned}
\tag{3.3}
$$

Combining (3.1) and (3.3), we complete the proof.

4 Thermocontrol Problems with Time Delay: Existence and Uniqueness of Solutions

4.1 Setting of the problem

Let $w(x,t)$ be the temperature at the point $x \in Q$ at the moment $t \ge 0$ satisfying the heat equation

$$
w_t(x,t) = Pw(x,t) + F(x,t,w(x,t),u(t)) \quad ((x,t) \in Q_T), \tag{4.1}
$$

where $F(x,t,w,u)$ and the control function $u(t)$ are specified below.

The initial condition is given by

$$
w(x,0) = \varphi(x) \quad (x \in Q). \tag{4.2}
$$

The boundary condition also contains the control function $u(t)$ which regulates the temperature on the boundary, the heat flux through the boundary, or the ambient temperature:

$$
-\gamma \frac{\partial w}{\partial \nu} = \sigma(x) w(x,t) + k_0(x) u(t) + k_1(x) \quad ((x,t) \in \Gamma_T), \tag{4.3}
$$

where γ and σ are the same as above, and $k_0, k_1 \in C^\infty(\mathbb{R}^n)$ are real-valued functions.

As before, for any functions $\psi(x)$ and $v(x,t)$, we denote

$$\psi_m = \int_Q m(x)\psi(x)\,dx, \qquad v_m(t) = \int_Q m(x)v(x,t)\,dx \quad (t \ge 0),$$

where $m \in L_\infty(Q)$ is a given function, $m(x) \not\equiv 0$.

To define the control function $u(t)$, we fix an arbitrary $\xi \in \mathcal{R}$ and introduce the Preisach operator $\mathcal{H} : C[0,T] \to L_\infty(0,T) \cap C_r[0,T)$ given by

$$\mathcal{H}(r)(t) = \mathcal{H}(r,\xi,0)(t), \qquad r \in C[0,T],\ t \in [0,T].$$

We assume that the control function $u(t)$ satisfies the following Cauchy problem:

$$au'(t) + u(t) = \mathcal{H}(w_m(\cdot - \tau))(t) \quad (t \in (0,T)), \tag{4.4}$$
$$u(0) = u_0, \tag{4.5}$$
$$w_m(t) = g(t) \quad (t \in [-\tau, 0)), \tag{4.6}$$

where $a > 0$, $u_0 \in \mathbb{R}$, $g \in C[-\tau, 0]$, and w is the function satisfying relations (4.1)–(4.3).

We assume that the consistency condition

$$g(0) = \varphi_m \tag{4.7}$$

holds for the initial data $g(t)$ and $\varphi(x)$ (which will ensure the continuity of the "mean" temperature at $t = 0$).

Further, we assume that

$$\begin{cases} F(\cdot,t,\psi(\cdot),u) \in L_2(Q) \qquad \forall t \in [0,T],\ \psi \in L_2(Q),\ u \in \mathbb{R}, \\ \|F(\cdot,t_2,\psi_2(\cdot),u_2,) - F(\cdot,t_1,\psi_1(\cdot),u_1)\|_{L_2(Q)} \\ \quad \le L\left(|t_2-t_1|^{1/2} + \|\psi_2-\psi_1\|_{L_2(Q)} + |u_2-u_1|\right) \\ \quad\quad \forall t \in [0,T],\ \psi_j \in L_2(Q),\ u_j \in \mathbb{R},\ j = 1,2, \end{cases} \tag{4.8}$$

$$\|F(\cdot,t,\psi(\cdot),u)\|_{L_2(Q)} \le \hat{F}(u) \tag{4.9}$$
$$\forall t \in [0,T],\ \psi \in L_2(Q),\ u \in \mathbb{R},$$

where $L > 0$ does not depend on the arguments of F and $\hat{F}(u)$ is bounded on bounded intervals.

Definition 4.1 *A pair of functions (w,u) is called a* (strong) solution *of problem* (4.1)–(4.6) *(in Q_T with the initial configuration $\xi \in \mathcal{R}$) if $w \in W_2^{2,1}(Q_T) \cap C([0,T], W_2^1(Q))$ satisfies Eq.* (4.1) *a.e. in Q_T and conditions* (4.2), (4.3) *in the sense of traces and $u \in C^1[0,T]$ satisfies Eq.* (4.4) *in $(0,T)$ and condition* (4.5), *whereas the function $w_m(t)$ for $t \in [-\tau, 0)$ is given by* (4.6).

In this subsection, we prove the existence and uniqueness of the solution for problem (4.1)–(4.6).

First, we reduce the problem to that with the homogeneous boundary condition and prove the existence and uniqueness of the so-called mild solution (see Definition 4.2 below).

Consider a function $U \in W_2^2(Q)$ such that

$$-\gamma \frac{\partial U}{\partial \nu} = \sigma(x)U(x) + 1 \quad (x \in \Gamma). \tag{4.10}$$

We assume that

$$U(x) = 0 \quad (x \in \Gamma) \qquad \text{if } \gamma > 0. \tag{4.11}$$

The existence of such a function U follows from Lemma 2.2 in [16, Chap. 2].

Set

$$v_0(x,t) = [k_0(x)u(t) + k_1(x)]U(x). \tag{4.12}$$

It follows from (4.1)–(4.3), (4.10), and (4.11) that the function $v = w - v_0$ satisfies the relations

$$v_t(x,t) = Pv(x,t) + f(x,t,v) \quad ((x,t) \in Q_T), \tag{4.13}$$

$$v(x,0) = \varphi(x) + \varphi_0(x) \quad (x \in Q), \tag{4.14}$$

$$\gamma \frac{\partial v}{\partial \nu} + \sigma(x)v(x,t) = 0 \quad ((x,t) \in \Gamma_T), \tag{4.15}$$

where

$$\begin{gathered} f(x,t,v) = P[k_0(x)U(x)]u(t) + P[k_1(x)U(x)] - k_0(x)U(x)u'(t) + F(x,t,v+v_0(x,t),u(t)), \\ \varphi_0(x) = -(k_0(x)u_0 + k_1(x))U(x). \end{gathered} \tag{4.16}$$

We introduce the analytic semigroup of contraction $\mathbf{T}_t : L_2(Q)\times\mathbb{R} \to L_2(Q)\times\mathbb{R}$, $t \ge 0$, defined as follows: for any $(\psi_0, u_0) \in L_2(Q)\times\mathbb{R}$

$$\mathbf{T}_t(\psi_0, u_0) = (\mathbf{S}_t\psi_0, e^{-t/a}u_0). \tag{4.17}$$

Definition 4.2 *A pair of functions $w \in C([0,T], L_2(Q))$, $u \in C^1[0,T]$ is called a* mild solution *of problem* (4.1)–(4.6) *(in Q_T with the initial configuration $\xi \in \mathcal{R}$) if $w = v_0 + v$, where v_0 is given by* (4.12),

$$(v(\cdot,t), u(t)) = \mathbf{T}_t(\varphi+\varphi_0, u_0) + \int_0^t \mathbf{T}_{t-s}\Big(f(\cdot,s,v(\cdot,s)), a^{-1}\mathcal{H}(w_m(\cdot-\tau))(s)\Big)\,ds, \tag{4.18}$$

and $w_m(t) = g(t)$ for $t \in [-\tau, 0)$; here f and φ_0 are given by (4.16).

Theorem 4.1 *Let Condition* 3.1 *hold. Suppose that* F *satisfies conditions* (4.8). *Then, for any initial data* $(\varphi, u_0, g) \in L_2(Q) \times \mathbb{R} \times C[-\tau, 0]$ *such that the consistency condition* (4.7) *holds, there exists a unique mild solution* (w, u) *of problem* (4.1)–(4.6) *in* Q_T.

PROOF. I. Let $t \in [0, \tau]$. Then $u(t)$ is given by

$$u(t) = u_0 e^{-t/a} + a^{-1} \int_0^t e^{-(t-s)/a} \mathcal{H}(g(\cdot - \tau))(s)\, ds \quad (t \geq 0) \tag{4.19}$$

Since $g(\cdot - \tau) \in C[0, \tau]$, we have $\mathcal{H}(g(\cdot - \tau)) \in C[0, \tau]$. Thus, it follows from (4.19) that $u \in C^1[0, \tau]$. Hence, using (4.8), we see that $f(\cdot, t, \psi(\cdot))$ as a function from $[0, \tau] \times L_2(Q)$ into $L_2(Q)$, is continuous in t and uniformly Lipschitz in ψ. Therefore, applying Theorem 1.2 in [18, Chap. 6], we obtain that there exists a unique mild solution of problem (4.1)–(4.6) in Q_τ.

II. Let $t \in [0, 2\tau]$. Then we can represent $u(t)$ as follows:

$$u(t) = u_0 e^{-t/a} + a^{-1} \int_0^t e^{-(t-s)/a} \mathcal{H}(w_m(\cdot - \tau))(s)\, ds \quad (t \in [0, 2\tau]),$$

where $w_m(t) = g(t)$ for $t \in [-\tau, 0)$ and $w_m(t)$ is uniquely defined for $t \in [0, \tau]$ in part I of the proof. It follows from the consistency condition (4.7) that $w_m(\cdot - \tau) \in C[0, 2\tau]$. Therefore, $\mathcal{H}(w_m(\cdot - \tau)) \in C[0, 2\tau]$, and, similarly to the above, we see that there exists a unique mild solution of problem (4.1)–(4.6) in $Q_{2\tau}$.

Repeating the above procedure finitely many times, we prove that there exists a unique mild solution of problem (4.1)–(4.6) in Q_T.

Now we introduce the set of initial data for which strong solutions of problem (4.1)–(4.6) exist. Let

$$\mathcal{V}_\tau = \Big\{(\varphi, u_0, g) \in W_2^1(Q) \times \mathbb{R} \times C[-\tau, 0] : \text{consistency condition (4.7) holds}\Big\}$$

if $\gamma > 0$ and

$$\begin{aligned} \mathcal{V}_\tau = \Big\{&(\varphi, u_0, g) \in W_2^1(Q) \times \mathbb{R} \times C[-\tau, 0] : \\ &\sigma(x)\varphi(x) + k_0(x)u_0 + k_1(x) = 0 \ (x \in \Gamma) \\ &\text{and consistency condition (4.7) holds}\Big\} \end{aligned}$$

if $\gamma = 0$.

Thus, if $(\varphi, u_0, g) \in \mathcal{V}_\tau$ and $\gamma = 0$, then $\varphi + \varphi_0 \in \mathring{W}_2^1(Q)$ (cf. (4.14)).

Theorem 4.2 *(1) Let Condition* 3.1 *hold. Suppose that* F *satisfies conditions* (4.8) *and* (4.9) *and* $(\varphi, u_0, g) \in \mathcal{V}_\tau$. *Then there exists a unique solution* (w, u) *of problem* (4.1)–(4.6) *in* Q_T *and*

$$\|w(\cdot,T)\|_{W_2^1(Q)} \le c_4 A(u_0) + e^{-\omega T}\|\varphi\|_{W_2^1(Q)}, \tag{4.20}$$

$$\|w\|_{W_2^{2,1}(Q_T)} + \|w\|_{C([0,T],W_2^1(Q))} \le c_5 B(\varphi, u_0), \tag{4.21}$$

$$\|u\|_{C[0,T]} \le \max(1, |u_0|), \tag{4.22}$$

$$|w_m(t_2) - w_m(t_1)| \le c_6 B(\varphi, u_0)|t_2 - t_1|^{1/2} \qquad \forall t_1, t_2 \in [0,T], \tag{4.23}$$

where

$$A(u_0) = \left[\max\left(1, 2a^{-1}, |u_0|, a^{-1}(1+|u_0|)\right) + \max_{|u_1|\le\max(1,|u_0|)} \hat{F}(u_1)\right], \tag{4.24}$$

$$B(\varphi, u_0) = A(u_0) + \|\varphi\|_{W_2^1(Q)}. \tag{4.25}$$

(2) If we additionally assume that Condition 3.2 *holds and* $T > \tau$, *then*

$$\|w(\cdot,T)\|_{W_2^2(Q)} \le c_7 B(\varphi, u_0). \tag{4.26}$$

Here $c_4, \dots, c_7 > 0$ *depend on* T, *do not depend on* φ, u_0, g, ξ, *and are bounded on any segment* $[T_1, T_2]$ $(0 < T_1 < T_2)$.

PROOF. I. Due to Theorem 4.1, there exists a unique mild solution (w, u) of problem (4.1)–(4.6) in Q_T. Since $u \in C^1[0,T]$, the function v_0 given by (4.12) belongs to $W_2^{2,1}(Q_T) \cap C([0,T], W_2^1(Q))$.

Using (4.8), we see that the function $f(x,t,v(x,t))$ given by (4.16) belongs to $L_2(Q_T)$. Therefore, due to Lemma 2.1, problem (4.13)–(4.15) has a unique solution $v \in W_2^{2,1}(Q_T) \cap C([0,T], W_2^1(Q))$ and the pair (v, u) satisfies (4.18). Thus, the mild solution (w, u) is also a (strong) solution of problem (4.1)–(4.6).

II. Since

$$0 \le \mathcal{H}(r) \le 1 \quad \forall r \in C[0,T] \tag{4.27}$$

due to the assumption $\mu(\mathcal{P}) = 1$, it follows from (4.27) and from the representation

$$u(t) = u_0 e^{-t/a} + a^{-1}\int_0^t e^{-(t-s)/a}\mathcal{H}(w_m(\cdot - \tau))(s)\, ds \quad (t \ge 0) \tag{4.28}$$

that

$$|u(t)| \le \max(1, |u_0|), \quad |u'(t)| \le a^{-1}(1+|u(t)|) \le a^{-1}\max(2, 1+|u_0|) \quad \forall t \ge 0, \tag{4.29}$$

which, in particular, yields (4.22).

It follows from relations (4.12) and (4.16), Corollary 2.1, and inequalities (4.29) that

$$\begin{aligned}
\|w(\cdot,T)\|_{W_2^1(Q)} &\le \|v_0(\cdot,T)\|_{W_2^1(Q)} + \|v(\cdot,T)\|_{W_2^1(Q)} \\
&\le k_1 \max(1,|u_0|) + c_1\|f\|_{L_2(Q_T)} + \|\varphi_0\|_{W_2^1(Q)} + e^{-\omega T}\|\varphi\|_{W_2^1(Q)} \\
&\le c_4 A(u_0) + e^{-\omega T}\|\varphi\|_{W_2^1(Q)},
\end{aligned}$$

where $A(u_0)$ is given by (4.24) and $k_1, k_2, \ldots > 0$ do not depend on φ, u_0, g, ξ. Inequality (4.20) is proved.

Inequality (4.21) can be proved analogously by using Lemma 2.1.

III. To prove inequality (4.23), we note that, due to (4.29),

$$|u(t_2) - u(t_1)| \le A(u_0)|t_2 - t_1| \qquad \forall t_1, t_2 \ge 0. \tag{4.30}$$

It follows from (4.12) and (4.30) that

$$\|v_0(\cdot,t_2) - v_0(\cdot,t_1)\|_{L_2(Q)} + |v_{0m}(t_2) - v_{0m}(t_1)| \le k_2 A(u_0)|t_2 - t_1|^{1/2} \tag{4.31}$$

for all $t_1, t_2 \in [0,T]$.

By Lemma 2.4 and inequalities (4.29), we have

$$\begin{aligned}
&\|v(\cdot,t_2) - v(\cdot,t_1)\|_{L_2(Q)} + |v_m(t_2) - v_m(t_1)| \\
&\qquad \le k_3(\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(Q)} + \|\varphi_0\|_{W_2^1(Q)})|t_2 - t_1|^{1/2} \\
&\qquad \le k_4 B(\varphi, u_0)|t_2 - t_1|^{1/2} \qquad \forall t_1, t_2 \in [0,T],
\end{aligned} \tag{4.32}$$

where $B(\varphi, u_0)$ is given by (4.25).

Combining (4.31) and (4.32), we obtain (4.23):

$$|w_m(t_2) - w_m(t_1)| \le k_5 B(\varphi, u_0)|t_2 - t_1|^{1/2} \qquad \forall t_1, t_2 \in [0,T]. \tag{4.33}$$

IV. It remains to prove inequality (4.26). Let $T = k\tau + \tau_1$, where $k \subset \mathbb{N}$ and $0 < \tau_1 \le \tau$.

Let $t_1, t_2 \in [T - \tau_1, T]$. Then $t_j - \tau \in [T - \tau - \tau_1, T - \tau] \subset [0,T]$, $j = 1,2$. Therefore, using the continuity of the function $w_m(t - \tau)$ on $[T - \tau_1, T]$, Lemma 3.4, and estimate (4.33), we have

$$|\mathcal{H}(w_m(\cdot - \tau))(t_2) - \mathcal{H}(w_m(\cdot - \tau))(t_1)| \le Ck_5 B(\varphi, u_0)|t_2 - t_1|^{1/2} \tag{4.34}$$

for all $t_1, t_2 \in [T - \tau_1, T]$.

By virtue of (4.4), (4.30), and (4.34), we have

$$|u'(t_2) - u'(t_1)| \le k_6 B(\varphi, u_0)|t_2 - t_1|^{1/2} \quad \forall t_1, t_2 \in [T - \tau_1, T].$$

Therefore, taking also (4.8), (4.9) (4.29)–(4.33) into account, we see that

$$\begin{aligned} \|f(\cdot, t, v(\cdot, t))\|_{L_2(Q)} &\le k_7 B(\varphi, u_0), \\ \|f(\cdot, t_2, v(\cdot, t_2)) - f(\cdot, t_1, v(\cdot, t_1))\|_{L_2(Q)} &\le k_7 B(\varphi, u_0)|t_2 - t_1|^{1/2} \end{aligned} \tag{4.35}$$

for all $t, t_1, t_2 \in [T - \tau_1, T]$.

Due to (4.35), we can apply Lemma 2.3; then, using relations (4.12), (4.16), and (4.29), we obtain

$$\begin{aligned} \|w(\cdot, T)\|_{W_2^2(Q)} &\le \|v_0(\cdot, T)\|_{W_2^2(Q)} + \|v(\cdot, T)\|_{W_2^2(Q)} \\ &\le k_8(A(u_0) + \|f\|_{L_2(Q_{T-\tau_1})} + \|\varphi_0\|_{L_2(Q)} + \|\varphi\|_{L_2(Q)} + B(u_0, \varphi)) \\ &\le k_9 B(u_0, \varphi), \end{aligned}$$

which completes the proof of (4.26).

5 Periodic Solutions of Thermocontrol Problems with Time Delay

In this section, we assume that the right-hand side $F(x, t, \psi, u)$ is T-periodic in t and prove the existence of a T-periodic solution (w, u) of problem (4.1), (4.3), (4.4).

Definition 5.1 *A pair* (w, u) *is called a* T-periodic solution of problem (4.1), (4.3), (4.4) (*with an initial configuration* $\xi \in \mathcal{R}$) *if there is a triple* $(\varphi, u_0, g) \in \mathcal{V}_\tau$ *such that the following holds:*

(1) (w, u) *is a solution of problem* (4.1)–(4.6) *in* Q_T *with the initial data* (φ, u_0, g) *and the initial configuration* ξ,
(2) $u(T) = u(0)$, $w(x, T) = \varphi(x)$, $w_m(t) = w_m(t - T)$ *for* $t \in [T - \tau, T)$, *and*

$$h_\rho(w_m(\cdot - \tau), \xi_\rho, 0)(T) = \xi_\rho \quad \forall \rho \in \mathcal{P}.$$

The last equality in Definition 5.1 means that, along with the control function u and the temperature w, the configuration of the Preisach operator at the moment $t = T$ is the same as at the moment $t = 0$. Only in this case, the solution will be T-periodic for all $t > 0$.

Lemma 5.1 *Let Condition* 3.1 *hold,* F *satisfy condition* (4.8), $(\varphi, u_0, g) \in \mathcal{V}_\tau$, *and the corresponding solution* (w, u) *of problem* (4.1)–(4.6) *be* T*-periodic* ($T > 0$). *Then the function* $w_m(t)$ *is Hölder-continuous* (*with exponent* $1/2$) *on the segment* $[-\tau, T]$.

PROOF. Consider a number $l \in \mathbb{N}$ such that $lT \geq \tau$. Let $(\tilde{w}, \tilde{u}) \in \mathcal{W}(Q_{(l+1)T})$ be a solution of problem (4.1)–(4.6) in $Q_{(l+1)T}$ with the same initial data (φ, u_0, g). It follows from Theorem 4.2 and Definition 5.1 that $(\tilde{w}, \tilde{u}) = (w, u)$ in Q_T and $\tilde{w}_m(t) = \tilde{w}_m(t + lT)$ for $t \in [-\tau, T]$. Since $\tilde{w}_m(t + lT)$ is Hölder-continuous for $t \in [-\tau, T]$ (due to Theorem 4.2 and the fact that $t + lT \in [lT - \tau, (l+1)T] \subset [0, (l+1)T]$), it follows that $w_m(t) = \tilde{w}_m(t) = \tilde{w}_m(t + lT)$ is also Hölder-continuous on the segment $[-\tau, T]$.

Theorem 5.1 *Let Conditions* 3.1 *and* 3.2 *hold. Suppose that the function* $F(x, t, \psi, u)$ *is* T*-periodic in* t *with some* $T > \tau$ *and satisfies conditions* (4.8) *and* (4.9). *Then the following assertions are true.*

(1) There is an initial configuration $\xi \in \mathcal{R}$ *such that there exists a* T*-periodic solution* (w, u) *of problem* (4.1), (4.3), (4.4) *with the initial configuration* ξ.

(2) For any T*-periodic solution* (w, u) *of problem* (4.1), (4.3), (4.4), *we have* $u(t) \in [0, 1]$ $(t \geq 0)$.

The proof will be based on the Schauder fixed-point theorem. One of the main difficulties here is that the configuration of the Preisach operator at the moment T may differ from its initial configuration even if the value of the control function u and the temperature w at the moment T coincide with their values at the initial moment. To overcome this difficulty, we will introduce another hysteresis operator with a longer "pre-history," prove the existence of a periodic solution in this case, and show that it coincides with a periodic solution of the problem with the original Preisach operator.

From now on, we assume that $T > \tau$. Introduce the space (cf. Sec. 3.2)

$$\tilde{C}_\tau[-T, T] = \tilde{C}[t_0, t_2], \quad \text{where } t_0 = -T,\ t_1 = -T + \tau,\ t_2 = T.$$

Fix an arbitrary $\zeta \in \mathcal{R}$ and consider the Preisach operator $\tilde{\mathcal{H}} : \tilde{C}_\tau[-T, T] \to \tilde{C}_\tau[-T, T]$ given by

$$\tilde{\mathcal{H}}(r)(t) = \mathcal{H}(r, \zeta, -T)(t), \qquad r \in \tilde{C}_\tau[-T, T].$$

To prove Theorem 5.1, we consider an auxiliary problem, namely, we replace relations (4.4)–(4.6) by the following ones:

$$au'(t) + u(t) = \tilde{\mathcal{H}}(w_m(\cdot - \tau))(t) \ (t \in (0, T)), \tag{5.1}$$

$$u(0) = u_0, \tag{5.2}$$

$$w_m(t) = g(t) \qquad (t \in [-T - \tau, 0)); \tag{5.3}$$

here

$$g \in \tilde{C}[-T - \tau, 0] = \tilde{C}[t_0, t_2], \quad \text{where } t_0 = -T - \tau,\ t_1 = -T,\ t_2 = 0.$$

In other words, we consider the "pre-history" functions $g(t)$ on the larger interval $[-T-\tau, 0]$ and allow them to be discontinuous at $t=-T$.

As before (cf. (4.7)), we assume that

$$g(0) = \varphi_m. \tag{5.4}$$

Similarly to Definition 4.1, one can define a (*strong*) *solution* (w, u) of problem (4.1)–(4.3), (5.1)–(5.3).

Remark 5.1 *If* (w, u) *is a solution of problem* (4.1)–(4.3), (5.1)–(5.3) *and consistency condition* (5.4) *holds, then the function* $w_m(t)$ *is continuous at the point* $t = 0$ *due to condition* (5.4). *This ensures that* $w_m(\cdot - \tau) \in \tilde{C}_\tau[-T, T]$ *and the Preisach operator* $\tilde{\mathcal{H}}$ *is well defined.*

Set

$$\tilde{\mathcal{V}}_\tau = \Big\{(\varphi, u_0, g) \in W_2^1(Q) \times \mathbb{R} \times \tilde{C}[-T-\tau, 0] : \text{condition (5.4) holds}\Big\}$$

if $\gamma > 0$ and

$$\begin{aligned}\tilde{\mathcal{V}}_\tau = \Big\{&(\varphi, u_0, g) \in W_2^1(Q) \times \mathbb{R} \times \tilde{C}[-T-\tau, 0] : \\ &\sigma(x)\varphi(x) + k_0(x)u_0 + k_1(x) = 0 \ (x \in \Gamma) \text{ and condition (5.4) holds}\Big\}\end{aligned}$$

if $\gamma = 0$.

Definition 5.2 *A pair* (w, u) *is called a* T*-periodic solution of problem* (4.1), (4.3), (5.1) *(with the initial configuration* $\zeta \in \mathcal{R}$*) if there is a triple* $(\varphi, u_0, g) \in \tilde{\mathcal{V}}_\tau$ *such that the following holds:*

(1) (w, u) *is a solution of problem* (4.1)–(4.3), (5.1)–(5.3) *in* Q_T *with the initial data* (φ, u_0, g) *and the initial configuration* ζ*,*
(2) $u(T) = u(0)$*,* $w(x, T) = \varphi(x)$*,* $w_m(t) = w_m(\cdot, t-T)$ *for* $t \in [-\tau, T)$*.*

Remark 5.2 *Unlike Definition 5.1, we do not require in Definition 5.2 that*

$$h_\rho(w_m(\cdot - \tau), \zeta_\rho, -T)(T) = h_\rho(w_m(\cdot - \tau), \zeta_\rho, -T)(0) \quad \forall \rho \in \mathcal{P} \tag{5.5}$$

because this relation is automatically fulfilled for $w_m(\cdot - \tau)$ *that is* T*-periodic on* $[-T, T]$*. This is basically the main reason why we have introduced the operator* $\tilde{\mathcal{H}}$*.*

Remark 5.3 *It follows from the definition of the operator* $\tilde{\mathcal{H}}$ *that*

$$h_\rho(w_m(\cdot - \tau), \zeta_\rho, -T)(t) = h_\rho(w_m(\cdot - \tau), \xi_\rho, 0)(t), \quad t \geq 0,$$

$$\tilde{\mathcal{H}}(w_m(\cdot - \tau), \zeta, -T)(t) = \mathcal{H}(w_m(\cdot - \tau), \xi, 0)(t), \quad t \geq 0,$$

where $\xi_\rho = h_\rho(w_m(\cdot - \tau), \zeta_\rho, -T)(0)$.

Therefore, the solution of problem (4.1)–(4.3), (5.1)–(5.3) *with the initial data* (φ, u_0, g) *and the initial configuration* ζ *coincides with the solution of problem* (4.1)–(4.6) *with the initial data* $(\varphi, u_0, g|_{[-\tau,0]})$ *and the initial configuration* ξ.

Similarly, taking into account Remark 5.2, we see that a T*-periodic solution of problem* (4.1), (4.3), (5.1) *with the initial configuration* ζ *is a* T*-periodic solution of problem* (4.1), (4.3), (4.4) *with the initial configuration* ξ.

Thus, assertion 1 of Theorem 5.1 is a consequence of the following result, which will be proved in this section.

Theorem 5.2 *Let the hypothesis of Theorem* 5.1 *hold. Then, for any* $\zeta \in \mathcal{R}$, *there is a* T*-periodic solution* (w, u) *of problem* (4.1), (4.3), (5.1).

We introduce the operator $G : \tilde{\mathcal{V}}_\tau \to \tilde{\mathcal{V}}_\tau$ given by

$$G(\varphi, u_0, g) = (w(\cdot, T), u(T), r)\,, \tag{5.6}$$

where (w, u) is the solution of problem (4.1)–(4.3), (5.1)–(5.3) with the initial data $(\varphi, u_0, g) \in \tilde{\mathcal{V}}_\tau$ and the initial configuration ζ and (see Fig. 5.1)

$$r(t) = \begin{cases} w_m(t + 2T), & t \in [-T - \tau, -T), \\ w_m(t + T), & t \in [-T, 0]. \end{cases} \tag{5.7}$$

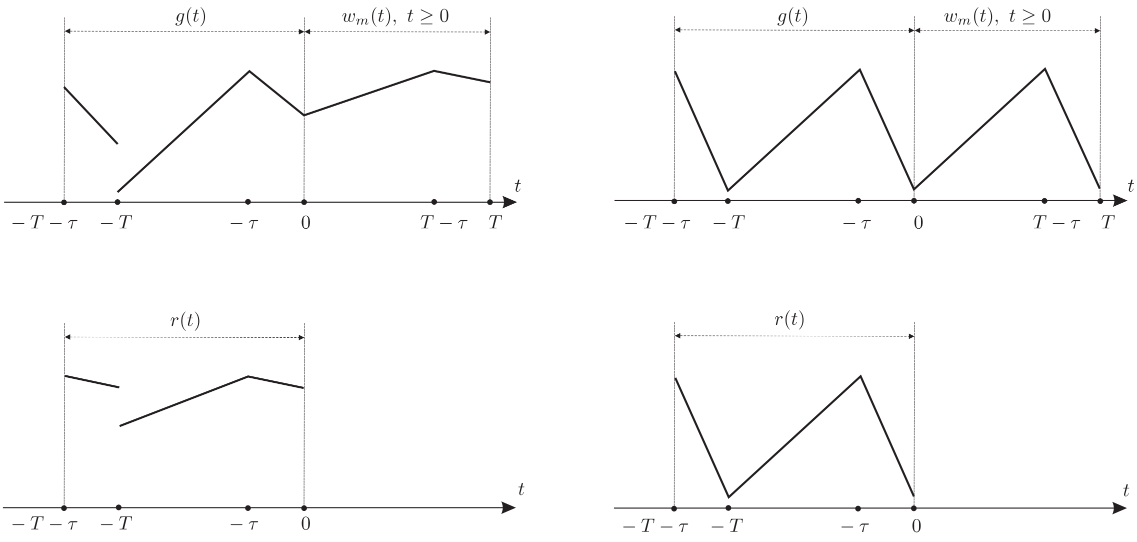
It follows from Theorem 4.2 that $w(\cdot, T) \in W_2^1(Q)$, $r \in \tilde{C}[-T - \tau, 0]$, and

$$\int_Q m(x) w(x, T)\, dx = w_m(T) = r(0).$$

Therefore, the image of the operator G indeed lies in $\tilde{\mathcal{V}}_\tau$; thus, G is well defined.

Remark 5.4 *If* $(\varphi, u_0, g) \in \tilde{\mathcal{V}}_\tau$ *is a fixed point of the operator* G, *then the function* $w_m(\cdot - \tau)$ *is continuous and* T*-periodic on* $[-T, T]$ *(Fig. 5.1.b). Hence, the corresponding solution* (w, u) *of problem* (4.1)–(4.3), (5.1)–(5.3) *is a* T*-periodic solution of problem* (4.1), (4.3), (5.1). *By Remark 5.3, it is a* T*-periodic solution of problem* (4.1), (4.3), (4.4).

Lemma 5.2 *Let the hypothesis of Theorem* 5.1 *hold. Then the operator* $G : \tilde{\mathcal{V}}_\tau \to \tilde{\mathcal{V}}_\tau$ *is continuous.*



$(a)\ g \neq r$ $\qquad$ $(b)\ g = r$

Fig. 5.1. The map $\tilde{C}[-T-\tau, 0] \ni g \mapsto r \in \tilde{C}[-T-\tau, 0]$.

PROOF. I. We fix an arbitrary $0 < \varepsilon < 1$ and consider two initial-data triples (φ, u_0, g) and $(\tilde{\varphi}, \tilde{u}_0, \tilde{g})$ from $\tilde{\mathcal{V}}_\tau$ such that

$$\|\tilde{\varphi} - \varphi\|_{W_2^1(Q)} + |\tilde{u}_0 - u_0| + \|\tilde{g} - g\|_{\tilde{C}[-T-\tau,0]} \leq \varepsilon. \tag{5.8}$$

Let (w, u) and $(\tilde{w}, \tilde{u})$ be the corresponding solutions of problem (4.1)–(4.3), (5.1)–(5.3) in Q_{T_1}, where $T_1 = N\tau \geq T$, $N \in \mathbb{N}$.

We represent the functions w and $\tilde{w}$ as follows:

$$w = v + v_0, \qquad \tilde{w} = \tilde{v} + \tilde{v}_0;$$

here v is a solution of problem (4.13)–(4.15) in Q_{T_1}, v_0 is given by (4.12), $\tilde{v}$ is a solution of the problem

$$\tilde{v}_t(x,t) = P\tilde{v}(x,t) + \tilde{f}(x,t,\tilde{v}) \quad ((x,t) \in Q_{T_1}), \tag{5.9}$$

$$\tilde{v}(x,0) = \tilde{\varphi}(x) + \tilde{\varphi}_0(x) \quad (x \in Q), \tag{5.10}$$

$$\gamma \frac{\partial \tilde{v}}{\partial \nu} + \sigma(x)\tilde{v}(x,t) = 0 \quad ((x,t) \in \Gamma_{T_1}), \tag{5.11}$$

where

$$\begin{aligned} \tilde{f}(x,t,\tilde{v}) &= P[k_0(x)U(x)]\tilde{u}(t) + P[k_1(x)U(x)] - k_0(x)U(x)\tilde{u}'(t) \\ &\quad + F(x,t,\tilde{v} + \tilde{v}_0(x,t), \tilde{u}(t)), \\ \tilde{\varphi}_0(x) &= -(k_0(x)\tilde{u}(0) + k_1(x))U(x), \end{aligned} \tag{5.12}$$

and

$$\tilde{v}_0(x,t) = [k_0(x)\tilde{u}(t) + k_1(x)]U(x).$$

II. Let $t \in [0, \tau]$. Using (4.4), (5.8), and Lemma 3.5, we obtain

$$
\begin{aligned}
&|\tilde{u}(t) - u(t)| \\
&\quad = \left| (\tilde{u}_0 - u_0)e^{-t/a} + a^{-1} \int_0^t e^{-(t-s)/a} [\tilde{\mathcal{H}}(\tilde{g}(\cdot - \tau))(s) - \tilde{\mathcal{H}}(g(\cdot - \tau))(s)]\, ds \right| \\
&\quad \le k_1 C_1(\varepsilon),
\end{aligned}
\tag{5.13}
$$

$$
|\tilde{u}'(t) - u'(t)| \le a|\tilde{u}(t) - u(t)| + |\tilde{\mathcal{H}}(\tilde{g}(\cdot - \tau))(t) - \tilde{\mathcal{H}}(g(\cdot - \tau))(t)| \le k_2 C_1(\varepsilon),
$$

where $k_1, k_2, \dots > 0$ and $C_1(\varepsilon) > 0$ do not depend on (φ, u_0, g), $(\tilde{\varphi}, \tilde{u}_0, \tilde{g})$, and t and $C_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore, taking (4.8) into account, we have

$$
\|\tilde{v}_0(\cdot, t) - v_0(\cdot, t)\|_{W_2^1(Q)} \le k_3 C_1(\varepsilon), \tag{5.14}
$$

$$
\|\tilde{f} - f\|_{L_2(Q_\tau)} + \|\tilde{\varphi}_0 - \varphi_0\|_{W_2^1(Q)} \le k_4 \left(C_1(\varepsilon) + \|\tilde{v} - v\|_{C([0,\tau], L_2(Q))} \right). \tag{5.15}
$$

Using the representation

$$
\tilde{v}(\cdot, t) - v(\cdot, t) = \mathbf{S}_t(\tilde{\varphi} + \tilde{\varphi}_0 - \varphi - \varphi_0) + \int_0^t \mathbf{S}_{t-s} \left(\tilde{f}(\cdot, s, \tilde{v}(\cdot, s)) - f(\cdot, s, v(\cdot, s)) \right) ds,
$$

the estimates

$$
\|\tilde{\varphi} - \varphi\|_{L_2(Q)} + \|\tilde{\varphi}_0 - \varphi_0\|_{L_2(Q)} \le k_5 C_1(\varepsilon),
$$

$$
\|\tilde{f}(\cdot, s) - f(\cdot, s)\|_{L_2(Q)} \le k_6 \left(C_1(\varepsilon) + \|\tilde{v}(\cdot, s) - v(\cdot, s)\|_{L_2(Q)} \right),
$$

and the Gronwall inequality, one obtains (cf. the proof of Theorem 1.2 in [18, Chap. 6])

$$
\|\tilde{v} - v\|_{C([0,\tau], L_2(Q))} \le k_7 C_1(\varepsilon).
$$

Combining this inequality with (5.15), we have

$$
\|\tilde{f} - f\|_{L_2(Q_\tau)} + \|\tilde{\varphi}_0 - \varphi_0\|_{W_2^1(Q)} \le k_8 C_1(\varepsilon). \tag{5.16}
$$

Further, due to Lemma 2.1 and inequalities (5.14), (5.15), and (5.8), we obtain

$$
\begin{aligned}
\|\tilde{w}(\cdot, t) - w(\cdot, t)\|_{W_2^1(Q)} &\le \|\tilde{v}_0(\cdot, t) - v_0(\cdot, t)\|_{W_2^1(Q)} + \|\tilde{v}(\cdot, t) - v(\cdot, t)\|_{W_2^1(Q)} \\
&\le k_3 C_1(\varepsilon) + k_9 \left(\|\tilde{f} - f\|_{L_2(Q_\tau)} + \|\tilde{\varphi}_0 - \varphi_0\|_{W_2^1(Q)} + \|\tilde{\varphi} - \varphi\|_{W_2^1(Q)} \right) \\
&\le k_{10} C_1(\varepsilon).
\end{aligned}
\tag{5.17}
$$

The latter estimate implies that

$$
\|\tilde{w}_m - w_m\|_{C[0,\tau]} \le k_{11} C_1(\varepsilon). \tag{5.18}
$$

III. Now assume that $t \in [\tau, 2\tau]$. Replacing the triples

$$
(\varphi, u_0, g), \quad (\tilde{\varphi}, \tilde{u}_0, \tilde{g})
$$

by

$$(w(\cdot,\tau),u(\tau),w_m(\cdot+\tau)),\quad (\tilde w(\cdot,\tau),\tilde u(\tau),\tilde w_m(\cdot+\tau)),$$

respectively, and using inequalities (5.13), (5.17) for $t=\tau$, and (5.18) instead of (5.8), we obtain similarly to the above that

$$\|\tilde w(\cdot,t)-w(\cdot,t)\|_{W_2^1(Q)}+|\tilde u(t)-u(t)|+\|\tilde w_m-w_m\|_{C[\tau,2\tau]}\le C_2(\varepsilon),$$

where $C_2(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Continuing this procedure, we see that if $t\in[(k-1)\tau,k\tau]$, $k=1,\dots,N$, then

$$\|\tilde w(\cdot,t)-w(\cdot,t)\|_{W_2^1(Q)}+|\tilde u(t)-u(t)|+\|\tilde w_m-w_m\|_{C[(k-1)\tau,k\tau]}\le C_k(\varepsilon),\quad (5.19)$$

where $C_k(\varepsilon)\to 0$ as $\varepsilon\to 0$. This proves the continuity of the operator $G:\tilde{\mathcal V}_\tau\to\tilde{\mathcal V}_\tau$.

Lemma 5.3 *Let the hypothesis of Theorem* 5.1 *hold. Then the operator* $G:\tilde{\mathcal V}_\tau\to\tilde{\mathcal V}_\tau$ *is compact.*

PROOF. Let $\mathcal B$ denote a bounded set in $\tilde{\mathcal V}_\tau$. Due to (4.26), we have

$$\|w(\cdot,T)\|_{W_2^2(Q)}\le k_1\qquad \forall(\varphi,u_0,g)\in\mathcal B,\qquad (5.20)$$

where $k_1=k_1(\mathcal B)>0$ does not depend on $(\varphi,u_0,g)\in\mathcal B$.

It follows from (4.23) that

$$|w_m(t_2)-w_m(t_1)|\le k_2|t_2-t_1|^{1/2}\qquad \forall(\varphi,u_0,g)\in\mathcal B,\ t_1,t_2\in[0,T],\quad (5.21)$$

where $k_2=k_2(\mathcal B)>0$ does not depend on $(\varphi,u_0,g)\in\mathcal B$.

Using inequalities (5.20) and (5.21), the uniform boundedness of the functions $w_m(t)$ on $[0,T]$ (cf. (4.21)), the Ascoli–Arzelà theorem, and the compactness of the embedding $W_2^2(Q)\subset W_2^1(Q)$, we see that the operator $G:\tilde{\mathcal V}_\tau\to\tilde{\mathcal V}_\tau$ is compact.

Now we will find a bounded closed convex set in $\tilde V_\tau$ which is mapped by the operator G into itself. Consider the set

$$\tilde{\mathcal V}_{\tau T}=\{(\varphi,u_0,g)\in\tilde{\mathcal V}_\tau:\|\varphi\|_{W_2^1(Q)}\le M_1,\ u_0\in[0,1],\ \|g\|_{\tilde C[-T-\tau,0]}\le M_2\},$$

where $M_1=M_1(T)>0$ and $M_2=M_2(T)>0$ are specified below.

Lemma 5.4 *Let the hypothesis of Theorem* 5.1 *hold. Then there exist positive numbers* $M_1=M_1(T)$ *and* $M_2=M_2(T)$ *such that the operator* G *maps* $\tilde{\mathcal V}_{\tau T}$ *into itself.*

PROOF. It follows from (4.22) that

$$u(T) \in [0,1] \qquad \forall(\varphi, u_0, g) \in \tilde{\mathcal{V}}_{\tau T}. \tag{5.22}$$

Denote

$$A_0 = \left[\max\left(1, 2a^{-1}\right) + \max_{|u_1| \le 1} \hat{F}(u_1) \right]. \tag{5.23}$$

Clearly, $A(u_0) = A_0$ for $u_0 \in [0,1]$, where $A(u_0)$ is given by (4.24). Set

$$M_1 = \frac{c_4 A_0}{1 - e^{-\omega T}},$$

where $c_4 = c_4(T)$ is the constant occurring in (4.20). Then, using (4.20), we obtain

$$\|w(\cdot, T)\|_{W_2^1(Q)} \le c_4 A_0 + e^{-\omega T} M_1 = M_1 \qquad \forall(\varphi, u_0, g) \in \tilde{\mathcal{V}}_{\tau T}. \tag{5.24}$$

It remains to choose $M_2 > 0$ such that

$$\|r\|_{\tilde{C}[-T-\tau, 0]} \le M_2 \qquad \forall(\varphi, u_0, g) \in \tilde{\mathcal{V}}_{\tau T}, \tag{5.25}$$

where $r(t)$ is given by (5.7). Using the Schwartz inequality and estimate (4.21), we have

$$\begin{aligned} \|r\|_{\tilde{C}[-T-\tau,0]} &= \|w_m\|_{C[0,T]} \le c_5 \|m\|_{L_2(Q)} \|w\|_{C([0,T], L_2(Q))} \\ &\le K c_5 \|m\|_{L_2(Q)} \|w\|_{C([0,T], W_2^1(Q))} \le K c_5 \|m\|_{L_2(Q)} (A_0 + M_1), \end{aligned}$$

where c_5 is the constant occurring in (4.21) and K is the norm of the embedding operator $W_2^1(Q) \to L_2(Q)$ (the norm in $W_2^1(Q)$ is given by (2.4)). By setting

$$M_2 = K c_5 \|m\|_{L_2(Q)} (A_0 + M_1),$$

we obtain (5.25).

PROOF. [Proof of Theorems 5.1 and 5.2] I. It follows from Lemmas 5.2–5.4 that the operator G maps a bounded closed convex set $\tilde{\mathcal{V}}_{\tau T}$ into itself and is compact. By the Schauder fixed-point theorem, the operator G has a fixed point $(\varphi, u_0, g) \in \tilde{\mathcal{V}}_{\tau T}$. Therefore, due to Remark 5.4, the corresponding solution (w, u) of problem (4.1)–(4.3), (5.1)–(5.3) is a T-periodic solution of problem (4.1), (4.3), (5.1). Theorem 5.2 and assertion 1 of Theorem 5.1 are proved.

II. Let (w, u) be an arbitrary T-periodic solution of problem (4.1), (4.3), (4.4). It follows from (4.27) and from the representation (4.28) that

$$u_0 e^{-t/a} \le u(t) \le (u_0 - 1)e^{-t/a} + 1 \quad \forall t \ge 0.$$

Since $u_0 e^{-t/a} \to 0$ and $(u_0 - 1)e^{-t/a} + 1 \to 1$ as $t \to +\infty$, it follows that the image of the periodic function $u(t)$ lies in the segment $[0, 1]$.

Remark 5.5 *Let F not depend on $w(x,t)$. Then any periodic solution (w,u) of problem* (4.1), (4.3), (4.4) *is uniquely determined by the initial configuration $\xi \in \mathcal{R}$, the "mean" temperature $w_m(t)$, and the control function $u(t)$. Namely, using the Banach fixed-point theorem, similarly to [10, Sec. 4], one can show the following. Let (w,u) be a T-periodic solution of problem* (4.1), (4.3), (4.4) *and $(\tilde{w}, u)$ be a solution of problem* (4.1)–(4.6) *in Q_T for all $T > 0$ such that $\tilde{w}_m(t) = w_m(t)$ for all $t \geq 0$. Then either $\tilde{w} = w$ or $\tilde{w}$ is not periodic in t and*

$$\|\tilde{w}(\cdot, t) - w(\cdot, t)\|_{W_2^1(Q)} \to 0 \quad as \quad t \to \infty.$$

6 Thermocontrol Problems Without Time Delay: Existence and Uniqueness of Solutions

6.1 Setting of the problem

Now we consider a thermocontrol problem without time delay. Let $w(x,t)$ be the temperature at the point $x \in Q$ at the moment $t \geq 0$ obeying the heat equation

$$w_t(x,t) = Pw(x,t) + F(x,t,w(x,t),u(t)) \quad ((x,t) \in Q_T), \qquad (6.1)$$

where $F(x,t,w,u)$ satisfies (4.8) and the control function $u(t)$ is to be defined below.

The initial condition has the form

$$w(x,0) = \varphi(x) \quad (x \in Q). \qquad (6.2)$$

The boundary condition is given by:

$$-\gamma \frac{\partial w}{\partial \nu} = \sigma(x) w(x,t) + k_0(x) u(t) + k_1(x) \quad ((x,t) \in \Gamma_T), \qquad (6.3)$$

where γ, σ, k_0, k_1 are the same as above.

Fix an arbitrary $\xi \in \mathcal{R}$ and consider the Preisach operator $\mathcal{H} : C[0,T] \to L_\infty(0,T) \cap C_r[0,T)$ given by

$$\mathcal{H}(r)(t) = \mathcal{H}(r, \xi, 0)(t), \qquad r \in C[0,T], \ t \in [0,T].$$

We assume that the control function $u(t)$ satisfies the following Cauchy problem:

$$au'(t) + u(t) = \mathcal{H}(w_m)(t) \quad (t \in (0,T)), \tag{6.4}$$
$$u(0) = u_0, \tag{6.5}$$

where $a > 0$, $u_0 \in \mathbb{R}$, and w is the function satisfying relations (6.1)–(6.3).

Definition 6.1 *A pair of functions* (w, u) *is called a* solution *of problem* (6.1)–(6.5) (*in* Q_T *with the initial configuration* $\xi \in \mathcal{R}$) *if*

$$w \in W_2^{2,1}(Q_T) \cap C([0,T], W_2^1(Q))$$

satisfies Eq. (6.1) *a.e. in* Q_T *and conditions* (6.2), (6.3) *in the sense of traces and* $u \in C^1[0,T]$ *satisfies Eq.* (6.4) *in* $(0,T)$ *and condition* (6.5).

6.2 Solvability and a priori estimates

In this subsection, we prove the existence and uniqueness of the solution for problem (4.1)–(4.6). As before, we reduce the problem to that with the homogeneous boundary condition. Consider a function $U \in W_2^2(Q)$ satisfying (4.10) and (4.11). Set

$$v_0(x,t) = [k_0(x)u(t) + k_1(x)]U(x). \tag{6.6}$$

Similarly to (4.13)–(4.15), we obtain that the function $v = w - v_0$ satisfies the relations

$$v_t(x,t) = Pv(x,t) + f(x,t,v) \quad ((x,t) \in Q_T), \tag{6.7}$$
$$v(x,0) = \varphi(x) + \varphi_0(x) \quad (x \in Q), \tag{6.8}$$
$$\gamma \frac{\partial v}{\partial \nu} + \sigma(x) v(x,t) = 0 \quad ((x,t) \in \Gamma_T), \tag{6.9}$$

where

$$\begin{aligned} f(x,t,v) &= P[k_0(x)U(x)]u(t) + P[k_1(x)U(x)] - k_0(x)U(x)u_1(t) \\ &\quad + F(x,t,v+v_0(x,t),u(t)), \\ u_1(t) &= a^{-1}(\mathcal{H}(w_m)(t) - u(t)), \\ \varphi_0(x) &= -(k_0(x)u_0 + k_1(x))U(x). \end{aligned} \tag{6.10}$$

Remark 6.1 *We shall now define a mild solution* (w,u)*, where* u *is a priori not supposed to be differentiable. That is why we have introduced* $u_1(t)$ *in* (6.10) *instead of writing* $u'(t)$*. Then we shall prove that* u *is differentiable and* $u'(t) \equiv u_1(t)$.

Consider the analytic semigroup of contraction $\mathbf{T}_t : L_2(Q) \times \mathbb{R} \to L_2(Q) \times \mathbb{R}$, $t \geq 0$, given by (4.17).

Definition 6.2 *A pair of functions* $(w, u) \in C([0,T], L_2(Q) \times \mathbb{R})$ *is called a* mild solution *of problem* (6.1)–(6.5) *(in* Q_T *with the initial configuration* $\xi \in \mathcal{R}$*) if* $w = v_0 + v$*, where* v_0 *is given by* (6.6) *and*

$$(v(\cdot,t), u(t)) = \mathbf{T}_t(\varphi + \varphi_0, u_0) + \int_0^t \mathbf{T}_{t-s}\left(f(\cdot, s, v(\cdot, s)), a^{-1}\mathcal{H}(w_m)(s)\right) ds; \tag{6.11}$$

here f *and* φ_0 *are given by* (6.10).

First, we prove the existence and uniqueness of a mild solution. We replace Condition 3.1 by a stronger one which ensures the Lipschitz continuity of the Preisach operator.

For any function $\psi \in \Psi$ (the class Ψ is described in Sec. 3) and any $\varepsilon > 0$, we define the set

$$G(\psi, \varepsilon) = \{(\rho_1, \rho_2) : \psi(\rho_2 - \rho_1) - \varepsilon \le \rho_1 + \rho_2 \le \psi(\rho_2 - \rho_1)\}.$$

Condition 6.1 *There is a constant* $L_1 > 0$ *such that*

$$\sup_{\psi \in \Psi} \mu(G(\psi, \varepsilon)) \le L_1 \varepsilon \qquad \forall \varepsilon > 0.$$

Lemma 6.1 (see Sec. 38.6 in [13]) *Condition* 6.1 *holds if and only if the operator* $\mathcal{H} : C[t_0, t_1] \to C[t_0, t_1]$ *is uniformly Lipschitz continuous, i.e., there is a constant* $L_2 > 0$ *(which does not depend on* t_0*,* t_1*, and* $\xi_j \in \mathcal{R}$*,* $r_j \in C[t_0, t_1]$*) such that*

$$\|\mathcal{H}(r_1, \xi_1, t_0) - \mathcal{H}(r_2, \xi_2, t_0)\|_{C[t_0,t_1]} \le L_2 \left(\|r_1 - r_2\|_{C[t_0,t_1]} + \int_{\mathcal{P}} |\xi_{1\rho} - \xi_{2\rho}| \, d\mu(\rho) \right).$$

Theorem 6.1 *Let Condition* 6.1 *hold. Suppose that* F *satisfies condition* (4.8)*. Then, for any initial data* $(\varphi, u_0) \in L_2(Q) \times \mathbb{R}$ *there exists a unique mild solution* (w, u) *of problem* (6.1)–(6.5) *in* Q_T*.*

PROOF. For a given pair $(\varphi, u_0) \in L_2(Q) \times \mathbb{R}$, we define the mapping

$$\mathbf{F} : C([0,T], L_2(Q) \times \mathbb{R}) \to C([0,T], L_2(Q) \times \mathbb{R})$$

by the formula

$$(\mathbf{F}\mathbf{x})(t) = \mathbf{T}_t(\varphi + \varphi_0, u_0) + \int_0^t \mathbf{T}_{t-s}\left(f(\cdot, s, v(\cdot, s)) a^{-1}\mathcal{H}(w_m)(s)\right) ds, \tag{6.12}$$

for all $\mathbf{x} = (v, u) \in C([0,T], L_2(Q) \times \mathbb{R})$ such that $\mathbf{x}(0) = (\varphi + \varphi_0, u_0)$, f and φ_0 are given by (6.10), $w = v + v_0$, and v_0 is defined in (6.6).

Denote by $\|\mathbf{x}\|_{[0,s]}$ the norm of $\mathbf{x}$ in $C([0,s], L_2(Q) \times \mathbb{R})$, $0 < s \le T$.

Let $\tilde{\mathbf{x}} = (\tilde{v}, \tilde{u}) \in C([0,T], L_2(Q) \times \mathbb{R})$, $\tilde{\mathbf{x}}(0) = (\varphi + \varphi_0, u_0)$.

We set $\tilde{w} = \tilde{v} + \tilde{v}_0$, $\tilde{v}_0 = (k_0(x)\tilde{u}(t) + k_1(x))U(x)$ (cf. (6.6)),

$$\begin{aligned}\tilde{f}(x,t,v) = {} & P[k_0(x)U(x)]\tilde{u}(t) + P[k_1(x)U(x)] - k_0(x)U(x)\tilde{u}_1(t) \\ & + F(x,t,\tilde{v} + \tilde{v}_0(x,t), \tilde{u}(t)),\end{aligned}$$

$$\tilde{u}_1(t) = a^{-1}(\mathcal{H}(\tilde{w}_m)(t) - \tilde{u}(t)).$$

By using (4.8) and Lemma 6.1, we obtain

$$\begin{aligned}&\left\| \left(f(\cdot,s,v(\cdot,s)), a^{-1}\mathcal{H}(w_m)(s)\right) - \left(\tilde{f}(\cdot,s,\tilde{v}(\cdot,s)), a^{-1}\mathcal{H}(\tilde{w}_m)(s)\right) \right\|_{L_2(Q)\times\mathbb{R}} \\ &\quad \le k_1 \|\mathbf{x} - \tilde{\mathbf{x}}\|_{[0,s]},\end{aligned} \tag{6.13}$$

where $k_1 > 0$ does not depend on s, $\mathbf{x}$, $\tilde{\mathbf{x}}$.

It follows from (6.12) and (6.13) that

$$\|(\mathbf{F}\mathbf{x})(t) - (\mathbf{F}\tilde{\mathbf{x}})(t)\|_{L_2(Q)\times\mathbb{R}} \le k_1 M t \|\mathbf{x} - \tilde{\mathbf{x}}\|_{[0,t]}, \tag{6.14}$$

where M is a bound of $\|\mathbf{T}_t\|$ on $[0,T]$ and $k_1, k_2, \ldots > 0$ do not depend on $t \in [0,T]$. Using (6.12)–(6.14), we obtain by induction

$$\|(\mathbf{F}^l\mathbf{x}) - (\mathbf{F}^l\tilde{\mathbf{x}})\|_{[0,T]} \le \frac{(k_1 M T)^l}{l!} \|\mathbf{x} - \tilde{\mathbf{x}}\|_{[0,T]}.$$

For a sufficiently large l, we have $(k_1MT)^l/l! < 1$. Therefore, by the generalized contraction principle, $\mathbf{F}$ has a unique fixed point $\mathbf{x} = (v,u)$. Clearly, the function $(w,u) = (v + v_0, u)$, where v_0 is given by (6.6), is a mild solution of problem (6.1)–(6.5).

Remark 6.2 *For any mild solution* (w,u) *of problem* (6.1)–(6.5)*, we have*

$$u(t) = u_0 e^{-t/a} + a^{-1} \int_0^t e^{-(t-s)/a} \mathcal{H}(w_m)(s)\, ds,$$

which implies that $u \in C^1[0,T]$ *and satisfies the Cauchy problem* (6.4)*,* (6.5)*.*

Let

$$\mathcal{V}_0 = W_2^1(Q) \times \mathbb{R}$$

if $\gamma > 0$ and

$$\mathcal{V}_0 = \left\{ (\varphi, u_0) \in W_2^1(Q) \times \mathbb{R} : \sigma(x)\varphi(x) + k_0(x)u_0 + k_1(x) = 0 \ (x \in \Gamma) \right\}$$

if $\gamma = 0$.

Theorem 6.2 *(1) Let Condition* 6.1 *hold. Suppose that F satisfies conditions* (4.8) *and* (4.9) *and* $(\varphi, u_0) \in \mathcal{V}_0$. *Then there exists a unique solution* (w, u) *of problem* (6.1)–(6.5) *in* Q_T *and*

$$\|w(\cdot, T)\|_{W_2^1(Q)} \le c_4 A(u_0) + e^{-\omega T}\|\varphi\|_{W_2^1(Q)}, \tag{6.15}$$

$$\|w\|_{C([0,T], W_2^1(Q))} \le c_5 B(\varphi, u_0), \tag{6.16}$$

$$\|u\|_{C[0,T]} \le \max(1, |u_0|), \tag{6.17}$$

$$|w_m(t_2) - w_m(t_1)| \le c_6 B(\varphi, u_0)|t_2 - t_1|^{1/2} \qquad \forall t_1, t_2 \in [0, T], \tag{6.18}$$

where $A(u_0)$ *is given by* (4.24) *and* $B(\varphi, u_0)$ *by* (4.25).

(2) If we additionally assume that Condition 3.2 *holds, then*

$$\|w(\cdot, T)\|_{W_2^2(Q)} \le c_7 B(\varphi, u_0). \tag{6.19}$$

Here $c_4, \dots, c_7 > 0$ *depend on* T, *do not depend on* φ, u_0, g, ξ, *and are bounded on any segment* $[T_1, T_2]$ $(0 < T_1 < T_2)$.

PROOF. I. Due to Theorem 6.1, there is a unique mild solution $(w, u) \in C([0, T], L_2(Q) \times \mathbb{R})$ of problem (6.1)–(6.5) in Q_T. Since

$$u(t) = u_0 e^{-t/a} + a^{-1}\int_0^t e^{-(t-s)/a}\mathcal{H}(w_m)(s)\, ds \quad (t \ge 0),$$

it follows that $u \in C^1[0, T]$ and satisfies (6.4), (6.5). Moreover, $v_0 \in W_2^{2,1}(Q_T)$ and $f(x, t, v(x, t))$ belongs to $L_2(Q_T)$. Therefore, due to Lemma 2.1, problem (6.7)–(6.9) has a unique solution $v \in W_2^{2,1}(Q_T) \cap C([0, T], W_2^1(Q))$ and the pair (v, u) satisfies (6.11). Thus, the mild solution (w, u) is also a (strong) solution of problem (6.1)–(6.5).

The proof of inequalities (6.15)–(6.19) is similar to the proof of inequalities (4.20)–(4.23) and (4.26) in Theorem 4.2.

Remark 6.3 *If the numbers* $\tau_0 < T$ *are fixed, then, for any* $\tau \in (0, \tau_0)$, *one can take the constants* $c_4, \dots, c_7$ *in Theorem 4.2 equal to the respective constants in Theorem 6.2.*

6.3 Continuous dependence of solutions upon the initial data and the delay τ

Now we investigate the dependence of solutions of problem (4.1)–(4.6) (with $\tau > 0$) and problem (6.1)–(6.5) (with $\tau = 0$) upon the initial data and the delay τ. In what follows, the phrase "(w, u) is a (mild) solution of the thermocontrol problem" means that (w, u) is a (mild) solution for problem (4.1)–(4.6) if $\tau > 0$ and problem (6.1)–(6.5) if $\tau = 0$.

Theorem 6.3 *Let Condition* 6.1 *hold. Suppose that* F *satisfies condition* (4.8)*. Let* (w_j, u_j)*,* $j = 1, 2$*, be a mild solution of the thermocontrol problem with initial configurations* $\xi_j \in \mathcal{R}$*, delay* $\tau_j \geq 0$*, and initial data*

$$\mathbf{v}_j = \begin{cases} (\varphi_j, u_{j0}, g_j) \in L_2(Q) \times \mathbb{R} \times C[-\tau_j, 0] & \text{if } \tau_j > 0, \\ (\varphi_j, u_{j0}) \in L_2(Q) \times \mathbb{R} & \text{if } \tau_j = 0. \end{cases}$$

We additionally assume that the following compatibility condition holds for $\tau_j > 0$:

$$g_j(0) = \varphi_{jm}, \quad j = 1, 2. \tag{6.20}$$

Then

$$\begin{aligned} &\|(w_1, u_1) - (w_2, u_2)\|_{[0,T]} \\ &\quad \leq c_8 \left(\int_{\mathcal{P}} |\xi_{1\rho} - \xi_{2\rho}| \, d\mu(\rho) + \|\varphi_1 - \varphi_2\|_{L_2(Q)} + |u_{10} - u_{20}| + C(\mathbf{v}_1, \mathbf{v}_2, \tau_1, \tau_2) \right), \end{aligned} \tag{6.21}$$

where

$$C(\mathbf{v}_1, \mathbf{v}_2, \tau_1, \tau_2) = \begin{cases} \sum\limits_{j=1,2} \left(\|g_j(\cdot) - g_j(0)\|_{C[-\tau_j,0]} + B(\varphi_j, u_{j0}) \tau_j^{1/2} \right) & \text{if } \tau_1 \neq \tau_2, \\ \|g_1 - g_2\|_{C[-\tau_1,0]} & \text{if } \tau_1 = \tau_2, \end{cases} \tag{6.22}$$

$B(\varphi_j, u_{j0})$ *is given by* (4.25)*, and* $c_8 = c_8(T) > 0$ *does not depend on* $\mathbf{v}_j, \xi_j, \tau_j$ *(if* $\tau_j = 0$*, we formally set* $g_j = 0$ *in* (6.22)*).*

PROOF. I. Denote $\xi = \xi_1 - \xi_2$, $w = w_1 - w_2$, $u = u_1 - u_2$, $u_0 = u_{10} - u_{20}$, $\varphi = \varphi_1 - \varphi_2$. We represent the function w as follows: $w = v + v_0$. Here

$$v_0(x, t) = k_0(x) u(t) \tag{6.23}$$

and v is the solution of the problem

$$v_t(x, t) = Pv(x, t) + f(x, t) \quad ((x, t) \in Q_T), \tag{6.24}$$

$$v(x, 0) = \varphi(x) + \varphi_0(x) \quad (x \in Q), \tag{6.25}$$

$$\gamma \frac{\partial v}{\partial \nu} + \sigma(x) v(x, t) = 0 \quad ((x, t) \in \Gamma_T), \tag{6.26}$$

where

$$\begin{aligned} f(x, t) &= P[k_0(x) U(x)] u(t) - k_0(x) U(x) u'(t) \\ &\quad + F_1(x, t, w_1(x, t), u_1(t)) - F_2(x, t, w_2(x, t), u_2(t)), \\ \varphi_0(x) &= -k_0(x) U(x) u_0. \end{aligned} \tag{6.27}$$

The function u belongs to $C^1[0, T]$ (by Remark 6.2) and satisfies

$$u'(t) = a^{-1} \Big(\mathcal{H}(\xi_1, w_{1m}(\cdot - \tau_1), 0)(t) - \mathcal{H}(\xi_2, w_{2m}(\cdot - \tau_2), 0)(t) - u(t) \Big) \tag{6.28}$$

$$u(0) = u_0. \tag{6.29}$$

Denote $\mathbf{x}(t) = (v(\cdot, t), u(t))$.

It follows from (6.27), (4.8), and (6.28) that

$$\begin{aligned} \|f(\cdot, s)\|_{L_2(Q)} \le k_1(\|\mathbf{x}(s)\|_{L_2(Q)\times\mathbb{R}} + |u'(s)|) \le k_2(\|\mathbf{x}(s)\|_{L_2(Q)\times\mathbb{R}} \\ + |\mathcal{H}(\xi_1, w_{1m}(\cdot - \tau_1), 0)(s) - \mathcal{H}(\xi_2, w_{2m}(\cdot - \tau_2), 0)(s)|), \end{aligned} \tag{6.30}$$

where $k_1, k_2, \ldots > 0$ depend on T but do not depend on $\mathbf{v}_j, \xi_j, \tau_j$.

Due to the compatibility condition (6.20), the Preisach operator $\mathcal{H}$ is considered for functions continuous on $[0, T]$. Therefore, using Lemma 6.1, we obtain

$$\begin{aligned} &|\mathcal{H}(\xi_1, w_{1m}(\cdot - \tau_1), 0)(s) - \mathcal{H}(\xi_2, w_{2m}(\cdot - \tau_2), 0)(s)| \\ &\qquad \le L_2 \left(\int_{\mathcal{P}} |\xi_\rho| \, d\mu(\rho) + \|w_{1m}(\cdot - \tau_1) - w_{2m}(\cdot - \tau_2)\|_{C[0,s]} \right). \end{aligned} \tag{6.31}$$

Assume that $s > \tau_j$, $j = 1, 2$ (the case where $s \le \tau_j$ is analogous but simpler).

If $\tau_1 \ne \tau_2$, then

$$\begin{aligned} &\|w_{1m}(\cdot - \tau_1) - w_{2m}(\cdot - \tau_2)\|_{C[0,s]} \\ &\quad \le \sum_{j=1,2} \|w_{jm}(\cdot - \tau_j) - w_{jm}(\cdot)\|_{C[0,s]} + \|w_{1m} - w_{2m}\|_{C[0,s]} \\ &\quad \le \sum_{j=1,2} \Big(\|g_j(\cdot - \tau_j) - g_j(0)\|_{C[0,\tau_j]} + \|w_{jm}(\cdot) - w_{jm}(0)\|_{C[0,\tau_j]} \\ &\quad + \|w_{jm}(\cdot - \tau_j) - w_{jm}(\cdot)\|_{C[\tau_j,s]} \Big) + \|w_{1m} - w_{2m}\|_{C[0,s]}. \end{aligned} \tag{6.32}$$

If $\tau_1 = \tau_2$, then

$$\|w_{1m}(\cdot - \tau_1) - w_{2m}(\cdot - \tau_2)\|_{C[0,s]} \le \|g_1 - g_2\|_{C[-\tau_1,0]} + \|w_{1m} - w_{2m}\|_{C[0,s]}. \tag{6.33}$$

Clearly, estimates (6.32) and (6.33) remain true for $s \le \tau_j$. If $\tau_j = 0$, they are also valid with $g_j = 0$.

Using the Schwartz inequality and Theorems 4.2 and 6.2, we have

$$\|w_{jm}(\cdot) - w_{jm}(0)\|_{C[0,\tau_j]} + \|w_{jm}(\cdot - \tau_j) - w_{jm}(\cdot)\|_{C[\tau_j,s]} \le k_3 B(\varphi_j, u_{j0}) \tau_j^{1/2}, \tag{6.34}$$

$$\|w_{1m} - w_{2m}\|_{C[0,s]} \le k_4(\|v_0\|_{C([0,s],L_2(Q))} + \|v\|_{C([0,s],L_2(Q))}) \le k_5 \|\mathbf{x}\|_{[0,s]}. \tag{6.35}$$

Combining inequalities (6.31)–(6.35), we obtain

$$
\begin{aligned}
&|\mathcal{H}(\xi_1, w_{1m}(\cdot - \tau_1), 0)(s) - \mathcal{H}(\xi_2, w_{2m}(\cdot - \tau_2), 0)(s)| \\
&\quad \le k_6 \left(\int_{\mathcal{P}} |\xi_\rho| \, d\mu(\rho) + C + \|\mathbf{x}\|_{[0,s]} \right),
\end{aligned}
\tag{6.36}
$$

where $C = C(\mathbf{v}_1, \mathbf{v}_2, \tau_1, \tau_2)$ is given by (6.22).

Now we can estimate the solution v of problem (6.24)–(6.26) and the solution u of problem (6.28), (6.29). Using representations (2.5) and

$$
\begin{aligned}
u(t) &= u_0 e^{-t/a} \\
&\quad + a^{-1} \int_0^t e^{-(t-s)/a} \big(\mathcal{H}(\xi_1, w_{1m}(\cdot - \tau_1), 0)(s) - \mathcal{H}(\xi_2, w_{2m}(\cdot - \tau_2), 0)(s) \big) \, ds
\end{aligned}
$$

and inequalities (6.30) and (6.36), we obtain

$$
\|\mathbf{x}(\cdot, t)\|_{L_2(Q)\times\mathbb{R}} \le k_7 \left(\int_{\mathcal{P}} |\xi_\rho| \, d\mu(\rho) + \|\varphi\|_{L_2(Q)} + |u_0| + C + \int_0^t \|\mathbf{x}\|_{[0,s]} \, ds \right). \tag{6.37}
$$

Using (6.37) and the Gronwall inequality, we obtain

$$
\|\mathbf{x}\|_{[0,t]} \le k_7 e^{k_7 t} \left(\int_{\mathcal{P}} |\xi_\rho| \, d\mu(\rho) + \|\varphi\|_{L_2(Q)} + |u_0| + C \right). \tag{6.38}
$$

In particular, it follows from (6.38) that

$$
\|v_0(\cdot, t)\|_{L_2(Q)} \le k_8 e^{k_7 t} \left(\int_{\mathcal{P}} |\xi_\rho| \, d\mu(\rho) + \|\varphi\|_{L_2(Q)} + |u_0| + C \right). \tag{6.39}
$$

Combining (6.38) and (6.39), we derive the desired estimate.

Theorem 6.4 *Let Condition* 6.1 *hold. Suppose that F satisfies condition* (4.8)*. Let (w_j, u_j), $j = 1, 2$, be a solution of the thermocontrol problem with initial configurations $\xi_j \in \mathcal{R}$, delay $\tau_j \ge 0$, and initial data*

$$
\mathbf{v}_j = \begin{cases} (\varphi_j, u_{j0}, g_j) \in \mathcal{V}_{\tau_j} & \text{if } \tau_j > 0, \\ (\varphi_j, u_{j0}) \in \mathcal{V}_0 & \text{if } \tau_j = 0. \end{cases}
$$

Then

$$
\begin{aligned}
&\|w_1 - w_2\|_{W_2^{2,1}(Q_T)} + \|w_1 - w_2\|_{C([0,T], W_2^1(Q))} + \|u_1 - u_2\|_{C^1[0,T]} \\
&\quad \le c_9 \Bigg(\int_{\mathcal{P}} |\xi_{1\rho} - \xi_{2\rho}| \, d\mu(\rho) + \|\varphi_1 - \varphi_2\|_{W_2^1(Q)} + |u_{10} - u_{20}| \\
&\qquad + C(\mathbf{v}_1, \mathbf{v}_2, \tau_1, \tau_2) \Bigg),
\end{aligned}
\tag{6.40}
$$

where $C(\mathbf{v}_1, \mathbf{v}_2, \tau_1, \tau_2)$ *is given by* (6.22) *and* $c_9 = c_9(T) > 0$ *does not depend on* $\mathbf{v}_j, \xi_j, \tau_j$.

PROOF. We keep the notation of the proof of Theorem 6.3. Due to this theorem, it suffices to estimate the solution v of problem (6.24)–(6.26) and the first derivative of the solution u of problem (6.28), (6.29).

It follows from (6.28) that

$$|u_1'(t) - u_2'(t)| \le a^{-1}(|u_1(t) - u_2(t)| \\ + |\mathcal{H}(\xi_1, w_{1m}(\cdot - \tau_1), 0)(t) - \mathcal{H}(\xi_2, w_{2m}(\cdot - \tau_2), 0)(t)|).$$

Using Lemma 2.1, we have

$$\|v\|_{W_2^{2,1}(Q_T)} + \|v\|_{C([0,T],W_2^1(Q))} \le k_1(\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(Q)} + \|\varphi_0\|_{W_2^1(Q)}).$$

Combining these two inequalities with estimates (6.30), (6.36), and (6.38), we obtain (6.40).

In particular, Theorems 6.3 and 6.4 ensure the Lipschitz dependence of the corresponding solution of the thermocontrol problem on initial configuration $\xi \in \mathcal{R}$ and initial data $\mathbf{v}$, provided that $\tau \ge 0$ is fixed.

Another consequence of Theorem 6.4 is the following result about the relation between the solutions of the thermocontrol problem for $\tau > 0$ and $\tau = 0$ respectively.

Corollary 6.1 *Let Condition* 6.1 *hold. Suppose that F satisfies condition* (4.8). *Let (w_τ, u_τ) be a solution of the thermocontrol problem with initial configuration $\xi_\tau \in \mathcal{R}$, delay $\tau \in [0,1]$, and initial data*

$$\mathbf{v}_\tau = \begin{cases} (\varphi_\tau, u_{\tau 0}, g_\tau) \in \mathcal{V}_\tau & \text{if } \tau > 0, \\ (\varphi_0, u_{00}) \in \mathcal{V}_0 & \text{if } \tau = 0. \end{cases}$$

Assume that

$$\int_{\mathcal{P}} |\xi_{\tau\rho} - \xi_{0\rho}|\, d\mu(\rho) \to 0, \quad \|\varphi_\tau - \varphi_0\|_{W_2^1(Q)} \to 0,$$

$$|u_{\tau 0} - u_{00}| \to 0, \quad \|g_\tau(\cdot) - g_\tau(0)\|_{C[-\tau,0]} \to 0$$

as $\tau \to 0$. *Then*

$$\|w_\tau - w_0\|_{W_2^{2,1}(Q_T)} + \|w_\tau - w_0\|_{C([0,T],W_2^1(Q))} + \|u_\tau - u_0\|_{C^1[0,T]} \to 0 \quad \text{as } \tau \to 0.$$

Remark 6.4 *The condition*

$$\|g_\tau(\cdot) - g_\tau(0)\|_{C[-\tau,0]} \to 0 \quad \text{as } \tau \to 0$$

holds if, e.g., there is a function $g \in C[-1,0]$ *such that*

$$g_\tau(t) = g(t) \quad (t \in [-\tau, 0]).$$

Another situation in which this condition holds is described in Sec. 7 (*see the proof of Theorem* 7.1).

7 Periodic Solutions of Thermocontrol Problems Without Time Delay

7.1 Existence of periodic solutions

In this section, we will prove the existence of a periodic solution of the thermocontrol problem with $\tau = 0$. Moreover, we will show that this solution is a limit (as $\tau \to 0$) of periodic solutions of the thermocontrol problems with $\tau > 0$.

We assume that the right-hand side $F(x, t, \psi, u)$ is T-periodic in t with some $T > 0$ and prove the existence of a T-periodic solution (w, u) of problem (6.1), (6.3), (6.4).

Definition 7.1 *A pair* (w, u) *is called a* T-periodic solution of problem (6.1), (6.3), (6.4) *(with an initial configuration* $\xi \in \mathcal{R}$*) if there is a couple* $(\varphi, u_0) \in \mathcal{V}_0$ *such that the following holds:*

(1) (w, u) *is a solution of problem* (6.1)–(6.5) *in* Q_T *with the initial data* (φ, u_0) *and the initial configuration* ξ,
(2) $u(T) = u_0$, $w(x, T) = \varphi(x)$, *and* $h_\rho(w_m, \xi_\rho, 0)(T) = \xi_\rho$ *for* $\rho \in \mathcal{P}$.

Theorem 7.1 *Let Conditions* 6.1 *and* 3.2 *hold. Suppose that the function* $F(x, t, \psi, u)$ *is* T*-periodic in* t *with some* $T > 0$ *and satisfies conditions* (4.8) *and* (4.9). *Then the following assertions are true.*

(1) There exists a T*-periodic solution* (w_0, u_0) *of problem* (6.1), (6.3), (6.4), *which is the limit in* $W_2^{2,1}(Q_T) \times C^1[0, T]$ *and* $C([0, T], W_2^1(Q) \times \mathbb{R})$ (*as* $\tau \to 0$) *of* T*-periodic solutions of problems* (4.1), (4.3), (4.4).
(2) For any T*-periodic solution* (w, u) *of problem* (6.1), (6.3), (6.4), *we have* $u(t) \in [0, 1]$ $(t \geq 0)$.

PROOF. I. Fix an arbitrary $\zeta \in \mathcal{R}$ and consider the operator $G : \tilde{\mathcal{V}}_\tau \to \tilde{\mathcal{V}}_\tau$, $\tau \in (0, T)$, given by (5.6). Due to Lemmas 5.2–5.4 and the Schauder fixed-point theorem, for any $\tau \in (0, T)$ it has a fixed point $(\varphi_\tau, u_{\tau 0}, g_\tau) \in \tilde{V}_{\tau T}$ and $g_\tau(t)$ is continuous on $[-T-\tau, 0]$. Due to Remark 5.4, the corresponding solution (w_τ, u_τ) of problem (4.1)–(4.3), (5.1)–(5.3) is a T-periodic solution of problem (4.1), (4.3), (5.1).

Further, due to Remark 5.3, the pair (w_τ, u_τ) is a periodic solution of problem (4.1), (4.3), (4.4) with the initial configuration

$$\xi_{\tau\rho} = h_\rho(g_\tau(\cdot - \tau), \zeta_\rho, -T)(0).$$

By Lemma 5.4, we have

$$\|\varphi_\tau\|_{W_2^1(Q)} \le k_1, \quad u_{\tau 0} \in [0,1], \quad \|g_\tau\|_{C[-T-\tau,0]} \le k_2, \tag{7.1}$$

where $k_1, k_2, \ldots > 0$ do not depend on τ.

Therefore, applying Theorem 4.2 and taking into account the uniform boundedness of $B(\varphi_\tau, u_{\tau 0})$, we obtain

$$\|\varphi_\tau\|_{W_2^2(Q)} = \|w_\tau(\cdot, T)\|_{W_2^2(Q)} \le k_3, \tag{7.2}$$

$$|g_\tau(t_1) - g_\tau(t_2)| = |w_{\tau m}(2T + t_1) - w_{\tau m}(2T + t_2)| \le k_4 |t_1 - t_2|^{1/2} \tag{7.3}$$

for all $t_1, t_2 \in [-T-\tau, 0]$.

It follows from estimates (7.1)–(7.3), from the compactness of the embedding $W_2^2(Q) \subset W_2^1(Q)$, and from the Ascoli–Arzelà theorem that there exist functions $\varphi \in W_2^1(Q)$ and $g \in C[-T, 0]$ and a number $u_{00} \in [0,1]$ such that

$$\|\varphi_\tau - \varphi_0\|_{W_2^1(Q)} \to 0, \quad \|g_\tau - g_0\|_{C[-T,0]} \to 0, \quad |u_{\tau 0} - u_{00}| \to 0 \quad \text{as } \tau \to 0. \tag{7.4}$$

We set

$$\xi_{0\rho} = h_\rho(g_0, \zeta_\rho, -T)(0).$$

Let us show that

$$\int_{\mathcal{P}} |\xi_{\tau\rho} - \xi_{0\rho}| \, d\mu(\rho) \to 0 \quad \text{as } \tau \to 0. \tag{7.5}$$

By using (7.3), we have

$$\begin{aligned}\|g_\tau(\cdot - \tau) - g_0(\cdot)\|_{C[-T,0]} &\le \|g_\tau(\cdot - \tau) - g_\tau(\cdot)\|_{C[-T,0]} + \|g_\tau - g_0\|_{C[-T,0]} \\ &\le k_4 \tau^{1/2} + \|g_\tau - g_0\|_{C[-T,0]}.\end{aligned}$$

Combining this estimate with (7.4), we see that

$$\|g_\tau(\cdot - \tau) - g_0(\cdot)\|_{C[-T,0]} \to 0 \quad \text{as } \tau \to 0.$$

This implies (7.5) (cf. (3.2)).

II. Due to (7.4), we have $(\varphi_0, u_{00}) \in \mathcal{V}_0$. Therefore, by Theorem 6.2, there is a unique solution (w_0, u_0) of problem (6.1)–(6.4) with the initial configuration ξ_0 and the initial data (φ_0, u_{00}).

We claim that (w_0, u_0) is the desired T-periodic solution of problem (6.1), (6.3), (6.4). Indeed, it follows from (7.3)–(7.5) and from Corollary 6.1 that

$$\|w_\tau - w_0\|_{W_2^{2,1}(Q_T)} + \|w_\tau - w_0\|_{C([0,T],W_2^1(Q))} + \|u_\tau - u_0\|_{C^1[0,T]} \to 0 \quad \text{as } \tau \to 0, \tag{7.6}$$

which proves that (w_0, u_0) is the limit of T-periodic solutions of problems (4.1), (4.3), (4.4) for $\tau > 0$.

Further, using (7.4) and (7.6), we obtain

$$u_0(T) = u_{00}, \quad w_0(x, T) = \varphi_0.$$

Thus, to show that (w_0, u_0) is a T-periodic solution of problem (6.1), (6.3), (6.4), it remains to prove that

$$h_\rho(w_{0m}, \xi_\rho, 0)(T) = \xi_\rho \quad (\rho \in \mathcal{P}). \tag{7.7}$$

Denote

$$\hat{w}_{0m} = \begin{cases} g_0(t), & t \in [-T, 0), \\ w_{0m}(t), & t \in [0, T]. \end{cases}$$

Due to (7.4) and (7.6), the function $\hat{w}_{0m}$ is continuous and T-periodic on $[-T, T]$. By Lemma 3.1 (the semigroup property), we have

$$h_\rho(w_{0m}, \xi_\rho, 0)(t) = h_\rho(\hat{w}_{0m}, \zeta_\rho, -T)(t) \quad \forall t \in [0, T]. \tag{7.8}$$

Using (7.8) and Remark 5.2, we obtain (7.7):

$$\begin{aligned} h_\rho(w_{0m}, \xi_\rho, 0)(T) &= h_\rho(\hat{w}_{0m}, \zeta_\rho, -T)(T) = h_\rho(\hat{w}_{0m}, \zeta_\rho, -T)(0) \\ &= h_\rho(\hat{w}_{0m}, \xi_\rho, 0)(0) = \xi_\rho. \end{aligned}$$

Statement 2 of the theorem is proved similarly to that in Theorem 5.1.

Remark 7.1 *Let F not depend on $w(x, t)$. Then any periodic solution (w, u) of problem* (6.1), (6.3), (6.4) *is uniquely determined by the initial configuration $\xi \in \mathcal{R}$, the "mean" temperature $w_m(t)$, and the control function $u(t)$ (cf. Remark 5.5).*

In this subsection, we consider the particular case $F(x,t,\psi,u) \equiv f(x,u)$ and prove that the thermocontrol problem has a stationary solution. The stability of the stationary solution as well as the case where f depends on ψ will not be studied in this paper.

Fix some initial configuration $\xi \in \mathcal{R}$ of the operator $\mathcal{H}$.

Definition 7.2 *We say that a pair* $(\psi, u_0) \in W_2^2(Q) \times \mathbb{R}$ *is a* stationary solution *of the thermocontrol problem with the right-hand side* $F(x,t,\psi,u) \equiv f(x,u)$ $(f \in C(\overline{Q} \times \mathbb{R}))$ *if it satisfies the relations*

$$
\begin{gathered}
-P\psi = f(x,u_0) \quad (x \in Q), \\
-\gamma \frac{\partial \psi}{\partial \nu} = \sigma(x)\psi(x) + k_0(x)u_0 + k_1(x) \quad (x \in \Gamma), \\
u_0 = \mathcal{H}(\psi_m).
\end{gathered}
$$

Theorem 7.2 *(1) Let Condition* 3.1 *hold, and let* $F(x,t,\psi(x),u) \equiv f(x,u)$, *where* $f \in C(\overline{Q} \times \mathbb{R})$. *Then the thermocontrol problem has a stationary solution* (ψ, u_0). *Moreover,* $u_0 \in [0,1]$.

(2) If we additionally assume that $m(x) \ge 0$, $k_0(x) \le 0$, *the function* $f(x,u)$ *is nondecreasing in* u *on the segment* $[0,1]$, *and*

$$
f(\cdot, u_0) \in C^\infty(\overline{Q}) \quad (u_0 \in [0,1]),
$$

then the above stationary solution is unique.

PROOF. I. We construct a function $U(u_0)$ in the following way. For any $u_0 \in \mathbb{R}$, denote by $\psi \in W_2^2(Q)$ the solution of the problem

$$
\begin{gathered}
-P\psi = f(x,u_0) \quad (x \in Q), \\
-\gamma \frac{\partial \psi}{\partial \nu} = \sigma(x)\psi(x) + k_0(x)u_0 + k_1(x) \quad (x \in \Gamma)
\end{gathered}
\tag{7.9}
$$

(which exists and is unique due to the assumptions about the elliptic operator P and the boundary conditions). Set

$$
U(u_0) = \mathcal{H}(\psi_m).
$$

It follows from the continuity of the function f and from the continuity of the operator $\mathcal{H}$ (see Lemma 3.3) that the function $U(u_0)$ is continuous. Due to (4.27), it maps $\mathbb{R}$ into $[0,1]$ and, therefore, has a fixed point (which belongs to the segment $[0,1]$).

Let u_0 be the above fixed point and ψ the corresponding solution of problem (7.9). Clearly, the pair (ψ, u_0) is a stationary solution of the thermocontrol problem.

II. Let us prove the uniqueness of the stationary solution under the additional assumptions about m, k_0, and f. To do so, it suffices to show that the function $U(u_0)$ is nonincreasing for $u_0 \in [0,1]$.

Let $0 \le u_0 \le \tilde{u}_0 \le 1$. Denote by ψ the solution of problem (7.9) and by $\tilde{\psi}$ the solution of the problem

$$-P\tilde{\psi} = f(x, \tilde{u}_0) \quad (x \in Q),$$
$$-\gamma \frac{\partial \tilde{\psi}}{\partial \nu} = \sigma(x)\tilde{\psi}(x) + k_0(x)\tilde{u}_0 + k_1(x) \quad (x \in \Gamma).$$

Clearly, the function $v = \tilde{\psi} - \psi$ satisfies the relations

$$-Pv = f(x, \tilde{u}_0) - f(x, u_0) \ge 0 \quad (x \in Q),$$
$$\gamma \frac{\partial v}{\partial \nu} + \sigma(x)v(x) = -k_0(x)(\tilde{u}_0 - u_0) \ge 0 \quad (x \in \Gamma).$$

It follows from the theorem on the regularity of solutions of elliptic problems, from the maximum principle, and from the Hopf lemma (if $\gamma > 0$) that $v(x) \ge 0$ for $x \in \overline{Q}$. Since $m(x) \ge 0$, we have $\tilde{\psi}_m \ge \psi_m$. Using the monotonicity of the operator $\mathcal{H}$ (see Lemma 3.2), we obtain $U(\tilde{u}_0) \le U(u_0)$.

Consider problem (4.1)–(4.3) with $F = 0$ and particular boundary conditions (cf. [5, 10]):

$$w_t(x,t) = Pw(x,t) \quad ((x,t) \in Q_T) \tag{7.10}$$

$$w(x,0) = \varphi(x) \quad (x \in Q), \tag{7.11}$$

$$\gamma \frac{\partial w}{\partial \nu} + \sigma(x)(w(x,t) - w_e(x)) = K(x)(u(t) - u_c) \quad ((x,t) \in \Gamma_T), \tag{7.12}$$

where γ and $\sigma(x)$ are the same as above, $w_e \in C^\infty(\Gamma)$ is the ambient temperature, $u_c \in \mathbb{R}$ is a "critical" value of the control function $u(t)$, $K \in C^\infty(\Gamma)$ is an amplification coefficient.

The following corollary directly results from Theorem 7.2.

Corollary 7.1 *Let Condition* 3.1 *hold. Then problem* (7.10), (7.12), (4.4) *(with* $\tau \ge 0$*) has a stationary solution* (ψ, u_0)*; moreover,* $u_0 \in [0,1]$*. If* $m(x) \ge 0$ *and* $K(x) \ge 0$*, then the stationary solution is unique.*

In conclusion of this section, we note that the thermocontrol problem may have no periodic solutions (different from stationary ones). As an example, we consider the following particular case of relations (7.10), (7.12):

$$w_t(x,t) = \Delta w(x,t) - p_0 w(x,t) \quad ((x,t) \in Q_T), \tag{7.13}$$

$$\frac{\partial w}{\partial \nu} = K(x)(u(t) - u_c) \quad ((x,t) \in \Gamma_T), \tag{7.14}$$

where p_0 is a positive constant. Let $m(x) \equiv 1$ and $\int_\Gamma K(x)\,dx \geq 0$.

Denote

$$K_0 = \int_\Gamma K(x)\,dx.$$

By assumption, $K_0 \geq 0$. First, we show that *if* (w,u) *is a periodic solution of problem* (7.13)*,* (7.14)*,* (4.4) (*with* $\tau \geq 0$)*, then*

$$-K_0 u_c/p_0 \leq w_m(t) \leq -K_0 u_c/p_0 + K_0/p_0 \quad (t \geq 0). \tag{7.15}$$

Indeed, assume that (w,u) is a T-periodic solution of problem (7.13), (7.14), (4.4). We extend the solution (w,u) for all $t \geq 0$ by periodicity. It follows from Theorems 4.2 and 6.2 (uniqueness of solutions) and from the periodicity of (w,u) that Eq. (4.4) (with $\tau \geq 0$) holds for all $t \geq 0$:

$$au'(t) + u(t) = \mathcal{H}(w_m(\cdot - \tau)) \quad (t \geq 0). \tag{7.16}$$

Further, integrating Eq. (7.13) over Q and using the integration-by-parts formula and relation (7.14), we obtain the following ordinary differential equation for $w_m(t)$:

$$w_m'(t) + p_0 w_m(t) = K_0(u(t) - u_c) \quad (t \geq 0). \tag{7.17}$$

In particular, it follows from (7.17) and (7.16) that w_m is twice continuously differentiable. A periodic solution of Eq. (7.17) has the form

$$w_m(t) = K_0 \int_0^\infty e^{-p_0 s}(u(t-s) - u_c)\,ds,$$

where $u(t)$ is extended to $\mathbb{R}$ by periodicity. Combining this formula with the relations $0 \leq u(t) \leq 1$ (see Theorems 5.1 and 7.1), we obtain (7.15).

Denote

$$\alpha_1 = \inf_{\rho \in \operatorname{supp}\mu} \rho_1, \quad \alpha_2 = \sup_{\rho \in \operatorname{supp}\mu} \rho_2,$$
$$\beta_1 = \sup_{\rho \in \operatorname{supp}\mu} \rho_1, \quad \beta_2 = \inf_{\rho \in \operatorname{supp}\mu} \rho_2.$$

We claim that *if*

$$\alpha_1 > -\infty \quad \textit{and} \quad -K_0 u_c/p_0 + K_0/p_0 \leq \alpha_1 \tag{7.18}$$

or

$$\alpha_2 < +\infty \quad \text{and} \quad -K_0u_c/p_0 \geq \alpha_2, \tag{7.19}$$

or

$$\beta_1 \leq -K_0u_c/p_0 \leq -K_0u_c/p_0 + K_0/p_0 \leq \beta_2, \tag{7.20}$$

then any periodic solution of problem (7.13), (7.14), (4.4) *is a stationary solution.*

Indeed, in each of these cases, we have $\mathcal{H}(w_m(\cdot - \tau))(t) = \text{const}$ for $t \geq \tau$ due to (7.15). Therefore, $u(t) = u_0 = \text{const}$ for $t \geq \tau$ (and hence for $t \geq 0$) due to (7.16), where $u_0 \in [0, 1]$. Substituting $u(t) \equiv u_0$ into Eq. (7.17), we see that $w_m(t) = -K_0u_c/p_0 + K_0u_0/p_0$ for $t \geq 0$.

On the other hand, if $\psi(x)$ is the solution of the elliptic problem

$$\begin{aligned} &\Delta\psi(x) - p_0\psi(x) = 0 \quad (x \in Q), \\ &\frac{\partial\psi}{\partial\nu} = K(x)(u_0 - u_c) \quad (x \in \Gamma), \end{aligned}$$

then $(\psi(x), u_0)$ is a T-periodic solution (with any T) of problem (7.13), (7.14), (4.4) with the mean temperature $\psi_m = -K_0u_c/p_0 + K_0u_0/p_0$. Due to Remarks 5.5 and 7.1 , we have $w(x,t) \equiv \psi(x)$.

It is easy to check that if condition (7.18) holds, then $u_0 = 1$; if condition (7.19) holds, then $u_0 = 0$; if condition (7.20) holds, then $u_0 \in [0, 1]$. In all these cases, we have $\mathcal{H}(\psi_m) = u_0$.

8 Large Time Behavior of Solutions of Thermocontrol Problems

8.1 Existence of a global B-attractor

As before, the term “thermocontrol problem” refers to problem (4.1)–(4.6) if $\tau > 0$ and problem (6.1)–(6.5) if $\tau = 0$.

In this section, we study the large-time behavior of solutions for the thermocontrol problem (with $\tau \geq 0$) under the following assumptions.

Condition 8.1 *(1) If $\tau > 0$, then Condition* 3.1 *holds*; *if $\tau = 0$, then Condition* 6.1 *holds.*

(2) Condition 3.2 *holds.*

(3) The right-hand side F satisfies (4.8) *and the relation*

$$\hat{F} = \sup_{t \geq 0,\, \psi \in L_2(Q),\, u \in \mathbb{R}} \|F(\cdot, t, \psi(\cdot), u)\|_{L_2(Q)} < \infty. \tag{8.1}$$

We fix an initial configuration $\xi \in \mathcal{R}$ of the operator $\mathcal{H}$ and an arbitrary number $t_0 > \tau$ and consider the family $\{\mathbf{V}_t\}_{t\ge t_0} = \{\mathbf{V}_{t,\xi,F}\}_{t\ge t_0}$ of nonlinear operators $\mathbf{V}_t = \mathbf{V}_{t,\xi,F} : \mathcal{V}_\tau \to \mathcal{V}_\tau$ given by

$$\mathbf{V}_t(\mathbf{v}) = \mathbf{V}_{t,\xi,F}(\mathbf{v}) = \begin{cases} \big(w(\cdot,t), u(t), w_m(\cdot+t)|_{[-\tau,0]}\big) & \text{for } \tau > 0, \\ (w(\cdot,t), u(t)) & \text{for } \tau = 0, \end{cases}$$

where (w,u) is the solution of the thermocontrol problem with the initial data $\mathbf{v} \in \mathcal{V}_\tau$, right-hand side $F(x,t,w(x,t),u)$, and the initial configuration $\xi \in \mathcal{R}$.

Remark 8.1 *We have*

$$\mathbf{V}_{t_1+t_2,\xi,F}(\mathbf{v}) = \mathbf{V}_{t_2,\zeta,G}(\mathbf{V}_{t_1,\xi,F}(\mathbf{v})) \quad \forall t_1, t_2 \ge t_0,$$

where $\zeta_\rho = h_\rho(w_m,\xi_\rho,0)(t_1)$ and $G(x,t,\psi,q) = F(x,t+t_1,\psi,q)$. Thus, the family $\{\mathbf{V}_{t,\xi,F}\}_{t\ge t_0}$ does not form a semigroup even if $F \equiv 0$ (since ζ need not coincide with ξ).

Definition 8.1 *The family $\{\mathbf{V}_t\}_{t\ge t_0}$ is said to be* continuous *if the mapping $[t_0,\infty) \times \mathcal{V}_\tau \ni (t,\mathbf{v}) \mapsto \mathbf{V}_t(\mathbf{v}) \in \mathcal{V}_\tau$ is continuous.*

Lemma 8.1 *Let Condition* 8.1 *hold. Then the following assertions are true.*

(1) The operators $\mathbf{V}_t : \mathcal{V}_\tau \to \mathcal{V}_\tau$ are continuous and compact.
(2) The family $\{\mathbf{V}_t\}_{t\ge t_0}$ is continuous.

PROOF. I. The continuity of the operators $\mathbf{V}_t : \mathcal{V}_\tau \to \mathcal{V}_\tau$ follows from Theorem 6.4. The compactness follows from (4.20), (4.22), (4.23), and (4.26) if $\tau > 0$ and from (6.17) and (6.19) if $\tau = 0$; in both cases, the compactness of the embedding $W_2^2(Q) \subset W_2^1(Q)$ and the Ascoli–Arzelà theorem should be applied.

II. To prove the continuity of the family $\{\mathbf{V}_t\}_{t\ge t_0}$, we fix $(t_1,\mathbf{v}_1) \in [t_0,\infty)\times\mathcal{V}_\tau$ and $\varepsilon > 0$. Consider an arbitrary element $(t,\mathbf{v}) \in [t_0,\infty)\times\mathcal{V}_\tau$.

We have

$$\|\mathbf{V}_t(\mathbf{v}) - \mathbf{V}_{t_1}(\mathbf{v}_1)\|_{\mathcal{V}_\tau} \le \|\mathbf{V}_t(\mathbf{v}_1) - \mathbf{V}_{t_1}(\mathbf{v}_1)\|_{\mathcal{V}_\tau} + \|\mathbf{V}_t(\mathbf{v}_1) - \mathbf{V}_t(\mathbf{v})\|_{\mathcal{V}_\tau}. \tag{8.2}$$

Due to Theorems 4.2 and 6.2 with $T = 2t_1$,

$$\|\mathbf{V}_t(\mathbf{v}_1) - \mathbf{V}_{t_1}(\mathbf{v}_1)\|_{\mathcal{V}_\tau} \le \varepsilon/2 \tag{8.3}$$

whenever $|t - t_1| \le \delta_1$, where $\delta_1 = \delta_1(t_1,\mathbf{v}_1) > 0$ is sufficiently small.

On the other hand, Theorem 6.4 implies that

$$\|\mathbf{V}_t(\mathbf{v}_1) - \mathbf{V}_t(\mathbf{v})\|_{\mathcal{V}_\tau} \le \varepsilon/2 \tag{8.4}$$

whenever $t \le 2t_1$ and $\|\mathbf{v} - \mathbf{v}_1\|_{\mathcal{V}_\tau} \le \delta_2$, where $\delta_2 = \delta_2(t_1) > 0$ is sufficiently small.

Inequalities (8.2)–(8.4) prove assertion 2.

Definition 8.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathcal{V}_\tau$. We say that $\mathcal{A}$* attracts *$\mathcal{B}$ if for every $\varepsilon > 0$ there exists a number $t_1 = t_1(\varepsilon, \mathcal{B}) > t_0$ such that $\mathbf{V}_t(\mathcal{B})$ lies in the ε-neighborhood of $\mathcal{A}$ for all $t \ge t_1$.*

The set $\mathcal{A}$ is called a global B-attractor *(of the family $\{\mathbf{V}_t\}_{t\ge t_0}$) if $\mathcal{A}$ attracts each bounded set $\mathcal{B}$.*

The family $\{\mathbf{V}_t\}_{t\ge t_0}$ is called B-dissipative *if it has a bounded global B-attractor.*

Lemma 8.2 *Let Condition* 8.1 *hold. Then the family $\{\mathbf{V}_t\}_{t\ge t_0}$ is B-dissipative.*

PROOF. I. First, we note that any solution $u(t)$ of Eq. (4.4) satisfies the inequalities

$$u_0 e^{-t/a} \le u(t) \le (u_0 - 1)e^{-t/a} + 1 \quad \forall t \ge 0,$$

which follows from (4.27) and from the representation (4.28). Therefore, for any $\varepsilon > 0$, there is $t'(\varepsilon) \ge 0$ such that

$$-\varepsilon \le u(t) \le 1 + \varepsilon \qquad \forall t \ge t'(\varepsilon). \tag{8.5}$$

Now we fix an arbitrary $\varepsilon \in (0,1)$ and an arbitrary $M_1 > 0$ and consider a bounded set $\mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon(M_1) \subset \mathcal{V}_\tau$ consisting of the elements

$$\mathbf{v} = \begin{cases} (\varphi, u_0, g) & \text{for } \tau > 0, \\ (\varphi, u_0) & \text{for } \tau = 0 \end{cases}$$

such that $\|\varphi\|_{W_2^1(Q)} \le M_1$, $-\varepsilon \le u_0 \le 1 + \varepsilon$, and $\|g\|_{C[-\tau,0]} \le M_1$ if $\tau > 0$.

We will find a set $\mathcal{A}_\varepsilon \subset \mathcal{V}_\tau$ which attracts the sets $\mathcal{B}_\varepsilon(M_1)$ for all $M_1 > 0$.

Denote

$$A_\varepsilon = \left[\max\left(1+\varepsilon, a^{-1}(2+\varepsilon)\right) + \hat{F}\right],$$

where $\hat{F}$ is given by (8.1). Clearly, $A(u_0) \le A_\varepsilon$ for $-\varepsilon \le u_0 \le 1 + \varepsilon$, where $A(u_0)$ is given by (4.24).

Set $c = \sup_{T\in[1,2]} c_4(T)$, where $c_4(T) > 0$ is the constant occurring in (4.20) and (6.15). It follows from Theorems 4.2 and 6.2 that $c < \infty$.

Set

$$M_\varepsilon = \frac{cA_\varepsilon}{1-e^{-\omega}}.$$

Let (w, u) be the solution of the thermocontrol problem with an initial data $\mathbf{v}_1 \in \mathcal{B}_\varepsilon$ and the initial configuration $\xi \in \mathcal{R}$.

Let $t \in [1, 2]$. Then, due to (4.20) and (6.15), we have

$$\|w(\cdot,t)\|_{W_2^1(Q)} \le cA_\varepsilon + e^{-\omega}M_1 = M_\varepsilon + e^{-\omega}(M_1 - M_\varepsilon) = M_2 \quad \forall t \in [1,2]. \quad (8.6)$$

Let $t \in [2, 3]$. Then $(w(\cdot, t), u(t))$ coincides with the solution of the thermocontrol problem, at the moment $t - 1 \in [1, 2]$, with the initial data

$$\mathbf{v}_2 = \begin{cases} \left(w(\cdot,1), u(1), w_m(\cdot+1)|_{[-\tau,0]}\right) & \text{for } \tau > 0, \\ (w(\cdot,1), u(1)) & \text{for } \tau = 0, \end{cases}$$

the right-hand side $G(x, t, \psi(x), q) = F(x, t + 1, \psi(x), q)$, and the initial configuration $\zeta_\rho = h_\rho(w_m, \xi_\rho, 0)(1)$.

Note that the constant c does not depend on the initial configuration of the operator $\mathcal{H}$ and does not increase under the change of F for G. Hence, using (8.5), (8.6), (4.20), and (6.15), we obtain

$$\|w(\cdot,t)\|_{W_2^1(Q)} \le cA_\varepsilon + e^{-\omega}M_2 = M_\varepsilon + e^{-\omega}(M_2 - M_\varepsilon) = M_3 \quad \forall t \in [2,3].$$

Continuing this process, we see that

$$\|w(\cdot,t)\|_{W_2^1(Q)} \le M_{k+1} \quad \forall t \in [k, k+1],\ k = 1, 2, 3, \dots, \quad (8.7)$$

where $M_{k+1} = M_\varepsilon + e^{-\omega}(M_k - M_\varepsilon)$.

Further, using the Schwartz inequality, we have

$$|w_m(t)| \le \|m\|_{L_2(Q)}\|w(\cdot,t)\|_{L_2(Q)} \le K\|m\|_{L_2(Q)}\|w(\cdot,t)\|_{W_2^1(Q)} \quad \forall t \ge 0, \quad (8.8)$$

where K is the norm of the embedding operator $W_2^1(Q) \to L_2(Q)$ (recall that the norm in $W_2^1(Q)$ is given by (2.4)).

Since $M_k \to M_\varepsilon$ as $k \to \infty$, it follows from (8.5), (8.7), and (8.8) that the set

$$\mathcal{A}_\varepsilon = \begin{cases} \{(\varphi, u_0, g) \in \mathcal{V}_\tau : \ \|\varphi\|_{W_2^1(Q)} \le M_\varepsilon, \ -\varepsilon \le u_0 \le 1+\varepsilon, \\ \qquad\qquad \|g\|_{C[-\tau,0]} \le KM_\varepsilon \|m\|_{L_2(Q)}\} & \text{for } \tau > 0, \\ \{(\varphi, u_0) \in \mathcal{V}_0 : \|\varphi\|_{W_2^1(Q)} \le M_\varepsilon, \ -\varepsilon \le u_0 \le 1+\varepsilon\} & \text{for } \tau = 0, \end{cases}$$

attracts the set $\mathcal{B}_\varepsilon$.

II. Now we consider an arbitrary bounded set $\mathcal{B} \subset \mathcal{V}_\tau$. Let (w, u) be the solution of the thermocontrol problem with the initial data $\mathbf{v}_1 \in \mathcal{B}$ and the initial configuration $\xi \in \mathcal{R}$. It follows from (8.5) that, for any $\varepsilon > 0$, there is a number $T = T(\mathcal{B}, \varepsilon) \ge t_0$ such that $-\varepsilon \le u(T) \le 1+\varepsilon$, i.e.,

$$\mathbf{v} = \mathbf{V}_T(\mathbf{v}_1) \in \mathcal{B}_\varepsilon,$$

where $\mathcal{B}_\varepsilon$ is the set considered in step I of the proof.

Since

$$\mathbf{V}_t(\mathbf{v}_1) = \mathbf{V}_{t-T,\zeta,G}(\mathbf{v}) \quad \forall t \ge T + t_0,$$

where $\zeta_\rho = h_\rho(w_m, \xi_\rho, 0)(T)$ and $G(x, t, \psi(x), q) = F(x, t+T, \psi(x), q)$, it follows that the set $\mathcal{A}_\varepsilon$ attracts the set $\mathcal{B}$.

Since $\varepsilon > 0$ is arbitrary, we see that the set $\mathcal{A}$ equal to

$$\{(\varphi, u_0, g) \in \mathcal{V}_\tau : \|\varphi\|_{W_2^1(Q)} \le M, \ 0 \le u_0 \le 1, \ \|g\|_{C[-\tau,0]} \le KM\|m\|_{L_2(Q)}\}$$

if $\tau > 0$ and to

$$\{(\varphi, u_0) \in \mathcal{V}_0 : \|\varphi\|_{W_2^1(Q)} \le M, \ 0 \le u_0 \le 1\}$$

if $\tau = 0$, where

$$M = \frac{cA}{1 - e^{-\omega}}, \qquad A = \left[\max\left(1, 2a^{-1}\right) + \hat{F}\right],$$

attracts the set $\mathcal{B}$.

8.2 Existence of a compact connected minimal global B-attractor

In this subsection, we prove the existence of a minimal global B-attractor.

Definition 8.3 *A set $\mathcal{A} \subset \mathcal{V}_\tau$ is called the* minimal global B-attractor (of the family $\{\mathbf{V}_t\}_{t \ge t_0}$) *if $\mathcal{A}$ is closed, is a global B-attractor, and any other closed global B-attractor contains $\mathcal{A}$.*

Let us introduce the notion of ω-limit sets.

Definition 8.4 *The ω-limit set $\omega(\mathcal{A})$ for a set $\mathcal{A} \subset \mathcal{V}_\tau$ is the set of the limits of all converging sequences of the form $\mathbf{V}_{t_k}(\mathbf{v}_k)$, where $\mathbf{v}_k \in \mathcal{A}$ and the sequence $\{t_k\}$ is increasing and converging to $+\infty$.*

Denote

$$\gamma_t(\mathbf{v}) = \{\mathbf{V}_\tau(\mathbf{v}) : \tau \geq t\}, \quad t \geq t_0, \ \mathbf{v} \in \mathcal{V}_\tau;$$
$$\gamma_t(\mathcal{A}) = \bigcup_{\mathbf{v}\in\mathcal{A}} \gamma_t(\mathbf{v}), \quad \mathcal{A} \subset \mathcal{V}_\tau.$$

One can easily verify that

$$\omega(\mathcal{A}) = \bigcap_{t\geq T} \overline{\gamma_t(\mathcal{A})}, \tag{8.9}$$

where $T \geq t_0$ is an arbitrary fixed number.

Theorem 8.1 *Let Condition* 8.1 *hold. Then the family $\{\mathbf{V}_t\}_{t\geq t_0}$ has a minimal global B-attractor, which is compact and connected.*

PROOF. I. The proof of this theorem is based on Lemmas 8.1 and 8.2 and exploits the ideas of [15, Part I], where attractors of semigroups are studied.

It follows from Lemma 8.2 that there exists a bounded set $\mathcal{B}_0 \subset \mathcal{V}_\tau$ such that for any bounded set $\mathcal{B} \subset \mathcal{V}_\tau$

$$\mathbf{V}_t(\mathcal{B}) \subset \mathcal{B}_0 \qquad \forall t \geq T, \tag{8.10}$$

where $T = T(\mathcal{B}) \geq t_0$ is sufficiently large.

We claim that $\omega(\mathcal{B}_0)$ is the minimal global B-attractor, which is compact and connected.

II. First, we note that, due to (8.10), we have $\mathbf{V}_t(\mathcal{B}_0) \subset \mathcal{B}_0$ for all $t \geq T_0$, where $T_0 \geq T$ is sufficiently large.

Suppose that the set $\mathcal{B}_0$ lies in the ball of radius R centered at the origin.

Fix a number $t_1 \geq t_0$. Let us show that the set $\gamma_{t_1+T_0}(\mathcal{B}_0)$ is precompact. Indeed,

$$\gamma_{t_1+T_0}(\mathcal{B}_0) = \bigcup_{s\geq T_0} \mathbf{V}_{t_1,\zeta_s,G_s}(\mathbf{V}_s(\mathcal{B}_0)),$$

where $\zeta_{s\rho} = h_\rho(w_m, \xi_\rho, 0)(s)$ and $G_s(x,t,\psi(x),u) = F(x,t+s,\psi(x),u)$. Therefore, for any element

$$\mathbf{w} = \begin{cases} (\psi, u_1, r) & \text{for } \tau > 0, \\ (\psi, u_1) & \text{for } \tau = 0 \end{cases}$$

from $\gamma_{t_1+T_0}(\mathcal{B}_0)$, there is an element $\mathbf{v} \in \mathbf{V}_s(\mathcal{B}_0) \subset \mathcal{B}_0$ such that

$$\mathbf{w} = \mathbf{V}_{t_1,\zeta,G}(\mathbf{v}),$$

where $\zeta \in \mathcal{R}$ and the function G possesses the same properties as F does. Applying Theorems 4.2 and 6.2, we obtain

$$\|\psi\|_{W_2^2(Q)} \le k_1, \qquad |u_1| \le \max(1, R),$$

$$\|r\|_{C[-\tau,0]} \le k_2, \qquad |r(s_2) - r(s_1)| \le k_3|s_2 - s_1|^{1/2} \quad \forall s_1, s_2 \in [-\tau, 0]$$

(the latter two inequalities are absent if $\tau = 0$), where $k_1, k_2, k_3 > 0$ depend on t_1 and R but do not depend on $\mathbf{w} \in \gamma_{t_1+T_0}(\mathcal{B}_0)$. Using the compactness of the embedding $W_2^2(Q) \subset W_2^1(Q)$ and the Ascoli–Arzelà theorem, we see that the set $\gamma_{t_1+T_0}(\mathcal{B}_0)$ is precompact.

On the other hand, $\gamma_{t_2+T_0}(\mathcal{B}_0) \subset \gamma_{t_1+T_0}(\mathcal{B}_0)$ for all $t_2 > t_1 \ge t_0$. Therefore, the set

$$\omega(\mathcal{B}_0) = \bigcap_{t \ge t_0+T_0} \overline{\gamma_t(\mathcal{B}_0)}$$

is the intersection of the ordered family of compact sets. Hence, $\omega(\mathcal{B}_0)$ is nonempty, compact, and attracts $\mathcal{B}_0$. Due to (8.10), $\omega(\mathcal{B}_0)$ attracts any bounded set $\mathcal{B} \subset \mathcal{V}_\tau$, i.e., it is a global B-attractor.

The minimality of the global B-attractor $\omega(\mathcal{B}_0)$ and its connectedness are proved in the same way as in the proof of Theorem 2.1 in [15, Part I].

8.3 Physical interpretation

Fix some numbers $\rho_1^* < \rho_2^*$ and $\delta > 0$. Let the support $\operatorname{supp}\mu$ of the measure μ be a subset of the set

$$\mathcal{P}^* = \mathcal{P} \cap [\rho_1^* - \delta, \rho_1^* + \delta] \times [\rho_2^* - \delta, \rho_2^* + \delta].$$

The goal of this subsection is to formulate sufficient conditions under which the minimal global B-attractor has a nonempty intersection with the set

$$\{(\psi, u_0, r) \in \mathcal{V}_\tau : \psi_m \in (\rho_1^* - \delta, \rho_2^* + \delta)\}, \ \tau > 0$$
$$(\{(\psi, u_0) \in \mathcal{V}_0 : \psi_m \in (\rho_1^* - \delta, \rho_2^* + \delta)\}, \ \tau = 0)$$

and to provide the physical interpretation.

We assume that $F(x, t, \psi, q) \equiv f(x, t, q)$,

$$f(x, t, q) \text{ is uniformly continuous in } \overline{Q_T} \times \mathbb{R} \tag{8.11}$$

and there exist functions $F_1, F_2 \in L_2(Q)$ such that

$$\lim_{t\to\infty} \|f(\cdot,t,1) - F_1\|_{L_2(Q)} = 0, \qquad \lim_{t\to\infty} \|f(\cdot,t,0) - F_2\|_{L_2(Q)} = 0. \tag{8.12}$$

We consider the following two auxiliary elliptic boundary-value problems:

$$\begin{aligned} -Pw_1(x) &= F_1(x) && (x \in Q), \\ -\gamma \frac{\partial w_1}{\partial \nu} &= \sigma(x) w_1(x) + k_0(x) + k_1(x) && (x \in \Gamma) \end{aligned} \tag{8.13}$$

and

$$\begin{aligned} -Pw_2(x) &= F_2(x) && (x \in Q), \\ -\gamma \frac{\partial w_2}{\partial \nu} &= \sigma(x) w_1(x) + k_1(x) && (x \in \Gamma). \end{aligned} \tag{8.14}$$

Due to the assumptions about the elliptic operator P and about the coefficients in the boundary conditions, each of these problems has a unique solution from $W_2^2(Q)$.

Assume that

$$w_{1m} > \rho_1^* - \delta, \qquad w_{2m} < \rho_2^* + \delta. \tag{8.15}$$

Theorem 8.2 *Let* $\operatorname{supp}\mu \subset \mathcal{P}^*$, $F(x,t,\psi,q) \equiv f(x,t,q)$, *the function* f *satisfy conditions* (8.11) *and* (8.12), *and inequalities* (8.15) *hold. Then, for any* $T_1 > 0$, *there is a number* $T \geq T_1$ *such that* $w_m(T) \in (\rho_1^* - \delta, \rho_2^* + \delta)$, *where* (w,u) *is a solution of the thermocontrol problem.*

PROOF. Suppose that $w_m(t) \leq \rho_1^* - \delta$ for all sufficiently large t (the case $w_m(t) \geq \rho_2^* + \delta$ is treated analogously). Then $w_m(t) \leq \rho_1$ for any $\rho \in \mathcal{P}^*$. Therefore

$$\mathcal{H}(w_m(\cdot - \tau))(t) = \int_{\mathcal{P}^*} h_\rho(w_m(\cdot - \tau), \xi_\rho, 0)(t)\, d\mu(\rho) = \mu(\mathcal{P}^*) = 1$$

for all sufficiently large t. Hence, $u(t) \to 1$ as $t \to +\infty$. Reducing problem (4.1)–(4.3) to that with the homogeneous boundary conditions, taking into account that

$$\lim_{t\to\infty} \|f(\cdot,t,u(t)) - F_1(\cdot)\|_{L_2(Q)} = 0,$$

and applying Theorem 4.4 in [18, Chap. 4] (about the asymptotic behavior of solutions of abstract Cauchy problems), we obtain

$$\|w(\cdot,t) - w_1\|_{L_2(Q)} \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore, using the Schwartz inequality, we have

$$|w_m(t) - w_{1m}| \to 0 \quad \text{as} \quad t \to \infty.$$

Combining this relation with (8.15), we see that $w_m(t) > \rho_1^* - \delta$ for all sufficiently large t, and we have obtained the contradiction.

As an example, let us consider the thermocontrol problem with $F = 0$ and the particular boundary condition

$$\gamma \frac{\partial w}{\partial \nu} + \sigma(x)(w(x,t) - w_e(x)) = K(x)(u(t) - u_c) \quad ((x,t) \in \Gamma_T), \tag{8.16}$$

where γ and $\sigma(x)$ are the same as above, $w_e \in C^\infty(\Gamma)$ is the ambient temperature, $u_c \in (0,1)$ is a "critical" value of the control function $u(t)$, $K \in C^\infty(\Gamma)$ is a nonnegative amplification coefficient (cf. problem (7.10)–(7.12) and the papers [5, 10]). We also assume that $m(x) \geq 0$ and $m \neq 0$ (as an element of $L_\infty(Q)$).

Corollary 8.1 *There is a number $K_0 > 0$ such that if $K(x) \geq K_0$ $(x \in \Gamma)$, then for any $T_1 > 0$ there is a number $T \geq T_1$ such that $w_m(T) \in (\rho_1^* - \delta, \rho_2^* + \delta)$, where (w,u) is a solution of the thermocontrol problem with boundary condition* (8.16).

PROOF. In the particular case under consideration, auxiliary elliptic problems (8.13) and (8.14) reduce to the following problems:

$$\begin{aligned} -Pw_1(x) &= 0 && (x \in Q), \\ \gamma \frac{\partial w_1}{\partial \nu} + \sigma(x) w_1 &= \sigma(x) w_e + K(x)\,(1 - u_c) && (x \in \Gamma) \end{aligned} \tag{8.17}$$

and

$$\begin{aligned} -Pw_2(x) &= 0 && (x \in Q), \\ \gamma \frac{\partial w_2}{\partial \nu} + \sigma(x) w_2 &= \sigma(x) w_e - K(x) u_c && (x \in \Gamma). \end{aligned} \tag{8.18}$$

Clearly, one can choose $K_0 > 0$ in such a way that if $K(x) \geq K_0$ $(x \in \Gamma)$, then

$$\sigma(x) w_e(x) - K(x) u_c < 0 < \sigma(x) w_e(x) + K(x)\,(1 - u_c) \quad (x \in \Gamma). \tag{8.19}$$

In this case, the maximum principle and the Hopf lemma (if $\gamma > 0$) imply that $w_2(x) < 0 < w_1(x)$ for $x \in \overline{Q}$, where w_1 and w_2 are the solutions of problems (8.17) and (8.18), respectively. Therefore, if K_0 is sufficiently large and $K(x) \geq K_0$ $(x \in \Gamma)$, then inequalities (8.15) hold. Now it remains to apply Theorem 8.2.

Remark 8.2 *It follows from Corollary* 7.1 *that the thermocontrol problem with boundary condition* (8.16) *has a unique stationary solution (ψ, u_0) and $u_0 \in [0,1]$. Corollary* 8.1 *shows that if the amplification coefficient $K(x)$ is sufficiently large, then $\psi_m \in (\rho_1^* - \delta, \rho_2^* + \delta)$.*

9 Open questions

As a conclusion, we formulate some open questions which we did not address in the present work.

(1) We have shown the existence of stationary solutions of the thermocontrol problems, provided that the right-hand side of the parabolic equation does not depend on t. Find out if there are periodic solutions different from stationary solutions in this case.
(2) Study the stability of the above stationary solutions.
(3) Study the stability of periodic solutions.
(4) Investigate the structure of the global attractor in more detail.

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