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Nonlocal problems for elliptic equations in dihedral angles and the Green formula[1]

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Contents

Introduction

In the theory of nonlocal elliptic boundary value prolems in bounded domains, the most difficult case deals with the situation when support of nonlocal terms intersects with boundary of a domain (see [1]–[5]). This leads to apearance of degree singularities for solutions near some set. Therefore it is natural to consider nonlocal elliptic problems in weighted spaces (see [6]–[8]). In order to establish a priori estimates of solutions and construct a right regularizer for nonlocal problems in bounded domains, one must study nonlocal problems in dihedral and plane angles (see [4, 5]).

In paper [4], A.L. Skubachevskii found sufficient conditions of Fredholm solvability[2] for auxiliary nonlocal problems with parameter θ in plane angles and sufficient conditions of one–valued solvability for model nonlocal problems in dihedral angles. His consideration was based on a priori estimates of solutions and on using a right regularizer which needed some additional conditions on a corresponding "local" model problem.

In the present work, we use another approach. Instead of constructing a right regularizer, we obtain the Green formula and study adjoint nonlocal problems. This leads to nonlocal transmission problems in dihedral and plane angles. Similar problems were studied in [9, 10] for the case of smooth boundary of a domain, in [11] for the one-dimensional case, etc.

Our approach allows to establish 1) a necessary and sufficient condition of Fredholm solvability for auxiliary nonlocal problems with parameter θ in plane angles (Theorem 9.1); 2) necessary conditions of Fredholm solvability and sufficient conditions of one–valued solvability for model nonlocal problems in dihedral angles (Theorems 9.2, 9.3).

The paper is organized as follows. In §§1–3, we consider nonlocal boundary value problems in plane and dihedral angles. A priori estimates in weighted spaces are established. For reader's convinience, we formulate a number of results from the paper [4]. In §4, we obtain the Green formulas for nonlocal elliptic problems. The Green formulas generate nonlocal transmission problems, which are formally adjoint to nonlocal boundary value problems. Nonlocal transmission problems are studied in §§5–7. We prove the results that are analogous to those from §§1–3. §8 deals with operators that are adjoint to operators of nonlocal boundary value problems. Connection between adjoint operators and formally adjoint nonlocal transmission problems is considered. The main results are collected in §9 where we study solvability of nonlocal boundary value problems in plane and dihedral angles. §10 illustrates the results obtained in this work: we investigate the one–valued solvability of nonlocal problems for the Poisson equation in dihedral angles. The paper has two appendices.

[2] A closed operator $\mathcal{A}$ acting from a Hilbert space H_1 into a Hilbert space H_2 is said to be *Fredholm* if its range $\mathcal{R}(\mathcal{A})$ is closed, dimension of its kernel $\dim\ker(\mathcal{A})$ and codimension of its range $\operatorname{codim}\mathcal{R}(\mathcal{A})$ are finite. The number $\operatorname{ind}\mathcal{A} = \dim\ker(\mathcal{A}) - \operatorname{codim}\mathcal{R}(\mathcal{A})$ is called *index* of the Fredholm operator $\mathcal{A}$.

Appendix A deals with the operator that is adjoint to the operator of elliptic problem in $\mathbb{R}^n$ with additional conditions on the hyperplane $\{x_n = 0\}$. We prove a theorem concerning smoothness of solutions for the corresponding problem. This result is used in §8. In Appendix B, we prove some auxiliary properties of weighted spaces that are needed in the main part of the paper.

1 Nonlocal elliptic boundary value problems. Reduction to problems with homogeneous nonlocal conditions

1 Nonlocal problems in dihedral angles.

Introduce the sets

$$M = \{x = (y,\ z):\ y = 0,\ z \in \mathbb{R}^{n-2}\},$$

$$\Omega_j = \{x = (y,\ z):\ r > 0,\ b_{j1} < \varphi < b_{j,R_j+1},\ z \in \mathbb{R}^{n-2}\},$$

$$\Omega_{jt} = \{x = (y,\ z):\ r > 0,\ b_{jt} < \varphi < b_{j,t+1},\ z \in \mathbb{R}^{n-2}\}\ (t = 1,\ \dots,\ R_j),$$

$$\Gamma_{jq} = \{x = (y,\ z):\ r > 0,\ \varphi = b_{jq},\ z \in \mathbb{R}^{n-2}\}\ (q = 1,\ \dots,\ R_j + 1).$$

Here $x = (y,\ z) \in \mathbb{R}^n,\ y \in \mathbb{R}^2,\ z \in \mathbb{R}^{n-2}$; $r,\ \varphi$ are the polar coordinates of a point y; $R_j \geq 1$ is an integers; $0 < b_{j1} < \dots < b_{j,R_j+1} < 2\pi$; $j = 1,\ \dots,\ N$.

Denote by $\mathcal{P}_j(D_y,\ D_z)$, $B_{j\sigma\mu}(D_y,\ D_z)$, and $B_{j\sigma\mu kqs}(D_y,\ D_z)$ homogeneous differential operators with constant complex coefficients of orders $2m$, $m_{j\sigma\mu} \leq 2m-1$ and $m_{j\sigma\mu} \leq 2m-1$ correspondingly ($j,\ k = 1,\ \dots,\ N$; $\sigma = 1,\ R_j+1$; $\mu = 1,\ \dots,\ m$; $q = 2,\ \dots,\ R_j$ $s = 1,\ \dots,\ S_{j\sigma kq}$).

We shall assume that the following conditions hold (see [12, Chapter 2, §§1.2, 1.4]).

Condition 1.1. *For all $j = 1,\ \dots,\ N$, the operators $\mathcal{P}_j(D_y,\ D_z)$ are properly elliptic.*

Condition 1.2. *For all $j = 1,\ \dots,\ N$; $\sigma = 1,\ R_j + 1$, the system $\{B_{j\sigma\mu}(D_y,\ D_z)\}_{\mu=1}^m$ is normal and covers the operator $\mathcal{P}_j(D_y,\ D_z)$ on $\Gamma_{j\sigma}$.*

Consider the N equations for functions $U_1,\ \dots,\ U_N$

$$\mathcal{P}_j(D_y,\ D_z)U_j = f_j(x) \quad (x \in \Omega_j) \tag{1.1}$$

with the nonlocal conditions

$$\mathcal{B}_{j\sigma\mu}(D_y,\ D_z)U = B_{j\sigma\mu}(D_y,\ D_z)U_j|_{\Gamma_{j\sigma}}+$$
$$+\sum_{k,q,s}(B_{j\sigma\mu kqs}(D_y,\ D_z)U_k)(\mathcal{G}_{j\sigma kqs}y,\ z)|_{\Gamma_{j\sigma}} = g_{j\sigma\mu}(x) \quad (x\in\Gamma_{j\sigma}) \tag{1.2}$$

$$(j=1,\ \dots,\ N;\ \sigma=1,\ R_j+1;\ \mu=1,\ \dots,\ m).$$

Here and below the summation in the formula for $\mathcal{B}_{j\sigma\mu}(D_y,\ D_z)$ is taken over $k=1,\ \dots,\ N$; $q=2,\ \dots,\ R_k$; $s=1,\ \dots,\ S_{j\sigma kq}$; $U=(U_1,\ \dots,\ U_N)$; $(B_{j\sigma\mu kqs}(D_y,\ D_z)U_k)(\mathcal{G}_{j\sigma kqs}y,\ z)$ means that the expression $(B_{j\sigma\mu kqs}(D_{y'},\ D_{z'})U_k)(x')$ is calculated for $x'=(\mathcal{G}_{j\sigma kqs}y,\ z)$; $\mathcal{G}_{j\sigma kqs}$ is the operator of rotation by the angle $\varphi_{j\sigma kq}$ and expansion by $\chi_{j\sigma kqs}$ times in the plane $\{y\}$ such that $b_{k1}<b_{j\sigma}+\varphi_{j\sigma kq}=b_{kq}<b_{k,R_k+1}$, $0<\chi_{j\sigma kqs}$.

We introduce the space $H_a^l(\Omega)$ as a completion of the set $C_0^\infty(\bar\Omega\backslash M)$ in the norm

$$\|w\|_{H_a^l(\Omega)} = \left(\sum_{|\alpha|\le l}\int\limits_{\Omega} r^{2(a-l+|\alpha|)}|D_x^\alpha w(x)|^2dx\right)^{1/2},$$

where $\Omega=\{x=(y,\ z):\ \ r>0,\ 0<b_1<\varphi<b_2<2\pi,\ z\in\mathbb{R}^{n-2}\}$, $C_0^\infty(\bar\Omega\backslash M)$ is the set of infinitely differentiable functions in $\bar\Omega$ with compact supports belonging to $\bar\Omega\backslash M$; $a\in\mathbb{R}$, $l\ge 0$ is an integer. Denote by $H_a^{l-1/2}(\Gamma)$ (for $l\ge 1$) the space of traces on an $(n-1)$-dimensional half-plane $\Gamma\subset\bar\Omega$ with the norm

$$\|\psi\|_{H_a^{l-1/2}(\Gamma)} = \inf\|w\|_{H_a^l(\Omega)} \quad (w\in H_a^l(\Omega):\ w|_\Gamma=\psi).$$

Introduce the spaces of vector–functions

$$H_a^{l+2m,\,N}(\Omega)=\prod_{j=1}^N H_a^{l+2m}(\Omega_j),\ H_a^{l,\,N}(\Omega,\ \Gamma)=\prod_{j=1}^N H_a^l(\Omega_j,\ \Gamma_j),$$

$$H_a^l(\Omega_j,\ \Gamma_j)=H_a^l(\Omega_j)\times\prod_{\sigma=1,\,R_j+1}\prod_{\mu=1}^m H_a^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma}).$$

We study solutions $U=(U_1,\ \dots,\ U_N)\in H_a^{l+2m,\,N}(\Omega)$ for problem (1.1), (1.2) supposing that $f=\{f_j,\ g_{j\sigma\mu}\}\in H_a^{l,\,N}(\Omega,\ \Gamma)$. Introduce the bounded operator corresponding to problem (1.1), (1.2)

$$\mathcal{L}=\{\mathcal{P}_j(D_y,\ D_z),\ \mathcal{B}_{j\sigma\mu}(D_y,\ D_z)\}:H_a^{l+2m,\,N}(\Omega)\to H_a^{l,\,N}(\Omega,\ \Gamma).$$

Lemma 1.1. *For any* $g_{j\sigma\mu} \in H_a^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})$ $(j = 1, \dots, N;\ \sigma = 1,\ R_j + 1;\ \mu = 1, \dots, m)$, *there exists a vector–function* $U \in H_a^{l+2m,\,N}(\Omega)$ *such that*

$$\mathcal{B}_{j\sigma\mu}(D_y,\ D_z)U = g_{j\sigma\mu}(x) \quad (x \in \Gamma_{j\sigma}),$$

$$\|U\|_{H_a^{l+2m,\,N}(\Omega)} \le c \sum_{j,\sigma,\mu} \|g_{j\sigma\mu}\|_{H_a^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})},$$

where $c > 0$ *is independent of* $g_{j\sigma\mu}$.

Lemma 1.1 is proved in [4, §1].

Let $W^l(Q)$ be a Sobolev space, where $Q \subset \mathbb{R}^n$ is an open domain with Lipschitz boundary. By $W^{l-1/2}(\Gamma)$ (for $l \ge 1$) we denote the space of traces on an $(n-1)$-dimensional smooth manifold $\Gamma \subset \bar{Q}$. Further we shall need interpolation inequalities for Sobolev and weighted spaces.

Lemma 1.2. *Let* Q *be bounded; then for any* $w \in W^l(Q)$ *and* $\lambda \in \mathbb{C}$, *we have*

$$|\lambda|^{l-s} \|w\|_{W^s(Q)} \le c_{ls}(\|w\|_{W^l(Q)} + |\lambda|^l \|w\|_{L_2(Q)}). \tag{1.3}$$

Here $0 < s < l;\ c_{ls} > 0$ *is independent of* $w,\ \lambda$.

Lemma 1.3. *Let* Q *be bounded; then for any* $w \in W^1(Q)$ *and* $\lambda \in \mathbb{C}$, *we have*

$$|\lambda|^{1/2} \|w|_{\partial Q}\|_{L_2(\partial Q)} \le c(\|w\|_{W^1(Q)} + |\lambda| \|w\|_{L_2(Q)}). \tag{1.4}$$

Here $c > 0$ *is independent of* $w,\ \lambda$.

Lemmas 1.2, 1.3 are proved in [13, Chapter 1, §1]. Using lemma 1.2 and properties of weighted spaces, one can establish the following result (see [2, §1]).

Lemma 1.4. *For any* $w \in H_a^l(\Omega)$ *and* $\lambda \in \mathbb{C}$, *we have*

$$|\lambda|^s \|w\|_{H_{a-s}^{l-s}(\Omega)} \le c_{ls}(\|w\|_{H_a^l(\Omega)} + |\lambda|^l \|w\|_{H_{a-l}^0(\Omega)}). \tag{1.5}$$

Here $0 < s < l;\ c_{ls} > 0$ *is independent of* $w,\ \lambda$.

2 Nonlocal problems with parameter θ in plane angles.

Now we consider the case of the space $\mathbb{R}^2$. Put $K=\{y\in\mathbb{R}^2:\ r>0,\ 0<b_1<\varphi<b_2<2\pi\}$. As above, we introduce the spaces $H_a^l(K)$ and $H_a^{l-1/2}(\gamma)$, where $\gamma\subset\bar K$ is a ray.

Let us also introduce the space $E_a^l(K)$ as a completion of the set $C_0^\infty(\bar K\backslash\{0\})$ in the norm

$$\|w\|_{E_a^l(K)}=\left(\sum_{|\alpha|\le l}\int\limits_K r^{2a}(r^{2(|\alpha|-l)}+1)|D_y^\alpha w(y)|^2dy\right)^{1/2}.$$

By $E_a^{l-1/2}(\gamma)$ (for $l\ge 1$) we denote the space of traces on a ray $\gamma\subset\bar K$ with the norm

$$\|\psi\|_{E_a^{l-1/2}(\gamma)}=\inf\|w\|_{E_a^l(K)}\quad(w\in E_a^l(K):\ w|_\gamma=\psi).$$

One can find constructive definitions of the spaces $H_a^{l-1/2}(\Gamma)$ and $E_a^{l-1/2}(\gamma)$ in [7, §1].

Introduce the spaces of vector–functions

$$E_a^{l+2m,\,N}(K)=\prod_{j=1}^N E_a^{l+2m}(K_j),\ E_a^{l,\,N}(K,\ \gamma)=\prod_{j=1}^N E_a^l(K_j,\ \gamma_j),$$

$$E_a^l(K_j,\ \gamma_j)=E_a^l(K_j)\times\prod_{\sigma=1,\,R_j+1}\prod_{\mu=1}^m E_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}),$$

where $K_j=\{y:\ r>0,\ b_{j1}<\varphi<b_{j,R_j+1}\}$, $\gamma_{j\sigma}=\{y:\ r>0,\ \varphi=b_{j\sigma}\}$.

Consider the auxiliary problem for $u=(u_1,\ \dots,\ u_N)\in E_a^{l+2m,\,N}(K)$

$$\mathcal{P}_j(D_y,\ \theta)u_j=f_j(y)\quad(y\in K_j),\tag{1.6}$$

$$\begin{gathered}\mathcal{B}_{j\sigma\mu}(D_y,\ \theta)u=B_{j\sigma\mu}(D_y,\ \theta)u_j|_{\gamma_{j\sigma}}+\sum_{k,q,s}(B_{j\sigma\mu kqs}(D_y,\ \theta)u_k)(\mathcal{G}_{j\sigma kqs}y)|_{\gamma_{j\sigma}}=\\ =g_{j\sigma\mu}(y)\quad(y\in\gamma_{j\sigma})\\ (j=1,\ \dots,\ N;\ \sigma=1,\ R_j+1;\ \mu=1,\ \dots,\ m),\end{gathered}\tag{1.7}$$

where θ is an arbitrary point on a unit sphere $S^{n-3}=\{z\in\mathbb{R}^{n-2}:\ |z|=1\}$, $f=\{f_j,\ g_{j\sigma\mu}\}\in E_a^{l,\,N}(K,\ \gamma)$.

Introduce the bounded operator corresponding to problem (1.6), (1.7),

$$\mathcal{L}(\theta)=\{\mathcal{P}_j(D_y,\ \theta),\ \mathcal{B}_{j\sigma\mu}(D_y,\ \theta)\}:E_a^{l+2m,\,N}(K)\to E_a^{l,\,N}(K,\ \gamma).$$

Lemma 1.5. *For any* $g_{j\sigma\mu} \in E_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})$ $(j = 1, \ldots, N;\ \sigma = 1,\ R_j+1;\ \mu = 1, \ldots, m)$ *and* $\theta \in S^{n-3}$, *there exists a vector–function* $u \in E_a^{l+2m,\,N}(K)$ *such that*

$$\mathcal{B}_{j\sigma\mu}(D_y,\ \theta)u = g_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma}),$$

$$\|u\|_{E_a^{l+2m,\,N}(K)} \le c \sum_{j,\,\sigma,\,\mu} \|g_{j\sigma\mu}\|_{E_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})},$$

where $c > 0$ *is independent of* $g_{j\sigma\mu},\ \theta$.

Lemma 1.5 is proved in [4, §1].

2 Solvability of nonlocal boundary value problems in plane angles

We shall need the results of this section (obtained by A.L. Skubachevskii in [4, §2]) in §3 for study a priori estimates of solutions to nonlocal boundary value problems in dihedral angles.

1 Reduction of nonlocal problems in plane angles to nonlocal problems on arcs.

Consider the following nonlocal problem for $U = (U_1, \ldots, U_N) \in H_a^{l+2m,\,N}(K)$

$$\mathcal{P}_j(D_y,\ 0)U_j = f_j(x) \quad (y \in K_j), \tag{2.1}$$

$$\begin{gathered}\mathcal{B}_{j\sigma\mu}(D_y,\ 0)U = B_{j\sigma\mu}(D_y,\ 0)U_j|_{\gamma_{j\sigma}} + \\ + \sum_{k,q,s} (B_{j\sigma\mu kqs}(D_y,\ 0)U_k)(\mathcal{G}_{j\sigma kqs}y)|_{\gamma_{j\sigma}} = g_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma})\end{gathered} \tag{2.2}$$

$$(j = 1, \ldots, N;\ \sigma = 1,\ R_j+1;\ \mu = 1, \ldots, m),$$

where $f = \{f_j,\ g_{j\sigma\mu}\} \in H_a^{l,\,N}(K,\ \gamma)$.

We write the operators $\mathcal{P}_j(D_y,\ 0)$, $B_{j\sigma\mu}(D_y,\ 0)$, $B_{j\sigma\mu kqs}(D_y,\ 0)$ in the polar coordinates: $\mathcal{P}_j(D_y,\ 0) = r^{-2m}\tilde{\mathcal{P}}_j(\varphi,\ D_\varphi,\ rD_r)$, $B_{j\sigma\mu}(D_y,\ 0) = r^{-m_{j\sigma\mu}}\tilde{B}_{j\sigma\mu}(\varphi,\ D_\varphi,\ rD_r)$, $B_{j\sigma\mu kqs}(D_y,\ 0) = r^{-m_{j\sigma\mu}}\tilde{B}_{j\sigma\mu kqs}(\varphi,\ D_\varphi,\ rD_r)$, where $D_\varphi = -i\dfrac{\partial}{\partial\varphi}$, $D_r = -i\dfrac{\partial}{\partial r}$.

Put $\tau = \ln r$ and do the Fourier transform with respect to τ; then from (2.1), (2.2), we get

$$\tilde{\mathcal{P}}_j(\varphi,\ D_\varphi,\ \lambda)\tilde{U}_j(\varphi,\ \lambda) = \tilde{F}_j(\varphi,\ \lambda) \quad (b_{j1} < \varphi < b_{j,R_j+1}), \tag{2.3}$$

$$\begin{gathered}\tilde{\mathcal{B}}_{j\sigma\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{U}(\varphi,\ \lambda) = \tilde{B}_{j\sigma\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{U}_j(\varphi,\ \lambda)|_{\varphi=b_{j\sigma}}+\\ +\sum_{k,q,s} e^{(i\lambda-m_{j\sigma\mu})\ln\chi_{j\sigma kqs}}\tilde{B}_{j\sigma\mu kqs}(\varphi,\ D_\varphi,\ \lambda)\tilde{U}_k(\varphi+\varphi_{j\sigma kq},\ \lambda)|_{\varphi=b_{j\sigma}} =\\ = \tilde{G}_{j\sigma\mu}(\lambda),\end{gathered} \tag{2.4}$$

where $F_j(\varphi,\ \tau) = e^{2m\tau}f_j(\varphi,\ \tau)$, $G_{j\sigma\mu}(\tau) = e^{m_{j\sigma\mu}\tau}g_{j\sigma\mu}(\tau)$; $\tilde{U}_j(\varphi,\ \lambda) = (2\pi)^{-1/2}\int\limits_{-\infty}^{\infty} U_j(\varphi,\ \tau)e^{-i\lambda\tau}d\tau$.

This problem is a system of N ordinary differential equations (2.3) for functions $\tilde{U}_j \in W^{l+2m}(b_{j1},\ b_{j,R_j+1})$ with nonlocal conditions (2.4) connecting values of $\tilde{U}_j$ and their derivatives at the point $\varphi = b_{j\sigma}$ with values of $\tilde{U}_k$ and their derivatives at the points of the intervals $(b_{k1},\ b_{k,R_k+1})$.

2 Solvability of nonlocal problems with parameter λ on arcs.

Let us consider the operator–valued function

$$\begin{gathered}\tilde{\mathcal{L}}(\lambda) = \{\tilde{\mathcal{P}}_j(\varphi,\ D_\varphi,\ \lambda),\ \tilde{\mathcal{B}}_{j\sigma\mu}(\varphi,\ D_\varphi,\ \lambda)\} :\\ W^{l+2m,\,N}(b_1,\ b_2) \to W^{l,\,N}[b_1,\ b_2]\end{gathered}$$

corresponding to problem (2.3), (2.4). Here

$$\begin{gathered}W^{l+2m,\,N}(b_1,\ b_2) = \prod_{j=1}^N W^{l+2m}(b_{j1},\ b_{j,R_j+1}),\\ W^{l,\,N}[b_1,\ b_2] = \prod_{j=1}^N W^{l}[b_{j1},\ b_{j,R_j+1}],\end{gathered}$$

$$W^l[b_{j1},\ b_{j,R_j+1}] = W^l(b_{j1},\ b_{j,R_j+1}) \times \mathbb{C}^m \times \mathbb{C}^m.$$

Introduce the equivalent norms depending on the parameter λ ($|\lambda| \geq 1$) in the Hilbert spaces $W^l(b_{j1},\ b_{j,R_j+1})$ and $W^l[b_{j1},\ b_{j,R_j+1}]$:

$$|||\tilde{U}_j|||_{W^l(b_{j1},\ b_{j,R_j+1})} = \left(\|\tilde{U}_j\|^2_{W^l(b_{j1},\ b_{j,R_j+1})} + |\lambda|^{2l}\,\|\tilde{U}_j\|^2_{L_2(b_{j1},\ b_{j,R_j+1})}\right)^{1/2},$$

$$\begin{gathered}|||\{\tilde{F}_j,\ \tilde{G}_{j\sigma\mu}\}|||_{W^l[b_{j1},\ b_{j,R_j+1}]} = \Big(|||\tilde{F}_j|||^2_{W^l(b_{j1},\ b_{j,R_j+1})}+\\ +\sum_{\sigma,\,\mu}(1+|\lambda|^{2(l+2m-m_{j\sigma\mu}-1/2)})|\tilde{G}_{j\sigma\mu}|^2\Big)^{1/2},\end{gathered}$$

where $\tilde U_j \in W^l(b_{j1},\ b_{j,R_j+1})$, $\{\tilde F_j,\ \tilde G_{j\sigma\mu}\} \in W^l[b_{j1},\ b_{j,R_j+1}]$. And therefore we have

$$|||\tilde U|||_{W^{l+2m,\,N}(b_1,\ b_2)} = \Big(\sum_j |||\tilde U_j|||^2_{W^{l+2m}(b_{j1},\ b_{j,R_j+1})}\Big)^{1/2},$$

$$|||\tilde \Phi|||_{W^{l,\,N}[b_1,\ b_2]} = \Big(\sum_j |||\tilde \Phi_j|||^2_{W^{l}[b_{j1},\ b_{j,R_j+1}]}\Big)^{1/2},$$

where $\tilde U = (\tilde U_1,\ \dots,\ \tilde U_N) \in W^{l+2m,\,N}(b_1,\ b_2)$, $\tilde\Phi = (\tilde\Phi_1,\ \dots,\ \tilde\Phi_N) \in W^{l,\,N}[b_1,\ b_2]$.

The next two statements are proved in [4, §2].

Lemma 2.1. *For all $\lambda \in \mathbb{C}$, the operator $\tilde{\mathcal{L}}(\lambda) : W^{l+2m,\,N}(b_1,\ b_2) \to W^{l,\,N}[b_1,\ b_2]$ is Fredholm,* $\operatorname{ind}\tilde{\mathcal{L}}(\lambda) = 0$; *for any $h \in \mathbb{R}$, there exists a $q_0 > 0$ such that for $\lambda \in J_{h,\,q_0} = \{\lambda \in \mathbb{C} :\ \operatorname{Im}\lambda = h,\ |\operatorname{Re}\lambda| \ge q_0\}$, the operator $\tilde{\mathcal{L}}(\lambda)$ has the bounded inverse $\tilde{\mathcal{L}}^{-1}(\lambda) : W^{l,\,N}[b_1,\ b_2] \to W^{l+2m,\,N}(b_1,\ b_2)$ and*

$$|||\tilde{\mathcal{L}}^{-1}(\lambda)\tilde\Phi|||_{W^{l+2m,\,N}(b_1,\ b_2)} \le c|||\tilde\Phi|||_{W^{l,\,N}[b_1,\ b_2]}$$

for all $\tilde\Phi \in W^{l,\,N}[b_1,\ b_2]$, where $c > 0$ is independent of λ and $\tilde\Phi$. The operator–valued function $\tilde{\mathcal{L}}^{-1}(\lambda) : W^{l,\,N}[b_1,\ b_2] \to W^{l+2m,\,N}(b_1,\ b_2)$ is finitely meromorphic.

Lemma 2.2. *For any $0 < \varepsilon < 1/d$, there exists a $q > 1$ such that the set $\{\lambda \in \mathbb{C} :\ |\operatorname{Im}\lambda| \le \varepsilon \ln|\operatorname{Re}\lambda|,\ |\operatorname{Re}\lambda| \ge q\}$ contains no poles of the operator–valued function $\tilde{\mathcal{L}}^{-1}(\lambda)$, where $d = \max|\ln\chi_{j\sigma kqs}|$; for every pole λ_0 of the operator–valued function $\tilde{\mathcal{L}}^{-1}(\lambda)$, there exists a $\delta > 0$ such that the set $\{\lambda \in \mathbb{C} :\ 0 < |\operatorname{Im}\lambda - \operatorname{Im}\lambda_0| < \delta\}$ contains no poles of the operator–valued function $\tilde{\mathcal{L}}^{-1}(\lambda)$.*

3 One–valued solvability of nonlocal problems in plane angles.

The following theorem is obtained from Lemma 2.1.

Theorem 2.1. *Suppose the line $\operatorname{Im}\lambda = a+1-l-2m$ contains no poles of the operator–valued function $\tilde{\mathcal{L}}^{-1}(\lambda)$; then nonlocal boundary value problem (2.1), (2.2) has a unique solution $U \in H_a^{l+2m,\,N}(K)$ for every right-hand side $f \in H_a^{l,\,N}(K,\ \gamma)$ and*

$$\|U\|_{H_a^{l+2m,\,N}(K)} \le c\|f\|_{H_a^{l,\,N}(K,\ \gamma)},$$

where $c > 0$ does not depend on f.

One can find the proof of Theorem 2.1 in [4, §2].

3 A priori estimates of solutions for nonlocal boundary value problems

1 A priori estimates in dihedral angles.

Denote $d_1=\min\{1,\ \chi_{j\sigma kqs}\}/2,\ d_2=2\max\{1,\ \chi_{j\sigma kqs}\}$, $\Omega_j^p=\Omega_j\cap\{r_1d_1^{6-p}<r<r_2d_2^{6-p},\ |z|<2^{-p-1}\}$, where $j=1,\ \dots,\ N;\ p=0,\ \dots,\ 6;\ 0<r_1<r_2$.

Lemma 3.1. *Suppose* $U_j\in W^{2m}(\Omega_j^0)$,

$$\mathcal{P}_j(D_y,\ D_z)U_j\in W^l(\Omega_j^0),\ \mathcal{B}_{j\sigma\mu}(D_y,\ D_z)U\in W^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma}\cap\bar\Omega_j^0)$$
$$(j=1,\ \dots,\ N;\ \sigma=1,\ R_j+1;\ \mu=1,\ \dots,\ m); \tag{3.1}$$

then $U\in\prod\limits_j W^{l+2m}(\Omega_j^3)$ *and for* $|\lambda|\ge 1$,

$$\begin{gathered}\sum_j\|U_j\|_{W^{l+2m}(\Omega_j^6)}\le c\sum_j\big\{\|\mathcal{P}_j(D_y,\ D_z)U_j\|_{W^l(\Omega_j^3)}+\\ +\sum_{\sigma,\mu}\|\mathcal{B}_{j\sigma\mu}(D_y,\ D_z)U\|_{W^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma}\cap\bar\Omega_j^3)}+|\lambda|^{-1}\|U_j\|_{W^{l+2m}(\Omega_j^3)}+\\ +|\lambda|^{l+2m-1}\|U_j\|_{L_2(\Omega_j^3)}\big\},\end{gathered} \tag{3.2}$$

where $c>0$ *is independent of* λ *and* U.

Proof. Denote

$$\varepsilon=\min\{b_{j,q+1}-b_{jq}\}/4\ (j=1,\ \dots,\ N;\ q=1,\ \dots,\ R_j) \tag{3.3}$$

and introduce the functions $\zeta_{jq}\in C^\infty(\mathbb{R})$ such that

$$\zeta_{jq}(\varphi)=1 \text{ for } |b_{jq}-\varphi|<\varepsilon/2,\ \zeta_{jq}(\varphi)=0 \text{ for } |b_{jq}-\varphi|>\varepsilon \tag{3.4}$$
$$(j=1,\ \dots,\ N;\ q=1,\ \dots,\ R_j+1).$$

Put $\zeta_j(\varphi)=\zeta_{j1}(\varphi)+\zeta_{j,R_j+1}(\varphi)$. Since the functions ζ_j are the multiplicators in $W^l(\Omega_j^p)$, we have $(1-\zeta_j)U_j\in W^{2m}(\Omega_j^0)$. Apply theorem 5.1 [12, Chapter 2, §5.1] to the function $(1-\zeta_j)U_j$ and to the operator $\mathcal{P}_j(D_y,\ D_z)$; then from (3.1) and Leibniz' formula, we get

$$(1-\zeta_j)U_j\in W^{l+2m}(\Omega_j^1). \tag{3.5}$$

Denote $V_{j\sigma\mu} = \sum\limits_{k,\,q,\,s} (B_{j\sigma\mu kqs}(D_y,\ D_z)((1-\zeta_k)U_k))(\mathcal{G}_{j\sigma kqs}y,\ z)$. Clearly, we have

$$V_{j\sigma\mu}|_{\Gamma_{j\sigma}\cap\bar{\Omega}_j^2} = \sum_{k,\,q,\,s} (B_{j\sigma\mu kqs}(D_y,\ D_z)U_k)(\mathcal{G}_{j\sigma kqs}y,\ z)|_{\Gamma_{j\sigma}\cap\bar{\Omega}_j^2}. \tag{3.6}$$

From equality (3.6) and relations (3.1), (3.5), it follows that

$$\begin{gathered} B_{j\sigma\mu}(D_y,\ D_z)U_j|_{\Gamma_{j\sigma}\cap\bar{\Omega}_j^2} = \mathcal{B}_{j\sigma\mu}(D_y,\ D_z)U - V_{j\sigma\mu}|_{\Gamma_{j\sigma}\cap\bar{\Omega}_j^2} \in \\ \in W^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma}\cap\bar{\Omega}_j^2). \end{gathered} \tag{3.7}$$

Again applying theorem 5.1 [12, Chapter 2, §5.1] to the function U_j and to the operator $\{\mathcal{P}_j(D_y,\ D_z),\ B_{j\sigma\mu}(D_y,\ D_z)|_{\Gamma_{j\sigma}\cap\bar{\Omega}_j^2}\}$ $(\sigma = 1,\ R_j+1;\ \mu = 1,\ \dots,\ m)$ from (3.1), (3.7), we obtain $U_j \in W^{l+2m}(\Omega_j^3)$.

Now estimate (3.2) follows from lemma 3.1 [4, §3]. □

Let $W^l_{\text{loc}}(\bar{\Omega}_j\backslash M)$ be a set of functions belonging to the space W^l on any compactum in $\bar{\Omega}_j$ that does not intersect with M.

Theorem 3.1. *Let $U \in \prod\limits_j W^{2m}_{\text{loc}}(\bar{\Omega}_j\backslash M)$ be a solution for nonlocal boundary value problem (1.1), (1.2) such that $U \in H^{0,\,N}_{a-l-2m}(\Omega)$ and $f \in H^{l,\,N}_a(\Omega,\ \Gamma)$; then $U \in H^{l+2m,\,N}_a(\Omega)$ and*

$$\|U\|_{H^{l+2m,\,N}_a(\Omega)} \le c\big(\|f\|_{H^{l,\,N}_a(\Omega,\ \Gamma)} + \|U\|_{H^{0,\,N}_{a-l-2m}(\Omega)}\big), \tag{3.8}$$

where $c > 0$ is independent of U.

Proof. From Lemma 3.1, it follows that $U \in \prod\limits_j W^{l+2m}_{\text{loc}}(\bar{\Omega}_j\backslash M)$. Now lemma 3.2 [4, §3] implies that $U \in H^{l+2m,\,N}_a(\Omega)$ and a priori estimate (3.8) is valid. □

2 A priori estimates in plane angles.

Put $K_j^{ps} = K_j \cap \{r_1 d_1^{6-p}\cdot 2^s < r < r_2 d_2^{6-p}\cdot 2^s\}$, where $0 < r_1 < r_2$; $s \ge 1$; $j = 1,\ \dots,\ N$; $p = 0,\ \dots,\ 6$.

Lemma 3.2. *Suppose $s \ge 1$, $\theta \in S^{n-3}$. Assume that $u_j \in W^{2m}(K_j^{0s})$,*

$$\mathcal{P}_j(D_y,\ \theta)u_j \in W^l(K_j^{0s}),\ \mathcal{B}_{j\sigma\mu}(D_y,\ \theta)u = 0 \quad (y \in \gamma_{j\sigma} \cap \bar{K}_j^{0s})$$

$$(j = 1,\ \dots,\ N;\ \sigma = 1,\ R_j + 1;\ \mu = 1,\ \dots,\ m);$$

then $u \in \prod\limits_j W^{l+2m}(K_j^{3s})$ *and for* $|\lambda| \geq 1$,

$$\begin{gathered}\sum_j 2^{sa} \|u_j\|_{W^{l+2m}(K_j^{6s})} \leq c \sum_j \big\{ 2^{sa} \|\mathcal{P}_j(D_y,\ \theta) u_j\|_{W^l(K_j^{3s})} + \\ + |\lambda|^{-1} 2^{sa} \|u_j\|_{W^{l+2m}(K_j^{3s})} + |\lambda|^{l+2m-1} 2^{s(a-l-2m)} \|u_j\|_{L_2(K_j^{3s})} \big\},\end{gathered} \tag{3.9}$$

where $c > 0$ *is independent of* u, θ, λ, *and* s.

Proof. Repeating the proof of Lemma 3.1 and substituting K_j^{ps} for Ω_j^p and θ for D_z, we obtain $u \in \prod\limits_j W^{l+2m}(K_j^{3s})$. Now a priori estimate (3.9) follows from lemma 3.3 [4, §3]. □

Theorem 3.2. *Let* $u \in \prod\limits_j W^{2m}_{\text{loc}}(\bar{K}_j \backslash \{0\})$ *be a solution for problem (1.6), (1.7) such that* $u \in E^{0,\,N}_{a-l-2m}(K)$ *and* $f \in E^{l,\,N}_a(K,\ \gamma)$; *then* $u \in E^{l+2m,\,N}_a(K)$ *and*

$$\|u\|_{E^{l+2m,\,N}_a(K)} \leq c \big(\|f\|_{E^{l,\,N}_a(K,\ \gamma)} + \|u\|_{E^{0,\,N}_{a-l-2m}(K)} \big), \tag{3.10}$$

where $c > 0$ *is independent of* u, $\theta \in S^{n-3}$.

Proof. 1) By Lemma 1.5, it suffices to consider the case $g_{j\sigma\mu} = 0$. Since $f_j \in E^l_a(K_j) \subset W^l_{\text{loc}}(\bar{K}_j \backslash \{0\})$, as above, one can show that $u \in \prod\limits_j W^{l+2m}_{\text{loc}}(\bar{K}_j \backslash \{0\})$. Put $r_1 = d_1$, $r_2 = d_2$ and denote $K_j^{ps} = K_j \cap \{d_1^{7-p} \cdot 2^s < r < d_2^{7-p} \cdot 2^s\}$, where $s \geq 1$; $j = 1,\ \dots,\ N$; $p = 0,\ \dots,\ 6$. Let us also denote $K_j^{60} = K_j \cap \{r < d_2\}$. Introduce the functions $\psi \in C^\infty(\mathbb{R})$, $\psi(r) = 1$ for $r < d_2$, $\psi(r) = 0$ for $r > 2d_2$; $\hat{\psi} \in C^\infty(\mathbb{R})$, $\hat{\psi}(r) = 1$ for $r < 2d_2^2$, $\hat{\psi}(r) = 0$ for $r > 3d_2^2$.

Applying Theorem 3.1 to the operator $\{\mathcal{P}_j(D_y,\ 0),\ \mathcal{B}_{j\sigma\mu}(D_y,\ 0)\}$ (for $n = 2$), we get

$$\begin{gathered}\sum_j \|u_j\|_{E^{l+2m}_a(K_j^{60})} \leq k_1 \sum_j \|\psi u_j\|_{H^{l+2m}_a(K_j)} \leq \\ \leq k_2 \sum_j \big\{ \|\mathcal{P}_j(D_y,\ 0)(\psi u_j)\|_{H^l_a(K_j)} + \\ + \sum_{\sigma,\,\mu} \|\mathcal{B}_{j\sigma\mu}(D_y,\ 0)(\psi u_j)\|_{H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} + \|\psi u_j\|_{H^0_{a-l-2m}(K_j)} \big\}.\end{gathered} \tag{3.11}$$

Let us estimate $\|\mathcal{P}_j(D_y,\ 0)(\psi u_j)\|_{H_a^l(K_j)}$. Using Leibniz' formula, the condition $\theta \in S^{n-3}$, and limitations for supports of the functions $\psi,\ \hat{\psi}$, we obtain

$$
\begin{gathered}
\|\mathcal{P}_j(D_y,\ 0)(\psi u_j)\|_{H_a^l(K_j)} \le \\
\le k_3(\|\mathcal{P}_j(D_y,\ \theta)(\psi u_j)\|_{H_a^l(K_j)} + \|\psi u_j\|_{H_{a-1}^{l+2m-1}(K_j)}) \le \\
\le k_4(\|\mathcal{P}_j(D_y,\ \theta) u_j\|_{E_a^l(K_j)} + \|\hat{\psi} u_j\|_{H_{a-1}^{l+2m-1}(K_j)}).
\end{gathered} \tag{3.12}
$$

Let us estimate $\|\mathcal{B}_{j\sigma\mu}(D_y,\ 0)(\psi u_j)\|_{H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})}$. Using Leibniz' formula, the condition $\theta \in S^{n-3}$, limitations for supports of the functions $\psi,\ \hat{\psi}$, and the condition $g_{j\sigma\mu} = 0$, we get

$$
\begin{gathered}
\|\mathcal{B}_{j\sigma\mu}(D_y,\ 0)(\psi u_j)\|_{H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} \le \\
\le k_5(\|\mathcal{B}_{j\sigma\mu}(D_y,\ \theta)(\psi u_j)\|_{H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} + \|\hat{\psi} u_j\|_{H_{a-1}^{l+2m-1}(K_j)}) \le \\
\le k_6\big(\|\psi \mathcal{B}_{j\sigma\mu}(D_y,\ \theta) u_j\|_{H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} + \sum_{k,\,q,\,s} \|(\psi(\chi_{j\sigma kqs} y) - \\
-\psi(y))(B_{j\sigma\mu kqs}(D_y,\ \theta) u_k)(\mathcal{G}_{j\sigma kqs} y)|_{\gamma_{j\sigma}}\|_{H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} + \\
+\|\hat{\psi} u_j\|_{H_{a-1}^{l+2m-1}(K_j)}\big) \le k_7\big(\sum_k \|u_k\|_{W^{l+2m}(K_j \cap S_0)} + \|\hat{\psi} u_j\|_{H_{a-1}^{l+2m-1}(K_j)}\big),
\end{gathered} \tag{3.13}
$$

where $S_0 = \{y \in \mathbb{R}^2 :\ 1 < r < 2d_2/d_1\}$.

Inequalities (3.11)–(3.13), Lemma 3.1, and interpolation inequality (1.5) yield

$$
\begin{gathered}
\sum_j \|u_j\|_{E_a^{l+2m}(K_j^{60})} \le k_8 \sum_j \big\{\|f_j\|_{E_a^l(K_j)} + \\
+|\lambda|^{-1}\|u_j\|_{E_a^{l+2m}(K_j)} + |\lambda|^{l+2m-1}\|u_j\|_{E_{a-l-2m}^0(K_j)}\big\}
\end{gathered} \tag{3.14}
$$

2) By virtue of Lemma 3.2, for $s \ge 1$, we have

$$
\begin{gathered}
\sum_j \|u_j\|_{E_a^{l+2m}(K_j^{6s})} \le k_9 \sum_j \big\{\|f_j\|_{E_a^l(K_j^{3s})} + \\
+|\lambda|^{-1}\|u_j\|_{E_a^{l+2m}(K_j^{3s})} + |\lambda|^{l+2m-1}\|u_j\|_{E_{a-l-2m}^0(K_j^{3s})}\big\}.
\end{gathered} \tag{3.15}
$$

Summing up (3.14), (3.15) for all $s \ge 1$ and taking a sufficiently large $|\lambda|$, we obtain (3.10). □

From Theorem 2.1 and Lemma 3.2, one can also get the following result (see theorem 3.1 [4, §3]).

Theorem 3.3. *Suppose the line* $\operatorname{Im}\lambda = a+1-l-2m$ *contains no poles of the operator–valued function* $\tilde{\mathcal{L}}^{-1}(\lambda)$*; then for all solutions* $u \in E_a^{l+2m,\,N}(K)$ *to nonlocal boundary value problem (1.6), (1.7) and all* $\theta \in S^{n-3}$*, we have*

$$\|u\|_{E_a^{l+2m,\,N}(K)} \le c\big(\|f\|_{E_a^{l,\,N}(K,\;\gamma)} + \sum_j \|u_j\|_{L_2(K_j\cap S)}\big), \tag{3.16}$$

where $S = \{y \in \mathbb{R}^2:\ 0 < R_1 < r < R_2\}$*,* $c > 0$ *is independent of* θ *and* u*.*

If for any $\theta \in S^{n-3}$*, estimate (3.16) holds for all solutions to nonlocal boundary value problem (1.6), (1.7), then the line* $\operatorname{Im}\lambda = a+1-l-2m$ *contains no poles of the operator–valued function* $\tilde{\mathcal{L}}^{-1}(\lambda)$*.*

Theorem 3.3 implies that kernel of $\mathcal{L}(\theta)$ is of finite dimension and range of $\mathcal{L}(\theta)$ is closed. In order to prove that cokernel of $\mathcal{L}(\theta)$ is also of finite dimension, we shall obtain the Green formula for nonlocal problems and study problems that are adjoint to nonlocal boundary value problems with respect to the Green formula.

4 The Green formula for nonlocal elliptic problems

In this section, we obtain the Green formula, which connects nonlocal boundary value problems and nonlocal transmission problems in dihedral angles, plane angles, and on arcs. Nonlocal transmission problems will be studied in §§5–7.

1 The Green formula in dihedral angles.

Consider nonlocal boundary value problem (1.1), (1.2).

Let n_{kq} be the unit normal vector to Γ_{kq} directed inside Ω_{kq} ($q = 1, \dots, R_k$), n_{k,R_k+1} be the unit normal vector to Γ_{k,R_k+1}, directed inside Ω_{kR_k}.

Denote by $C^\infty(\bar\Omega_{jt}\backslash M)$ ($C^\infty(\bar\Omega_j\backslash M)$, $C^\infty(\Gamma_{jq}\backslash M)$) the set of infinitely differentiable in $\bar\Omega_{jt}\backslash M$ (in $\bar\Omega_j\backslash M$, in $\Gamma_{jq}\backslash M$) functions. We also denote by $C_0^\infty(\bar\Omega_{jt}\backslash M)$ ($C_0^\infty(\bar\Omega_j\backslash M)$, $C_0^\infty(\Gamma_{jq}\backslash M)$) the set of infinitely differentiable in $\bar\Omega_{jt}$ (in $\bar\Omega_j$, in Γ_{jq}) functions with compact support from $\bar\Omega_{jt}\backslash M$ (from $\bar\Omega_j\backslash M$, from $\Gamma_{jq}\backslash M$) ($j = 1, \dots N$; $t = 1, \dots, R_j$; $q = 1, \dots, R_j+1$).

For $U_{jt} \in C_0^\infty(\bar\Omega_{jt}\backslash M),\ V_{jt} \in C^\infty(\bar\Omega_{jt}\backslash M)$ (or $U_{jt} \in C^\infty(\bar\Omega_{jt}\backslash M),\ V_{jt} \in C_0^\infty(\bar\Omega_{jt}\backslash M)$), put

$$(U_{jt},\ V_{jt})_{\Omega_{jt}} = \int\limits_{\Omega_{jt}} U_{jt} \cdot \bar V_{jt}\, dx \quad (j = 1,\ \dots,\ N;\ t = 1,\ \dots,\ R_j).$$

For $U_{\Gamma_{jq}} \in C_0^\infty(\Gamma_{jq}),\ V_{\Gamma_{jq}} \in C^\infty(\Gamma_{jq})$ (or $U_{\Gamma_{jq}} \in C^\infty(\Gamma_{jq}),\ V_{\Gamma_{jq}} \in C_0^\infty(\Gamma_{jq})$), put

$$(U_{\Gamma_{jq}},\ V_{\Gamma_{jq}})_{\Gamma_{jq}} = \int\limits_{\Gamma_{jq}} U_{\Gamma_{jq}} \cdot \bar V_{\Gamma_{jq}}\, d\Gamma \quad (j = 1,\ \dots,\ N;\ q = 1,\ \dots,\ R_j + 1).$$

If we have functions $V_{jt}(x)$ defined in Ω_{jt}, then denote by $V_j(x)$ the function given by $V_j(x) \equiv V_{jt}(x)$ for $x \in \Omega_{jt}$.

For short, let us omit the arguments $(D_y,\ D_z)$ of differential operators. Denote by $\mathcal{Q}_j$ the operator that is formally adjoint to $\mathcal{P}_j$.

Theorem 4.1. *For the operators $\mathcal{P}_j$, $B_{j\sigma\mu}$, and $B_{j\sigma\mu kqs}$ defined in §1, there exist (not unique)*

1) a system $\{B'_{j\sigma\mu}\}_{\mu=1}^m$ of normal on $\Gamma_{j\sigma}$ operators of orders $2m-1-m'_{j\sigma\mu}$ with constant coefficients such that the system $\{B_{j\sigma\mu},\ B'_{j\sigma\mu}\}_{\mu=1}^m$ is a Dirichlet one on $\Gamma_{j\sigma}$[3] *of order $2m$ $(\sigma = 1,\ R_j + 1)$;*

2) a Dirichlet system $\{B_{jq\mu},\ B'_{jq\mu}\}_{\mu=1}^m$ on Γ_{jq} of order $2m$ such that the operators $B_{jq\mu}$ and $B'_{jq\mu}$ are of orders $2m - \mu$ and $m - \mu$ correspondingly $(q = 2,\ \dots,\ R_j)$.

If the choice has been done, then there exist operators $C_{j\sigma\mu},\ F_{j\sigma\mu},\ T_{jq\nu}$, and $T_{jq\nu k\sigma s}$ $(j,\ k = 1,\ \dots,\ N;\ \sigma = 1,\ R_j + 1$ for the operators $C_{j\sigma\mu}$ and $F_{j\sigma\mu}$, $\sigma = 1,\ R_k + 1$ for the operators $T_{jq\nu k\sigma s}$; $\mu = 1,\ \dots,\ m$; $q = 2,\ \dots,\ R_j$; $\nu = 1,\ \dots,\ 2m$; $s = 1,\ \dots,\ S'_{jqk\sigma} = S_{k\sigma jq})$ with constant coefficients such that

I) the operators $C_{j\sigma\mu},\ F_{j\sigma\mu},\ T_{jq\nu}$, and $T_{jq\nu k\sigma s}$ are of orders $m'_{j\sigma\mu},\ 2m - 1 - m_{j\sigma\mu},\ \nu - 1$, and $\nu - 1$ correspondingly;

II) the system $\{C_{j\sigma\mu},\ F_{j\sigma\mu}\}_{\mu=1}^m$ is a Dirichlet one on $\Gamma_{j\sigma}$ of order $2m$ $(\sigma = 1,\ R_j + 1)$,

the system $\{C_{j\sigma\mu}\}_{\mu=1}^m$ covers the operator $\mathcal{Q}_j$ on $\Gamma_{j\sigma}$ $(\sigma = 1,\ R_j + 1)$,

the system $\{T_{jq\nu}\}_{\nu=1}^{2m}$ is a Dirichlet one on Γ_{jq} of order $2m$ $(q = 2,\ \dots,\ R_j)$;

[3] See [12, Chapter 2, §2.2] for the definition of a Dirichlet system.

III) for all $U_j \in C_0^\infty(\bar\Omega_j\backslash M)$, $V_{jt} \in C^\infty(\bar\Omega_{jt}\backslash M)$ (or $U_j \in C^\infty(\bar\Omega_j\backslash M)$, $V_{jt} \in C_0^\infty(\bar\Omega_{jt}\backslash M)$), the following Green formula is valid:

$$\begin{gathered}\sum_j\Big\{\sum_t(\mathcal{P}_jU_j,\ V_{jt})_{\Omega_{jt}}+\sum_{\sigma,\mu}(\mathcal{B}_{j\sigma\mu}U,\ F_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}}+\\ +\sum_{q,\mu}(B_{jq\mu}U_j|_{\Gamma_{jq}},\ \mathcal{T}_{jq\mu}V)_{\Gamma_{jq}}\Big\}=\sum_j\Big\{\sum_t(U_j,\ \mathcal{Q}_jV_{jt})_{\Omega_{jt}}+\\ +\sum_{\sigma,\mu}(B'_{j\sigma\mu}U_j|_{\Gamma_{j\sigma}},\ C_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}}+\sum_{q,\mu}(B'_{jq\mu}U_j|_{\Gamma_{jq}},\ \mathcal{T}_{jq,m+\mu}V)_{\Gamma_{jq}}\Big\}.\end{gathered}\tag{4.1}$$

In the Green formulas (here and below), the summation is taken over $j = 1,\ \dots,\ N$; $t = 1,\ \dots,\ R_j$; $\sigma = 1,\ R_j + 1$; $q = 2,\ \dots,\ R_j$; $\mu = 1,\ \dots,\ m$; $\mathcal{B}_{j\sigma\mu}$ is given by (1.2);

$$\mathcal{T}_{jq\nu}V = T_{jq\nu}V_{j,q-1}|_{\Gamma_{jq}} - T_{jq\nu}V_{jq}|_{\Gamma_{jq}} + \sum_{k,\sigma,s}(T_{jq\nu k\sigma s}V_k)(\mathcal{G}'_{jqk\sigma s}y,\ z)|_{\Gamma_{jq}}$$
$$(\nu = 1,\ \dots,\ 2m),$$

in the formula for $\mathcal{T}_{jq\nu}$ (here and below), the summation is taken over $k = 1,\ \dots,\ N$; $\sigma = 1,\ R_k + 1$; $s = 1,\ \dots,\ S'_{jqk\sigma} = S_{k\sigma jq}$; $\mathcal{G}'_{jqk\sigma s}$ is the operator of rotation by the angle $\varphi'_{jqk\sigma} = -\varphi_{k\sigma jq}$ and expansion by $\chi'_{jqk\sigma s} = 1/\chi_{k\sigma jqs}$ times in the plane $\{y\}$.

Proof. For $j = 1,\ \dots,\ N$, put $B'_{j\sigma\mu} = \left(-i\dfrac{\partial}{\partial n_{j\sigma}}\right)^{2m-1-m'_{j\sigma\mu}}$, $B_{jq\mu} = \left(-i\dfrac{\partial}{\partial n_{jq}}\right)^{2m-\mu}$, $B'_{jq\mu} = \left(-i\dfrac{\partial}{\partial n_{jq}}\right)^{m-\mu}$ ($\sigma = 1,\ R_j+1;\ q = 2,\ \dots, R_j;\ \mu = 1,\ \dots,\ m$), where $m'_{j\sigma\mu}$ are chosen so that the numbers $m_{j\sigma\mu}$ and $2m-1-m'_{j\sigma\mu}$ run over the set $0,\ 1,\ \dots,\ 2m-1$, while μ changes from 1 to $2m$.

By theorem 2.1 [12, Chapter 2, §2.2], there exist uniquely defined differential operators $F_{j\sigma\mu}$, $F'_{j\sigma\mu}$, $F_{jq\mu}$, and $F'_{jq\mu}$ ($j = 1,\ \dots,\ N;\ \sigma = 1,\ R_j+1;\ q = 2,\ \dots, R_j;\ \mu = 1,\ \dots,\ m$) of orders $2m-1-m_{j\sigma\mu}$, $m'_{j\sigma\mu}$, $\mu-1$, and $m+\mu-1$ correspondingly with constant coefficients such that

the system $\{F_{j\sigma\mu},\ F'_{j\sigma\mu}\}_{\mu=1}^m$ is a Dirichlet one on $\Gamma_{j\sigma}$ of order $2m$ ($\sigma = 1,\ R_j + 1$),

the system $\{F'_{j\sigma\mu}\}_{\mu=1}^m$ covers the operator Q_j on $\Gamma_{j\sigma}$ ($\sigma = 1,\ R_j + 1$),

the system $\{F_{jq\mu},\ F'_{jq\mu}\}_{\mu=1}^m$ is a Dirichlet one on Γ_{jq} of order $2m$ ($q = 2,\ \dots,\ R_j$),

for any $U_j \in C_0^\infty(\bar\Omega_j\backslash M)$, $V_{jt} \in C^\infty(\bar\Omega_{jt}\backslash M)$ (or $U_j \in C^\infty(\bar\Omega_j\backslash M)$, $V_{jt} \in$

$C_0^\infty(\bar{\Omega}_{jt}\backslash M))$, the following Green formulas are valid:

$$
\begin{gathered}
(\mathcal{P}_j U_j,\ V_{j1})_{\Omega_{j1}} + \sum_{\mu=1}^{m}(B_{j1\mu}U_j|_{\Gamma_{j1}},\ F_{j1\mu}V_{j1}|_{\Gamma_{j1}})_{\Gamma_{j1}} + \\
+ \sum_{\mu=1}^{m}(B_{j2\mu}U_j|_{\Gamma_{j2}},\ F_{j2\mu}V_{j1}|_{\Gamma_{j2}})_{\Gamma_{j2}} = (U_j,\ \mathcal{Q}_j V_{j1})_{\Omega_{j1}} + \\
+ \sum_{\mu=1}^{m}(B'_{j1\mu}U_j|_{\Gamma_{j1}},\ F'_{j1\mu}V_{j1}|_{\Gamma_{j1}})_{\Gamma_{j1}} + \sum_{\mu=1}^{m}(B'_{j2\mu}U_j|_{\Gamma_{j2}},\ F'_{j2\mu}V_{j1}|_{\Gamma_{j2}})_{\Gamma_{j2}},
\end{gathered}
$$

$$
\begin{gathered}
(\mathcal{P}_j U_j,\ V_{j2})_{\Omega_{j2}} - \sum_{\mu=1}^{m}(B_{j2\mu}U_j|_{\Gamma_{j2}},\ F_{j2\mu}V_{j2}|_{\Gamma_{j2}})_{\Gamma_{j2}} + \\
+ \sum_{\mu=1}^{m}(B_{j3\mu}U_j|_{\Gamma_{j3}},\ F_{j3\mu}V_{j2}|_{\Gamma_{j3}})_{\Gamma_{j3}} = (U_j,\ \mathcal{Q}_j V_{j2})_{\Omega_{j2}} - \\
- \sum_{\mu=1}^{m}(B'_{j2\mu}U_j|_{\Gamma_{j2}},\ F'_{j2\mu}V_{j2}|_{\Gamma_{j2}})_{\Gamma_{j2}} + \sum_{\mu=1}^{m}(B'_{j3\mu}U_j|_{\Gamma_{j3}},\ F'_{j3\mu}V_{j2}|_{\Gamma_{j3}})_{\Gamma_{j3}}, \qquad (4.2)
\end{gathered}
$$

$$\cdots,$$

$$
\begin{gathered}
(\mathcal{P}_j U_j,\ V_{jR_j})_{\Omega_{jR_j}} - \sum_{\mu=1}^{m}(B_{jR_j\mu}U_j|_{\Gamma_{jR_j}},\ F_{jR_j\mu}V_{jR_j}|_{\Gamma_{jR_j}})_{\Gamma_{jR_j}} + \\
+ \sum_{\mu=1}^{m}(B_{j,R_j+1,\mu}U_j|_{\Gamma_{j,R_j+1}},\ F_{j,R_j+1,\mu}V_{jR_j}|_{\Gamma_{j,R_j+1}})_{\Gamma_{j,R_j+1}} = \\
= (U_j,\ \mathcal{Q}_j V_{jR_j})_{\Omega_{jR_j}} - \sum_{\mu=1}^{m}(B'_{jR_j\mu}U_j|_{\Gamma_{jR_j}},\ F'_{jR_j\mu}V_{jR_j}|_{\Gamma_{jR_j}})_{\Gamma_{jR_j}} + \\
+ \sum_{\mu=1}^{m}(B'_{j,R_j+1,\mu}U_j|_{\Gamma_{j,R_j+1}},\ F'_{j,R_j+1,\mu}V_{jR_j}|_{\Gamma_{j,R_j+1}})_{\Gamma_{j,R_j+1}}.
\end{gathered}
$$

Adding equalities (4.2) together, we get

$$
\begin{gathered}
\sum_t(\mathcal{P}_j U_j,\ V_{jt})_{\Omega_{jt}} + \sum_{\sigma=1,\,R_j+1}\sum_{\mu=1}^{m}(B_{j\sigma\mu}U_j|_{\Gamma_{j\sigma}},\ F_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}} + \\
+ \sum_{q=2}^{R_j}\sum_{\mu=1}^{m}(B_{jq\mu}U_j|_{\Gamma_{jq}},\ F_{jq\mu}V_{j,q-1}|_{\Gamma_{jq}} - F_{jq\mu}V_{jq}|_{\Gamma_{jq}})_{\Gamma_{jq}} = \\
= \sum_t(U_j,\ \mathcal{Q}_j V_{jt})_{\Omega_{jt}} + \sum_{\sigma=1,\,R_j+1}\sum_{\mu=1}^{m}(B'_{j\sigma\mu}U_j|_{\Gamma_{j\sigma}},\ F'_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}} + \\
+ \sum_{q=2}^{R_j}\sum_{\mu=1}^{m}(B'_{jq\mu}U_j|_{\Gamma_{jq}},\ F'_{jq\mu}V_{j,q-1}|_{\Gamma_{jq}} - F'_{jq\mu}V_{jq}|_{\Gamma_{jq}})_{\Gamma_{jq}}.
\end{gathered} \qquad (4.3)
$$

Add $\sum\limits_{k=1}^{N}\sum\limits_{q=2}^{R_k}\sum\limits_{s=1}^{S_{j\sigma kq}}(B_{j\sigma\mu kqs}U_k)(\mathcal{G}_{j\sigma kqs}\cdot)$ to $B_{j\sigma\mu}U_j$ and subtract it in (4.3); then using change of variables $x'=(\mathcal{G}_{j\sigma kqs}y,\ z)$ in the integrals over $\Gamma_{j\sigma}$, we obtain

$$\begin{gathered}(B_{j\sigma\mu}U_j|_{\Gamma_{j\sigma}},\ F_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}}=(\mathcal{B}_{j\sigma\mu}U,\ F_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}}-\\ -(\sum_{k,q}\sum_{s=1}^{S_{j\sigma kq}}(B_{j\sigma\mu kqs}U_k)(\mathcal{G}_{j\sigma kqs}\cdot)|_{\Gamma_{j\sigma}},\ F_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}}=\\ =(\mathcal{B}_{j\sigma\mu}U,\ F_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}}+\\ +\sum_{k,q}\sum_{s=1}^{S'_{kqj\sigma}}(-\frac{1}{\chi_{j\sigma kqs}}B_{j\sigma\mu kqs}U_k|_{\Gamma_{kq}},\ (F_{j\sigma\mu}V_j)(\mathcal{G}'_{kqj\sigma s}\cdot)|_{\Gamma_{kq}})_{\Gamma_{kq}}.\end{gathered}\tag{4.4}$$

Here $S'_{kqj\sigma}=S_{j\sigma kq}$; $\mathcal{G}'_{kqj\sigma s}$ is the operator of rotation by the angle $\varphi'_{kqj\sigma}=-\varphi_{j\sigma kq}$ and expansion by $\chi'_{kqj\sigma s}=1/\chi_{j\sigma kqs}$ times in the plane $\{y\}$.

Clearly, we have

$$-\frac{1}{\chi_{j\sigma kqs}}B_{j\sigma\mu kqs}=\sum_{\alpha=1}^{m}\Lambda_{j\sigma\mu kqs\alpha}B_{kq\alpha}-\sum_{\alpha=1}^{m}\Lambda'_{j\sigma\mu kqs\alpha}B'_{kq\alpha}{}^{4}.\tag{4.5}$$

Here

$$\Lambda_{j\sigma\mu kqs\alpha}=\sum_{|\beta|+l=0}^{m_{j\sigma\mu}-(2m-\alpha)}a^{\beta l}_{j\sigma\mu kqs\alpha}D_z^\beta\left(\frac{\partial}{\partial y_{kq}}\right)^l,$$
$$\Lambda'_{j\sigma\mu kqs\alpha}=\sum_{|\beta|+l=0}^{m_{j\sigma\mu}-(m-\alpha)}a'^{\beta l}_{j\sigma\mu kqs\alpha}D_z^\beta\left(\frac{\partial}{\partial y_{kq}}\right)^l,$$

$a^{\beta l}_{j\sigma\mu kqs\alpha}$, $a'^{\beta l}_{j\sigma\mu kqs\alpha}\in\mathbb{C}$, y_{kq} is the coordinate on the half-axis $\Gamma_{kq}\cap\{z=0\}$. If $m_{j\sigma\mu}-(2m-\alpha)<0$ ($m_{j\sigma\mu}-(m-\alpha)<0$), then we put $\Lambda_{j\sigma\mu kqs\alpha}=0$ ($\Lambda'_{j\sigma\mu kqs\alpha}=0$).

Denote by $(\Lambda_{j\sigma\mu kqs\alpha})^*$, $(\Lambda'_{j\sigma\mu kqs\alpha})^*$ the operators that are formally adjoint to $\Lambda_{j\sigma\mu kqs\alpha}$, $\Lambda'_{j\sigma\mu kqs\alpha}$ correspondingly. Then (4.4) and (4.5) imply

$$\begin{gathered}(B_{j\sigma\mu}U_j|_{\Gamma_{j\sigma}},\ F_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}}=(\mathcal{B}_{j\sigma\mu}U,\ F_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}}+\\ +\sum_{k,q}\sum_{s=1}^{S'_{kqj\sigma}}\sum_{\alpha=1}^{m}(B_{kq\alpha}U_k|_{\Gamma_{kq}},\ (\Lambda_{j\sigma\mu kqs\alpha})^*[(F_{j\sigma\mu}V_j)(\mathcal{G}'_{kqj\sigma s}\cdot)|_{\Gamma_{kq}}])_{\Gamma_{kq}}-\\ -\sum_{k,q}\sum_{s=1}^{S'_{kqj\sigma}}\sum_{\alpha=1}^{m}(B'_{kq\alpha}U_k|_{\Gamma_{kq}},\ (\Lambda'_{j\sigma\mu kqs\alpha})^*[(F'_{j\sigma\mu}V_j)(\mathcal{G}'_{kqj\sigma s}\cdot)|_{\Gamma_{kq}}])_{\Gamma_{kq}}.\end{gathered}\tag{4.6}$$

[4] We choose the sign "minus" in right hand side of relation (4.5) just for convenience.

Substituting (4.6) into (4.3), summing over j, and grouping the summands containing $B_{jq\mu}U_j$, we get

$$
\begin{gathered}
\sum_j\Big\{\sum_t(\mathcal{P}_jU_j,\ V_{jt})_{\Omega_{jt}}+\sum_{\sigma=1,\,R_j+1}\sum_{\mu=1}^{m}(\mathcal{B}_{j\sigma\mu}U,\ F_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}}+\\
+\sum_{q=2}^{R_j}\sum_{\mu=1}^{m}(B_{jq\mu}U_j|_{\Gamma_{jq}},\ F_{jq\mu}V_{j,q-1}|_{\Gamma_{jq}}-F_{jq\mu}V_{jq}|_{\Gamma_{jq}}+\\
+\sum_k\sum_{\sigma=1,\,R_k+1}\sum_{s=1}^{S'_{jqk\sigma}}(\{\sum_{\alpha=1}^{m}(\hat\Lambda_{k\sigma\alpha jqs\mu})^*F_{k\sigma\alpha}\}V_k)(\mathcal{G}'_{jqk\sigma s}\cdot)|_{\Gamma_{jq}})_{\Gamma_{jq}}\Big\}=\\
=\sum_j\Big\{\sum_t(U_j,\ \mathcal{Q}_jV_{jt})_{\Omega_{jt}}+\sum_{\sigma=1,\,R_j+1}\sum_{\mu=1}^{m}(B'_{j\sigma\mu}U_j|_{\Gamma_{j\sigma}},\ F'_{j\sigma\mu}V_j|_{\Gamma_{j\sigma}})_{\Gamma_{j\sigma}}+\\
+\sum_{q=2}^{R_j}\sum_{\mu=1}^{m}(B'_{jq\mu}U_j|_{\Gamma_{jq}},\ F'_{jq\mu}V_{j,q-1}|_{\Gamma_{jq}}-F'_{jq\mu}V_{jq}|_{\Gamma_{jq}}+\\
+\sum_k\sum_{\sigma=1,\,R_k+1}\sum_{s=1}^{S'_{jqk\sigma}}(\{\sum_{\alpha=1}^{m}(\hat\Lambda'_{k\sigma\alpha jqs\mu})^*F'_{k\sigma\alpha}\}V_k)(\mathcal{G}'_{jqk\sigma s}\cdot)|_{\Gamma_{jq}})_{\Gamma_{jq}}\Big\},
\end{gathered}
\tag{4.7}
$$

where the operators $\hat\Lambda_{k\sigma\alpha jqs\mu}$ and $\hat\Lambda'_{k\sigma\alpha jqs\mu}$ are obtained from the operators $\Lambda_{k\sigma\alpha jqs\mu}$ and $\Lambda'_{k\sigma\alpha jqs\mu}$ by substituting $a^{\beta l}_{k\sigma\alpha jqs\mu}(\chi'_{jqk\sigma s})^l$ and ${a'}^{\beta l}_{k\sigma\alpha jqs\mu}(\chi'_{jqk\sigma s})^l$ for $a^{\beta l}_{k\sigma\alpha jqs\mu}$ and ${a'}^{\beta l}_{k\sigma\alpha jqs\mu}$ correspondingly.

Denoting

$$
\begin{gathered}
C_{j\sigma\mu}=F'_{j\sigma\mu}\ (j=1,\ \dots,\ N;\ \sigma=1,\ R_j+1;\ \mu=1,\ \dots,\ m),\\
T_{jq\nu}=F_{jq\nu}\ \text{for}\ \nu=1,\ \dots,\ m;\ T_{jq\nu}=F'_{jq,\nu-m}\ \text{for}\ \nu=m+1,\ \dots,\ 2m;\\
T_{jq\nu k\sigma s}=\sum_{\alpha=1}^{m}(\hat\Lambda_{k\sigma\alpha jqs\nu})^*F_{k\sigma\alpha}\ \text{for}\ \nu=1,\ \dots,\ m,\\
T_{jq\nu k\sigma s}=\sum_{\alpha=1}^{m}(\hat\Lambda'_{k\sigma\alpha jqs,\nu-m})^*F'_{k\sigma\alpha}\ \text{for}\ \nu=m+1,\ \dots,\ 2m\\
(j,\ k=1,\ \dots,\ N;\ q=2,\ \dots,\ R_j;\ \sigma=1,\ R_k+1;\ s=1,\ \dots,\ S'_{jqk\sigma}),
\end{gathered}
$$

we complete the proof. □

Remark 4.1. *Formula (4.1) can be extended by continuity for the case* $U_j\in H_a^{2m}(\Omega_j)$, $V_{jt}\in H^{2m}_{-a+2m}(\Omega_{jt})$. *Indeed,* $C_0^\infty(\bar\Omega_j\backslash M)$ *is dense in* $H_a^{2m}(\Omega_j)$, $C_0^\infty(\bar\Omega_{jt}\backslash M)$ *is dense in* $H^{2m}_{-a+2m}(\Omega_{jt})$; *therefore there exist sequences* $\{U_j^p\}_{p=1}^\infty\subset C_0^\infty(\bar\Omega_j\backslash\{0\})$ *and* $\{V_{jt}^q\}_{q=1}^\infty\subset C_0^\infty(\bar\Omega_{jt}\backslash\{0\})$ *that converge to* U_j *and* V_{jt} *in* $H_a^{2m}(\Omega_j)$ *and* $H^{2m}_{-a+2m}(\Omega_{jt})$ *correspondingly. Green* *formula (4.1) is valid for the functions U_j^p and V_{jt}^q; passing to the limit as $p,\ q \to \infty$, we obtain the Green formula for U_j and V_{jt} (we can pass to the limit by virtue of the Schwarz inequality and Theorem B.1).*

The following two examples illustrate the Green formula.

Example 4.1. For simplicity we assume that $n = 2,\ N = 1$. Put $K = \{y:\ \ r > 0,\ b_1 < \varphi < b_3\}$, $K_t = \{y:\ \ r > 0,\ b_t < \varphi < b_{t+1}\}$ $(t = 1,\ 2)$, $\gamma_q = \{y:\ \ r > 0,\ \varphi = b_q\}$ $(q = 1,\ 2,\ 3)$, where $y = (y_1,\ y_2) \in \mathbb{R}^2$; $0 < b_1 < b_2 < b_3 < 2\pi$.

Let n_1 be the unit normal vector to γ_1 directed inside K_1 and n_2, n_3 be the unit normal vectors to γ_2, γ_3 correspondingly directed inside K_2.

Consider the nonlocal problem

$$-\triangle U = f(y) \quad (y \in K), \tag{4.8}$$

$$\begin{gathered} U|_{\gamma_1} + \alpha U(\chi_{12} r,\ \varphi + \varphi_{12})|_{\gamma_1} = g_1(y) \quad\ (y \in \gamma_1), \\ U|_{\gamma_3} = g_3(y) \quad (y \in \gamma_3). \end{gathered} \tag{4.9}$$

Here $U(r,\ \varphi)$ is the function $U(y)$ written in the polar coordinates; $b_1 + \varphi_{12} = b_2$, $\chi_{12} > 0$; $\alpha \in \mathbb{R}$.

Take $U \in C_0^\infty(\bar{K}\backslash\{0\})$, $V_t \in C^\infty(\bar{K}_t\backslash\{0\})$. Multiply $-\triangle U$ by $\bar{V}_t$ and integrate over K_t, $t = 1,\ 2$; then using the formula of integration by parts, we get

$$\begin{aligned} &\int\limits_{K_1} (-\triangle U) \cdot \bar{V}_1\, dy + \int\limits_{\gamma_1} U|_{\gamma_1} \cdot \frac{\partial \bar{V}_1}{\partial n_1}\Big|_{\gamma_1} d\gamma - \int\limits_{\gamma_2} U|_{\gamma_2} \cdot \frac{\partial \bar{V}_1}{\partial n_2}\Big|_{\gamma_2} d\gamma = \\ &= \int\limits_{K_1} U \cdot (-\triangle \bar{V}_1)\, dy + \int\limits_{\gamma_1} \frac{\partial U}{\partial n_1}\Big|_{\gamma_1} \cdot \bar{V}_1|_{\gamma_1}\, d\gamma - \int\limits_{\gamma_2} \frac{\partial U}{\partial n_2}\Big|_{\gamma_2} \cdot \bar{V}_1|_{\gamma_2}\, d\gamma, \end{aligned}$$

$$\begin{aligned} &\int\limits_{K_2} (-\triangle U) \cdot \bar{V}_2\, dy + \int\limits_{\gamma_2} U|_{\gamma_2} \cdot \frac{\partial \bar{V}_2}{\partial n_2}\Big|_{\gamma_2} d\gamma + \int\limits_{\gamma_3} U|_{\gamma_3} \cdot \frac{\partial \bar{V}_2}{\partial n_3}\Big|_{\gamma_3} d\gamma = \\ &= \int\limits_{K_2} U \cdot (-\triangle \bar{V}_2)\, dy + \int\limits_{\gamma_2} \frac{\partial U}{\partial n_2}\Big|_{\gamma_2} \cdot \bar{V}_2|_{\gamma_2}\, d\gamma + \int\limits_{\gamma_3} \frac{\partial U}{\partial n_3}\Big|_{\gamma_3} \cdot \bar{V}_2|_{\gamma_3}\, d\gamma. \end{aligned}$$

Adding the last two equalities together, we obtain

$$\begin{gathered}
\sum_t \int_{K_t} (-\triangle U)\cdot \bar V_t\,dy + \int_{\gamma_1} U|_{\gamma_1}\cdot \frac{\partial \bar V_1}{\partial n_1}\Big|_{\gamma_1} d\gamma + \\
+\int_{\gamma_3} U|_{\gamma_3}\cdot \frac{\partial \bar V_2}{\partial n_3}\Big|_{\gamma_3} d\gamma + \int_{\gamma_2} U|_{\gamma_2}\cdot \Big(\frac{\partial \bar V_2}{\partial n_2}\Big|_{\gamma_2} - \frac{\partial \bar V_1}{\partial n_2}\Big|_{\gamma_2}\Big)\,d\gamma = \\
= \sum_t \int_{K_t} U\cdot(-\triangle \bar V_t)\,dy + \int_{\gamma_1} \frac{\partial U}{\partial n_1}\Big|_{\gamma_1}\cdot \bar V_1|_{\gamma_1}\,d\gamma + \int_{\gamma_3} \frac{\partial U}{\partial n_3}\Big|_{\gamma_3}\cdot \bar V_2|_{\gamma_3}\,d\gamma + \\
+\int_{\gamma_2} \frac{\partial U}{\partial n_2}\Big|_{\gamma_2}\cdot(\bar V_2|_{\gamma_2} - \bar V_1|_{\gamma_2})\,d\gamma.
\end{gathered} \tag{4.10}$$

But we have

$$\begin{gathered}
\int_{\gamma_1} U|_{\gamma_1}\cdot \frac{\partial \bar V_1}{\partial n_1}\Big|_{\gamma_1} d\gamma = \int_{\gamma_1} (U|_{\gamma_1} + \alpha U(\chi_{12} r,\ \varphi+\varphi_{12})|_{\gamma_1})\cdot \frac{\partial \bar V_1}{\partial n_1}\Big|_{\gamma_1} d\gamma - \\
-\int_{\gamma_1} \alpha U(\chi_{12} r,\ \varphi+\varphi_{12})|_{\gamma_1}\cdot \frac{\partial \bar V_1}{\partial n_1}\Big|_{\gamma_1} d\gamma = \\
= \int_{\gamma_1} (U|_{\gamma_1} + \alpha U(\chi_{12} r,\ \varphi+\varphi_{12})|_{\gamma_1})\cdot \frac{\partial \bar V_1}{\partial n_1}\Big|_{\gamma_1} d\gamma - \\
-\int_{\gamma_2} U|_{\gamma_2}\cdot \alpha\chi'_{21}\frac{\partial \bar V_1}{\partial n_1}(\chi'_{21} r,\ \varphi+\varphi'_{21})\Big|_{\gamma_2} d\gamma,
\end{gathered}$$

where $\chi'_{21} = 1/\chi_{12},\ \varphi'_{21} = -\varphi_{12}$. This and (4.10) finally yield

$$\begin{gathered}
\sum_t \int_{K_t} (-\triangle U)\cdot \bar V_t\,dy + \int_{\gamma_1} (U|_{\gamma_1} + \alpha U(\chi_{12} r,\ \varphi+\varphi_{12})|_{\gamma_1})\cdot \frac{\partial \bar V_1}{\partial n_1}\Big|_{\gamma_1} d\gamma + \\
+\int_{\gamma_3} U|_{\gamma_3}\cdot \frac{\partial \bar V_2}{\partial n_3}\Big|_{\gamma_3} d\gamma + \int_{\gamma_2} \frac{\partial U}{\partial n_2}\Big|_{\gamma_2}\cdot(\bar V_1|_{\gamma_2} - \bar V_2|_{\gamma_2})\,d\gamma = \\
= \sum_t \int_{K_t} U\cdot(-\triangle \bar V_t)\,dy + \int_{\gamma_1} \frac{\partial U}{\partial n_1}\Big|_{\gamma_1}\cdot \bar V_1|_{\gamma_1}\,d\gamma + \int_{\gamma_3} \frac{\partial U}{\partial n_3}\Big|_{\gamma_3}\cdot \bar V_2|_{\gamma_3}\,d\gamma + \\
\int_{\gamma_2} U|_{\gamma_2}\cdot \Big(\frac{\partial \bar V_1}{\partial n_2}\Big|_{\gamma_2} - \frac{\partial \bar V_2}{\partial n_2}\Big|_{\gamma_2} + \alpha\chi'_{21}\frac{\partial \bar V_1}{\partial n_1}(\chi'_{21} r,\ \varphi+\varphi'_{21})\Big|_{\gamma_2}\Big)\,d\gamma.
\end{gathered}$$

Example 4.2. Using denotations of Example 4.1, consider the nonlocal problem

$$-\triangle U = f(y) \quad (y\in K), \tag{4.11}$$

$$\begin{gathered}
\frac{\partial U}{\partial n_1}|_{\gamma_1} + \alpha\frac{\partial U}{\partial r}(\chi_{12} r,\ \varphi+\varphi_{12})|_{\gamma_1} = g_1(y) \quad (y\in\gamma_1), \\
\frac{\partial U}{\partial n_3}|_{\gamma_3} = g_3(y) \quad (y\in\gamma_3).
\end{gathered} \tag{4.12}$$

From formula (4.10) and equality

$$\int\limits_{\gamma_1} \frac{\partial U}{\partial n_1}\Big|_{\gamma_1} \cdot \bar V_1|_{\gamma_1}\, d\gamma = \int\limits_{\gamma_1} \Big(\frac{\partial U}{\partial n_1}\Big|_{\gamma_1} + \alpha \frac{\partial U}{\partial r}(\chi_{12} r,\ \varphi + \varphi_{12})\Big|_{\gamma_1} \Big) \cdot \bar V_1|_{\gamma_1}\, d\gamma +$$
$$+ \int\limits_{\gamma_2} U|_{\gamma_2} \cdot \alpha (\chi'_{21})^2 \frac{\partial \bar V_1}{\partial r}(\chi'_{21} r,\ \varphi + \varphi'_{21})\Big|_{\gamma_2} d\gamma$$

(where $\chi'_{21} = 1/\chi_{12},\ \varphi'_{21} = -\varphi_{12}$), we get the following Green formula:

$$\sum_t \int\limits_{K_t} (-\triangle U) \cdot \bar V_t\, dy + \int\limits_{\gamma_1} \Big(\frac{\partial U}{\partial n_1}\Big|_{\gamma_1} + \alpha \frac{\partial U}{\partial r}(\chi_{12} r,\ \varphi + \varphi_{12})\Big|_{\gamma_1} \Big) \cdot (-\bar V_1)|_{\gamma_1}\, d\gamma +$$
$$+ \int\limits_{\gamma_3} \frac{\partial U}{\partial n_3}\Big|_{\gamma_3} \cdot (-\bar V_2)|_{\gamma_3}\, d\gamma + \int\limits_{\gamma_2} \frac{\partial U}{\partial n_2}\Big|_{\gamma_2} \cdot (\bar V_1|_{\gamma_2} - \bar V_2|_{\gamma_2})\, d\gamma =$$

$$= \sum_t \int\limits_{K_t} U \cdot (-\triangle \bar V_t)\, dy + \int\limits_{\gamma_1} (-U)|_{\gamma_1} \cdot \frac{\partial \bar V_1}{\partial n_1}\Big|_{\gamma_1} d\gamma + \int\limits_{\gamma_3} (-U)|_{\gamma_3} \cdot \frac{\partial \bar V_2}{\partial n_3}\Big|_{\gamma_3} d\gamma +$$
$$+ \int\limits_{\gamma_2} U|_{\gamma_2} \cdot \Big(\frac{\partial \bar V_1}{\partial n_2}\Big|_{\gamma_2} - \frac{\partial \bar V_2}{\partial n_2}\Big|_{\gamma_2} + \alpha (\chi'_{21})^2 \frac{\partial \bar V_1}{\partial r}(\chi'_{21} r,\ \varphi + \varphi'_{21})\Big|_{\gamma_2} \Big) d\gamma.$$

2 The Green formula with parameter η in plane angles.

For $n = 2,\ j = 1,\ \dots,\ N$, put

$$K_j = \{y:\ r > 0,\ b_{j1} < \varphi < b_{j,R_j+1}\},$$
$$K_{jt} = \{y:\ r > 0,\ b_{jt} < \varphi < b_{j,t+1}\}\ (t = 1,\ \dots,\ R_j),$$
$$\gamma_{jq} = \{y:\ r > 0,\ \varphi = b_{jq}\}\ (q = 1,\ \dots,\ R_j + 1).$$

Replace D_z by η in differential operators and consider the auxiliary nonlocal boundary value problem with parameter $\eta \in \mathbb{R}^{n-2}$ for $u = (u_1,\ \dots,\ u_N)$

$$\mathcal{P}_j(D_y,\ \eta) u_j = f_j(y) \quad (y \in K_j), \tag{4.13}$$

$$\mathcal{B}_{j\sigma\mu}(D_y,\ \eta) u = B_{j\sigma\mu}(D_y,\ \eta) u_j|_{\gamma_{j\sigma}} + \\ + \sum_{k,q,s} (B_{j\sigma\mu kqs}(D_y,\ \eta) u_k)(\mathcal{G}_{j\sigma kqs} y)|_{\gamma_{j\sigma}} = g_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma}) \tag{4.14}$$

$$(j = 1,\ \dots,\ N;\ \sigma = 1,\ R_j + 1;\ \mu = 1,\ \dots,\ m).$$

For $u_{jt} \in C_0^\infty(\bar K_{jt} \backslash \{0\}),\ v_{jt} \in C^\infty(\bar K_{jt} \backslash \{0\})$ (or $u_{jt} \in C^\infty(\bar K_{jt} \backslash \{0\}),\ v_{jt} \in C_0^\infty(\bar K_{jt} \backslash \{0\})$), put

$$(u_{jt},\ v_{jt})_{K_{jt}} = \int\limits_{K_{jt}} u_{jt} \cdot \bar v_{jt}\, dy \quad (j = 1,\ \dots,\ N;\ t = 1,\ \dots,\ R_j).$$

For $u_{\gamma_{jq}} \in C_0^\infty(\gamma_{jq}),\ v_{\gamma_{jq}} \in C^\infty(\gamma_{jq})$ (or $u_{\gamma_{jq}} \in C^\infty(\gamma_{jq}),\ v_{\gamma_{jq}} \in C_0^\infty(\gamma_{jq})$), put

$$(u_{\gamma_{jq}},\ v_{\gamma_{jq}})_{\gamma_{jq}} = \int\limits_{\gamma_{jq}} u_{\gamma_{jq}} \cdot \bar{v}_{\gamma_{jq}}\, d\gamma \quad (j = 1,\ \ldots,\ N;\ q = 1,\ \ldots,\ R_j + 1).$$

If we have functions $v_{jt}(y)$ defined in K_{jt}, then denote by $v_j(y)$ the function given by $v_j(y) \equiv v_{jt}(y)$ for $y \in K_{jt}$.

Theorem 4.2. *Let $\mathcal{P}_j$, $B_{j\sigma\mu}$, etc., be the operators from Theorem 4.1. Then for all $u_j \in C_0^\infty(\bar{K}_j\backslash\{0\})$, $v_{jt} \in C^\infty(\bar{K}_{jt}\backslash\{0\})$ (or $u_j \in C^\infty(\bar{K}_j\backslash\{0\})$, $v_{jt} \in C_0^\infty(\bar{K}_{jt}\backslash\{0\})$), the following Green formula with parameter η is valid:*

$$\begin{gathered}
\sum_j \Big\{ \sum_t (\mathcal{P}_j(D_y,\ \eta)u_j,\ v_{jt})_{K_{jt}} + \\
+ \sum_{\sigma,\mu} (\mathcal{B}_{j\sigma\mu}(D_y,\ \eta)u,\ F_{j\sigma\mu}(D_y,\ \eta)v_j|_{\gamma_{j\sigma}})_{\gamma_{j\sigma}} + \\
+ \sum_{q,\mu} (B_{jq\mu}(D_y,\ \eta)u_j|_{\gamma_{jq}},\ \mathcal{T}_{jq\mu}(D_y,\ \eta)v)_{\gamma_{jq}} \Big\} = \\
= \sum_j \Big\{ \sum_t (u_j,\ \mathcal{Q}_j(D_y,\ \eta)v_{jt})_{K_{jt}} + \\
+ \sum_{\sigma,\mu} (B'_{j\sigma\mu}(D_y,\ \eta)u_j|_{\gamma_{j\sigma}},\ C_{j\sigma\mu}(D_y,\ \eta)v_j|_{\gamma_{j\sigma}})_{\gamma_{j\sigma}} + \\
+ \sum_{q,\mu} (B'_{jq\mu}(D_y,\ \eta)u_j|_{\gamma_{jq}},\ \mathcal{T}_{jq,m+\mu}(D_y,\ \eta)v)_{\gamma_{jq}} \Big\}.
\end{gathered} \tag{4.15}$$

Here $\mathcal{B}_{j\sigma\mu}(D_y,\ \eta)$ is given by (4.14);

$$\begin{gathered}
\mathcal{T}_{jq\nu}(D_y,\ \eta)v = T_{jq\nu}(D_y,\ \eta)v_{j,q-1}|_{\gamma_{jq}} - T_{jq\nu}(D_y,\ \eta)v_{jq}|_{\gamma_{jq}} + \\
+ \sum_{k,\sigma,s} (T_{jq\nu k\sigma s}(D_y,\ \eta)v_k)(\mathcal{G}'_{jqk\sigma s}y)|_{\gamma_{jq}} \\
(\nu = 1,\ \ldots,\ 2m);
\end{gathered}$$

$\mathcal{G}'_{jqk\sigma s}$ is the transformation defined in Theorem 4.1.

Proof. Introduce the functions $\psi_1,\ \psi_2 \in C_0^\infty(\mathbb{R}^{n-2})$ such that

$$\psi_1(z) = 0 \text{ for } |z| > 1, \int\limits_{\mathbb{R}^{n-2}} \psi_1(z)dz = 1,$$

$$\psi_2(z) = 1 \text{ for } |z| < 1,\ \psi_2(z) = 0 \text{ for } |z| > 2.$$

Substituting $U_j(y,\ z) = e^{i(\eta,\ z)}\psi_1(z)u_j(y)$, $V_{jt}(y,\ z) = e^{i(\eta,\ z)}\psi_2(z)v_{jt}(y)$ into equality (4.1), we get (4.15). □

Remark 4.2. *Replacing in Remark 4.1* $H_a^{2m}(\cdot)$ *and* $H_{-a+2m}^{2m}(\cdot)$ *by* $E_a^{2m}(\cdot)$ *and* $E_{-a+2m}^{2m}(\cdot)$ *correspondingly and Theorem B.1 by Theorem B.2, we see that formula (4.15) can be extended by continuity for the case* $u_j \in E_a^{2m}(K_j)$, $v_{jt} \in E_{-a+2m}^{2m}(K_{jt})$.

3 The Green formula with parameter λ on arcs.

Put $\Pi_j = \{(\varphi,\ \tau):\ b_{j1} < \varphi < b_{j,R_j+1},\ \tau \in \mathbb{R}\}$, $\Pi_{jt} = \{(\varphi,\ \tau):\ b_{jt} < \varphi < b_{j,t+1},\ \tau \in \mathbb{R}\}$ $(t = 1,\ \dots,\ R_j)$.

For $u_{jt} \in C_0^\infty(\bar{\Pi}_{jt}),\ v_{jt} \in C^\infty(\bar{\Pi}_{jt})$ (or $u_{jt} \in C^\infty(\bar{\Pi}_{jt}),\ v_{jt} \in C_0^\infty(\bar{\Pi}_{jt})$), denote

$$(u_{jt},\ v_{jt})_{\Pi_{jt}} = \int\limits_{-\infty}^{\infty} \int\limits_{b_{jt}}^{b_{j,t+1}} u_{jt}(\varphi,\ \tau) \cdot \overline{v_{jt}(\varphi,\ \tau)}\, d\varphi d\tau$$
$$(j = 1,\ \dots,\ N;\ t = 1,\ \dots,\ R_j).$$

For $\psi \in C_0^\infty(\mathbb{R}),\ \xi \in C^\infty(\mathbb{R})$ (or $\psi \in C^\infty(\mathbb{R}),\ \xi \in C_0^\infty(\mathbb{R})$), denote $(\psi,\ \xi)_{\mathbb{R}} = \int\limits_{-\infty}^{\infty} \psi(\tau) \cdot \overline{\xi(\tau)}\, d\tau$. For $\tilde{U}_{jt},\ \tilde{V}_{jt} \in C^\infty([b_{jt},\ b_{j,t+1}])$, we also denote

$$(\tilde{U}_{jt},\ \tilde{V}_{jt})_{(b_{jt},\ b_{j,t+1})} = \int\limits_{b_{jt}}^{b_{j,t+1}} \tilde{U}_{jt}(\varphi) \cdot \overline{\tilde{V}_{jt}(\varphi)}\, d\varphi \quad (j = 1,\ \dots,\ N;\ t = 1,\ \dots,\ R_j).$$

And finally for $d,\ e \in \mathbb{C}$, we put $(d,\ e)_{\mathbb{C}} = d \cdot \bar{e}$.

If we have functions $\tilde{V}_{jt}(\varphi)$ defined in $[b_{jt},\ b_{j,t+1}]$, then denote by $\tilde{V}_j(\varphi)$ the function given by $\tilde{V}_j(\varphi) \equiv \tilde{V}_{jt}(\varphi)$ for $\varphi \in (b_{jt},\ b_{j,t+1})$.

Put $D_z = 0$ and write the differential operators in the polar coordinates: $\mathcal{P}_j(D_y,\ 0) = r^{-2m}\tilde{\mathcal{P}}_j(\varphi,\ D_\varphi,\ rD_r)$, $B_{j\sigma\mu}(D_y,\ 0) = r^{-m_{j\sigma\mu}}\tilde{B}_{j\sigma\mu}(\varphi,\ D_\varphi,\ rD_r)$, ets.

Consider nonlocal boundary value problem (2.3), (2.4) with parameter λ.

Theorem 4.3. *Let* $\mathcal{P}_j$, $B_{j\sigma\mu}$, *etc., be the operators from Theorem 4.1. Then for all* $\tilde{U}_j \in C^\infty([b_{j1},\ b_{j,R_j+1}])$, $\tilde{V}_{jt} \in C^\infty([b_{jt},\ b_{j,t+1}])$, *the following Green* *formula with parameter λ is valid:*

$$\begin{gathered}
\sum_j\big\{\sum_t(\tilde{\mathcal{P}}_j(\varphi,\ D_\varphi,\ \lambda)\tilde{U}_j,\ \tilde{V}_{jt})_{(b_{jt},\ b_{j,t+1})}+\\
+\sum_{\sigma,\mu}(\tilde{\mathcal{B}}_{j\sigma\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{U},\ \tilde{F}_{j\sigma\mu}(\varphi,\ D_\varphi,\ \lambda')\tilde{V}_j|_{\varphi=b_{j\sigma}})_{\mathbb{C}}+\\
+\sum_{q,\mu}(\tilde{B}_{jq\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{U}_j|_{\varphi=b_{jq}},\ \tilde{\mathcal{T}}_{jq\mu}(\varphi,\ D_\varphi,\ \lambda')\tilde{V})_{\mathbb{C}}\big\}=\\
=\sum_j\big\{\sum_t(\tilde{U}_j,\ \tilde{\mathcal{Q}}_j(\varphi,\ D_\varphi,\ \lambda')\tilde{V}_{jt})_{(b_{jt},\ b_{j,t+1})}+\\
+\sum_{\sigma,\mu}(\tilde{B}'_{j\sigma\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{U}_j|_{\varphi=b_{j\sigma}},\tilde{C}_{j\sigma\mu}(\varphi,\ D_\varphi,\ \lambda')\tilde{V}_j|_{\varphi=b_{j\sigma}})_{\mathbb{C}}+\\
+\sum_{q,\mu}(\tilde{B}'_{jq\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{U}_j|_{\varphi=b_{jq}},\ \tilde{\mathcal{T}}_{jq,m+\mu}(\varphi,\ D_\varphi,\ \lambda')\tilde{V})_{\mathbb{C}}\big\}.
\end{gathered}\tag{4.16}$$

Here $\tilde{\mathcal{B}}_{j\sigma\mu}(\varphi,\ D_\varphi,\ \lambda)$ is given by (2.4);

$$\begin{gathered}
\tilde{\mathcal{T}}_{jq\nu}(\varphi,\ D_\varphi,\ \lambda')\tilde{V}=\tilde{T}_{jq\nu}(\varphi,\ D_\varphi,\ \lambda')\tilde{V}_{j,q-1}(\varphi)|_{\varphi=b_{jq}}-\\
-\tilde{T}_{jq\nu}(\varphi,\ D_\varphi,\ \lambda')\tilde{V}_{jq}(\varphi)|_{\varphi=b_{jq}}+\\
+\sum_{k,\sigma,s}e^{(i\lambda'-(\nu-1))\ln\chi'_{jqk\sigma s}}\tilde{T}_{jq\nu k\sigma s}(\varphi,\ D_\varphi,\ \lambda')\tilde{V}_k(\varphi+\varphi'_{jqk\sigma})|_{\varphi=b_{jq}};
\end{gathered}$$

$\lambda'=\bar{\lambda}-2i(m-1)$; $\varphi'_{jqk\sigma}$ and $\chi'_{jqk\sigma s}$ are the rotation angles and the expansion coefficients correspondingly defined in Theorem 4.1.

Proof. Put $r=e^\tau,\ v_{jt}=r^{2m-2}w_{jt},\ w_j(\varphi,\ \tau)\equiv w_{jt}(\varphi,\ \tau)$ for $(\varphi,\ \tau)\in\Pi_{jt}$. Then from formula (4.15) for $\eta=0$, we obtain

$$\begin{gathered}
\sum_j\Big\{\sum_t\Big(\tilde{\mathcal{P}}_j(\varphi,\ D_\varphi,\ D_\tau)u_j,\ w_{jt}\Big)_{\Pi_{jt}}+\\
+\sum_{\sigma,\mu}\Big(\tilde{\mathcal{B}}_{j\sigma\mu}(\varphi,\ D_\varphi,\ D_\tau)u,\tilde{F}_{j\sigma\mu}(\varphi,\ D_\varphi,\ D_\tau-2i(m-1))w_j|_{\varphi=b_{j\sigma}}\Big)_{\mathbb{R}}+\\
+\sum_{q,\mu}\Big(\tilde{B}_{jq\mu}(\varphi,\ D_\varphi,\ D_\tau)u_j|_{\varphi=b_{j\sigma}},\tilde{\mathcal{T}}_{jq\mu}(\varphi,\ D_\varphi,\ D_\tau-2i(m-1))w\Big)_{\mathbb{R}}\Big\}=\\
=\sum_j\Big\{\sum_t\Big(u_j,\ \tilde{\mathcal{Q}}_j(\varphi,\ D_\varphi,\ D_\tau-2i(m-1))w_{jt}\Big)_{\Pi_{jt}}+\\
\sum_{\sigma,\mu}\Big(\tilde{B}'_{j\sigma\mu}(\varphi,\ D_\varphi,\ D_\tau)u_j|_{\varphi=b_{j\sigma}},\tilde{C}_{j\sigma\mu}(\varphi,\ D_\varphi,\ D_\tau-2i(m-1))w_j|_{\varphi=b_{j\sigma}}\Big)_{\mathbb{R}}+\\
+\sum_{q,\mu}\Big(\tilde{B}'_{jq\mu}(\varphi,\ D_\varphi,\ D_\tau)w_j|_{\varphi=b_{jq}},\tilde{\mathcal{T}}_{jq,m+\mu}(\varphi,\ D_\varphi,\ D_\tau-2i(m-1))w\Big)_{\mathbb{R}}\Big\},
\end{gathered}\tag{4.17}$$

where

$$\tilde{\mathcal{B}}_{j\sigma\mu}(\varphi,\ D_\varphi,\ D_\tau)u = \tilde{B}_{j\sigma\mu}(\varphi,\ D_\varphi,\ D_\tau)u_j|_{\varphi=b_{j\sigma}}+$$
$$+\sum_{k,q,s} e^{-m_{j\sigma\mu}\ln\chi_{j\sigma kqs}}\tilde{B}_{j\sigma\mu kqs}(\varphi,\ D_\varphi,\ D_\tau)u_k(\varphi+\varphi_{j\sigma kq},\ \tau+\ln\chi_{j\sigma kqs})|_{\varphi=b_{j\sigma}},$$

$$\tilde{\mathcal{T}}_{jq\nu}(\varphi,\ D_\varphi,\ D_\tau-2i(m-1))w =$$
$$=\tilde{T}_{jq\nu}(\varphi,\ D_\varphi,\ D_\tau-2i(m-1))w_{j,q-1}|_{\varphi=b_{jq}}-$$
$$-\tilde{T}_{jq\nu}(\varphi,\ D_\varphi,\ D_\tau-2i(m-1))w_{jq}|_{\varphi=b_{jq}}+\sum_{k,\sigma,s} e^{(2(m-1)-(\nu-1))\ln\chi'_{jqk\sigma s}}\times$$
$$\times\tilde{T}_{jq\nu k\sigma s}(\varphi,\ D_\varphi,\ D_\tau-2i(m-1))w_k(\varphi+\varphi'_{jqk\sigma},\ \tau+\ln\chi'_{jqk\sigma s})|_{\varphi=b_{jq}}.$$

Introduce the functions $\psi_1,\ \psi_2\in C_0^\infty(\mathbb{R})$ such that

$$\psi_1(\tau)=0 \text{ for } |\tau|>1,\ \int\limits_{-\infty}^{\infty}\psi_1(\tau)d\tau=1,$$

$$\psi_2(\tau)=1 \text{ for } |\tau|<1,\ \psi_2(\tau)=0 \text{ for } |\tau|>2.$$

Substituting $u_j(\varphi,\ \tau)=e^{i\lambda\tau}\psi_1(\tau)\tilde{U}_j(\varphi)$, $w_{jt}(\varphi,\ \tau)=e^{i\bar{\lambda}\tau}\psi_2(\tau)\tilde{V}_{jt}(\varphi)$ into equality (4.17), we obtain (4.16). □

Remark 4.3. *Formula (4.16) can be extended by continuity for the case* $\tilde{U}_j\in W^{2m}(b_{j1},\ b_{j,R_j+1})$, $\tilde{V}_{jt}\in W^{2m}(b_{jt},\ b_{j,t+1})$ *(see remark 2.2 [12, Chapter 2, §2.3]).*

5 Nonlocal elliptic transmission problems. Reduction to problems with homogeneous nonlocal and boundary conditions

1 Nonlocal problems in dihedral angles.

Put $V=(V_1,\ \dots,\ V_N)$, $f=(f_1,\ \dots,\ f_N)$. Here the functions $V_j(x)$ ($f_j(x)$) are defined in Ω_j ($j=1,\ \dots,\ N$). As before, we shall denote by V_{jt} (f_{jt}) the restriction of V_j (f_j) to Ω_{jt}. Then we see that Green formula (4.1) generates the problem, which is formally adjoint to problem (1.1), (1.2)

$$\mathcal{Q}_j(D_y,\ D_z)V_{jt}=f_{jt}(x)\quad (x\in\Omega_{jt};\ t=1,\ \dots,\ R_j), \tag{5.1}$$

$$
\begin{gathered}
\mathcal{C}_{j1\mu}(D_y,\ D_z)V = C_{j1\mu}(D_y,\ D_z)V_{j1}(x)|_{\Gamma_{j1}} = g_{j1\mu}(x) \quad (x \in \Gamma_{j1}),\\
\mathcal{C}_{j,R_j+1,\mu}(D_y,\ D_z)V = C_{j,R_j+1,\mu}(D_y,\ D_z)V_{jR_j}(x)|_{\Gamma_{j,R_j+1}} =\\
= g_{j,R_j+1,\mu}(x) \quad (x \in \Gamma_{j,R_j+1}),
\end{gathered} \tag{5.2}
$$

$$
\begin{gathered}
\mathcal{T}_{jq\nu}(D_y,\ D_z)V = T_{jq\nu}(D_y,\ D_z)V_{j,q-1}(x)|_{\Gamma_{jq}} - T_{jq\nu}(D_y,\ D_z)V_{jq}(x)|_{\Gamma_{jq}} +\\
+ \sum_{k,\sigma,s} (T_{jq\nu k\sigma s}(D_y,\ D_z)V_k)(\mathcal{G}'_{jqk\sigma s}y,\ z)|_{\Gamma_{jq}} = h_{jq\nu}(x) \quad (x \in \Gamma_{jq})\\
(j = 1,\ \dots,\ N;\ \mu = 1,\ \dots,\ m;\ q = 2,\ \dots,\ R_j;\ \nu = 1,\ \dots,\ 2m).
\end{gathered} \tag{5.3}
$$

Here $\mathcal{Q}_j$ is formally adjoint to $\mathcal{P}_j$; the operators $C_{j\sigma\mu}$, $T_{jq\nu}$, $T_{jq\nu k\sigma s}$ are of orders $m'_{j\sigma\mu}$, $\nu-1$, $\nu-1$ correspondingly; $\mathcal{G}'_{jqk\sigma s}$ is the operator of rotation by the angle $\varphi'_{jqk\sigma} = -\varphi_{k\sigma jq}$ and expansion by $\chi'_{jqk\sigma s} = 1/\chi_{k\sigma jqs}$ times in the plane $\{y\}$ such that $b_{jq} + \varphi'_{jqk\sigma} = b_{k\sigma}$, $0 < \chi'_{jqk\sigma s}$; $j,\ k = 1,\ \dots,\ N$; $q = 2,\ \dots,\ R_j$; $\sigma = 1,\ R_k+1$; $s = 1,\ \dots,\ S'_{jqk\sigma} = S_{k\sigma jq}$.

Problem (5.1)–(5.3) is a system of $R_1 + \dots + R_N$ equations for functions V_{jt} with boundary conditions (5.2) and nonlocal transmission conditions (5.3). We shall say that problem (5.1)–(5.3) is *a nonlocal transmission problem*.

Let us write the nonlocal transmission problems, which are formally adjoint to nonlocal boundary value problems of Examples 4.1 and 4.2.

Example 5.1. From Example 4.1, it follows that the problem

$$
-\triangle V_t = f_t(y) \quad (y \in K_t;\ t = 1,\ 2),
$$

$$
\begin{gathered}
V_1|_{\gamma_1} = g_1(y) \quad (y \in \gamma_1),\\
V_2|_{\gamma_3} = g_3(y) \quad (y \in \gamma_3),
\end{gathered}
$$

$$
\begin{gathered}
V_1|_{\gamma_2} - V_2|_{\gamma_2} = h_{21}(y) \quad (y \in \gamma_2),\\
\frac{\partial V_1}{\partial n_2}\Big|_{\gamma_2} - \frac{\partial V_2}{\partial n_2}\Big|_{\gamma_2} + \alpha\chi'_{21}\frac{\partial V_1}{\partial n_1}(\chi'_{21}r,\ \varphi + \varphi'_{21})\Big|_{\gamma_2} = h_{22}(y) \quad (y \in \gamma_2)
\end{gathered}
$$

is formally adjoint to problem (4.8), (4.9).

Example 5.2. From Example 4.2, it follows that the problem

$$
-\triangle V_t = f_t(y) \quad (y \in K_t;\ t = 1,\ 2),
$$

$$
\begin{gathered}
\frac{\partial V_1}{\partial n_1}\Big|_{\gamma_1} = g_1(y) \quad (y \in \gamma_1),\\
\frac{\partial V_2}{\partial n_3}\Big|_{\gamma_1} = g_3(y) \quad (y \in \gamma_3),
\end{gathered}
$$

$$V_1|_{\gamma_2} - V_2|_{\gamma_2} = h_{21}(y) \quad (y \in \gamma_2),$$
$$\frac{\partial V_1}{\partial n_2}\Big|_{\gamma_2} - \frac{\partial V_2}{\partial n_2}\Big|_{\gamma_2} + \alpha(\chi'_{21})^2 \frac{\partial V_1}{\partial r}(\chi'_{21} r,\ \varphi + \varphi'_{21})\Big|_{\gamma_2} = h_{22}(y) \quad (y \in \gamma_2)$$

is formally adjoint to problem (4.11), (4.12).

From Theorem 4.1, it follows that the following conditions hold (see [12, Chapter 2, §§1.2, 1.4]).

Condition 5.1. *For all $j = 1,\ \ldots,\ N$, the operators $\mathcal{Q}_j(D_y,\ D_z)$ are properly elliptic.*

Condition 5.2. *For all $j = 1,\ \ldots,\ N$; $\sigma = 1,\ R_j + 1$, the system $\{C_{j\sigma\mu}(D_y,\ D_z)\}_{\mu=1}^m$ is normal and covers the operator $\mathcal{Q}_j(D_y,\ D_z)$ on $\Gamma_{j\sigma}$.*

Condition 5.3. *For all $j = 1,\ \ldots,\ N$; $q = 2,\ \ldots,\ R_j$, the system $\{T_{jq\nu}(D_y,\ D_z)\}_{\nu=1}^{2m}$ is normal on Γ_{jq}.*

Remark 5.1. *One can easily prove that under condition 5.3, the system $\{T_{jq\nu}(D_y,\ D_z),\ T_{jq\nu}(D_y,\ D_z)\}_{\nu=1}^{2m}$ jointly covers the operator $\mathcal{Q}_j(D_y,\ D_z)$ on Γ_{jq} in the sense of [9].*

Consider the space $\mathcal{H}_a^l(\Omega_j) = \bigoplus_{t=1}^{R_j} H_a^l(\Omega_{jt})$ with the norm $\|V_j\|_{\mathcal{H}_a^l(\Omega_j)} = \left(\sum_{t=1}^{R_j} \|V_{jt}\|^2_{H_a^l(\Omega_{jt})}\right)^{1/2}$.

Introduce the spaces of vector–functions

$$\mathcal{H}_a^{l+2m,\,N}(\Omega) = \prod_{j=1}^N \mathcal{H}_a^{l+2m}(\Omega_j),\ \mathcal{H}_a^{l,\,N}(\Omega,\ \Gamma) = \prod_{j=1}^N \mathcal{H}_a^l(\Omega_j,\ \Gamma_j),$$

$$\mathcal{H}_a^l(\Omega_j,\ \Gamma_j) = \mathcal{H}_a^l(\Omega_j) \times$$
$$\times \prod_{\sigma=1,\,R_j+1} \prod_{\mu=1}^m H_a^{l+2m-m'_{j\sigma\mu}-1/2}(\Gamma_{j\sigma}) \times \prod_{q=2}^{R_j} \prod_{\nu=1}^{2m} H_a^{l+2m-\nu+1/2}(\Gamma_{jq}).$$

We study solutions $V = (V_1,\ \ldots,\ V_N) \in \mathcal{H}_a^{l+2m,\,N}(\Omega)$ for problem (5.1)–(5.3) supposing that $f = \{f_j,\ g_{j\sigma\mu},\ h_{jq\nu}\} \in \mathcal{H}_a^{l,\,N}(\Omega,\ \Gamma)$. Introduce the bounded operator $\mathcal{M} : \mathcal{H}_a^{l+2m,\,N}(\Omega) \to \mathcal{H}_a^{l,\,N}(\Omega,\ \Gamma)$ corresponding to problem (5.1)–(5.3) and given by

$$\mathcal{M}V = \{W_j,\ \mathcal{C}_{j\sigma\mu}(D_y,\ D_z)V,\ \mathcal{T}_{jq\nu}(D_y,\ D_z)V\}$$

Here $\mathcal{C}_{j\sigma\mu}(D_y,\ D_z)V$ and $\mathcal{T}_{jq\nu}(D_y,\ D_z)V$ are given by (5.2) and (5.3) correspondingly; $W_j(x) \equiv \mathcal{Q}_j(D_y,\ D_z)V_{jt}(x)$ for $x \in \Omega_{jt}$. (Notice that we cannot write $W_j \equiv \mathcal{Q}_j(D_y,\ D_z)V_j$ for $x \in \Omega_j$ because $V_j \in \mathcal{H}_a^{l+2m}(\Omega_j)$ may have discontinuity on Γ_{jq}, $q = 2,\ \dots,\ R_j$.)

Lemma 5.1. *For any* $g_{j\sigma\mu} \in H_a^{l+2m-m'_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})$, $h_{jq\nu} \in H_a^{l+2m-\nu+1/2}(\Gamma_{jq})$ $(j = 1,\ \dots,\ N;\ \sigma = 1,\ R_j+1;\ \mu = 1,\ \dots,\ m;\ q = 2,\ \dots,\ R_j;\ \nu = 1,\ \dots,\ 2m)$*, there exists a vector–function* $V \in \mathcal{H}_a^{l+2m,\,N}(\Omega)$ *such that*

$$\mathcal{C}_{j\sigma\mu}(D_y,\ D_z)V = g_{j\sigma\mu}(x)\ (x \in \Gamma_{j\sigma}), \quad \mathcal{T}_{jq\nu}(D_y,\ D_z)V = h_{jq\nu}(x)\ (x \in \Gamma_{jq}),$$

$$\|V\|_{\mathcal{H}_a^{l+2m,\,N}(\Omega)} \le c\sum_j\Big\{\sum_{\sigma,\,\mu}\|g_{j\sigma\mu}\|_{H_a^{l+2m-m'_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} + \\ +\sum_{q,\,\nu}\|h_{jq\nu}\|_{H_a^{l+2m-\nu+1/2}(\Gamma_{jq})}\Big\},$$

where $c > 0$ *is independent of* $g_{j\sigma\mu},\ h_{jq\nu}$.

Proof. By virtue of condition 5.2 and lemma 3.1 [7], there exists a vector–function $W \in H_a^{l+2m,\,N}(\Omega)$ such that

$$\mathcal{C}_{j\sigma\mu}(D_y,\ D_z)W = g_{j\sigma\mu}(x) \quad (x \in \Gamma_{j\sigma}), \tag{5.4}$$

$$\|W\|_{H_a^{l+2m,\,N}(\Omega)} \le k_1\sum_{j,\,\sigma,\,\mu}\|g_{j\sigma\mu}\|_{H_a^{l+2m-m'_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})}. \tag{5.5}$$

By virtue of condition 5.3 and lemma 3.1 [7], for all $j = 1,\ \dots,\ N$ and $q = 2,\ \dots,\ R_j$ there exists a function $\hat{W}_{j,q-1} \in H_a^{l+2m}(\Omega_{j,q-1})$ such that

$$T_{jq\nu}(D_y,\ D_z)\hat{W}_{j,q-1}(x)|_{\Gamma_{jq}} = h_{jq\nu}(x)- \\ -\sum_{k,\,\sigma,\,s}(T_{jq\nu k\sigma s}(D_y,\ D_z)W_k)(\mathcal{G}'_{jqk\sigma s}y,\ z)|_{\Gamma_{jq}} \quad (x \in \Gamma_{jq}), \tag{5.6}$$

$$\|\hat{W}_{j,q-1}\|_{H_a^{l+2m}(\Omega_{j,q-1})} \le k_2\sum_{\nu}\|h_{jq\nu}(x)- \\ -\sum_{k,\,\sigma,\,s}(T_{jq\nu k\sigma s}(D_y,\ D_z)W_k)(\mathcal{G}'_{jqk\sigma s}y,\ z)|_{\Gamma_{jq}}\|_{H_a^{l+2m-\nu+1/2}(\Gamma_{jq})}. \tag{5.7}$$

Since the functions ζ_{jq} defined by formula (3.4) are the multiplicators in the spaces $\mathcal{H}_a^{l+2m}(\Omega_j)$, from (5.4)–(5.7), it follows that the functions

$$V_j(x) = \begin{cases} \zeta_{j1}W_{j1}(x) + \zeta_{j2}\hat{W}_{j1}(x) & \text{for } x \in \Omega_{j1}, \\ \zeta_{j,t+1}\hat{W}_{jt}(x) & \text{for } x \in \Omega_{jt}\ (t = 2,\ \dots,\ R_j - 1), \\ \zeta_{j,R_j+1}W_{jR_j}(x) & \text{for } x \in \Omega_{jR_j} \end{cases}$$

satisfy the conditions of the Lemma. □

2 Nonlocal problems with parameter θ in plane angles.

Put $v=(v_1,\ \dots,\ v_N)$, $f=(f_1,\ \dots,\ f_N)$. Here the functions $v_j(y)$ ($f_j(y)$) are defined in K_j ($j=1,\ \dots,\ N$). As before, we shall denote by v_{jt} (f_{jt}) the restriction of v_j (f_j) to K_{jt}. Then we see that Green formula (4.15) (for $\eta=\theta\in S^{n-3}=\{z\in\mathbb{R}^{n-2}:\ |z|=1\}$) generates the problem, which is formally adjoint to problem (1.6), (1.7)

$$\mathcal{Q}_j(D_y,\ \theta)v_{jt}=f_{jt}(y)\quad (y\in K_{jt};\ t=1,\ \dots,\ R_j), \tag{5.8}$$

$$\begin{gathered}\mathcal{C}_{j1\mu}(D_y,\ \theta)v=C_{j1\mu}(D_y,\ \theta)v_{j1}(y)|_{\gamma_{j1}}=g_{j1\mu}(y)\quad (y\in\gamma_{j1}),\\ \mathcal{C}_{j,R_j+1,\mu}(D_y,\ \theta)v=C_{j,R_j+1,\mu}(D_y,\ \theta)v_{jR_j}(y)|_{\gamma_{j,R_j+1}}=\\ =g_{j,R_j+1,\mu}(y)\quad (y\in\gamma_{j,R_j+1}),\end{gathered} \tag{5.9}$$

$$\begin{gathered}\mathcal{T}_{jq\nu}(D_y,\ \theta)v=T_{jq\nu}(D_y,\ \theta)v_{j,q-1}(y)|_{\gamma_{jq}}-T_{jq\nu}(D_y,\ \theta)v_{jq}(y)|_{\gamma_{jq}}+\\ +\sum_{k,\sigma,s}(T_{jq\nu k\sigma s}(D_y,\ \theta)v_k)(\mathcal{G}'_{jqk\sigma s}y)|_{\gamma_{jq}}=h_{jq\nu}(y)\quad (y\in\gamma_{jq})\end{gathered} \tag{5.10}$$

$$(j=1,\ \dots,\ N;\ \mu=1,\ \dots,\ m;\ q=2,\ \dots,\ R_j;\ \nu=1,\ \dots,\ 2m).$$

It is easy to see that problem (5.8)–(5.10) can be also obtained from problem (5.1)–(5.3) by substituting θ for D_z.

Consider the space $\mathcal{H}_a^l(K_j)=\bigoplus\limits_{t=1}^{R_j}H_a^l(K_{jt})$ with the norm $\|v_j\|_{\mathcal{H}_a^l(K_j)}=\left(\sum\limits_{t=1}^{R_j}\|v_{jt}\|^2_{H_a^l(K_{jt})}\right)^{1/2}$ and the space $\mathcal{E}_a^l(K_j)=\bigoplus\limits_{t=1}^{R_j}E_a^l(K_{jt})$ with the norm $\|v_j\|_{\mathcal{E}_a^l(K_j)}=\left(\sum\limits_{t=1}^{R_j}\|v_{jt}\|^2_{E_a^l(K_{jt})}\right)^{1/2}$.

Introduce the spaces of vector–functions

$$\mathcal{E}_a^{l+2m,\,N}(K)=\prod_{j=1}^{N}\mathcal{E}_a^{l+2m}(K_j),\ \mathcal{E}_a^{l,\,N}(K,\ \gamma)=\prod_{j=1}^{N}\mathcal{E}_a^{l}(K_j,\ \gamma_j),$$

$$\begin{gathered}\mathcal{E}_a^l(K_j,\ \gamma_j)=\mathcal{E}_a^l(K_j)\times\\ \times\prod\nolimits_{\sigma=1,\,R_j+1}\prod\nolimits_{\mu=1}^{m}E_a^{l+2m-m'_{j\sigma\mu}-1/2}(\gamma_{j\sigma})\times\prod\nolimits_{q=2}^{R_j}\prod\nolimits_{\nu=1}^{2m}E_a^{l+2m-\nu+1/2}(\gamma_{jq}).\end{gathered}$$

We study solutions $v=(v_1,\ \dots,\ v_N)\in\mathcal{E}_a^{l+2m,\,N}(\Omega)$ for problem (5.8)–(5.10) supposing that $f=\{f_j,\ g_{j\sigma\mu},\ h_{jq\nu}\}\in\mathcal{E}_a^{l,\,N}(\Omega,\ \Gamma)$. Introduce the bounded operator $\mathcal{M}(\theta) : \mathcal{E}_a^{l+2m,\,N}(\Omega) \to \mathcal{E}_a^{l,\,N}(\Omega,\ \Gamma)$ corresponding to problem (5.8)–(5.10) and given by

$$\mathcal{M}v = \{w_j,\ \mathcal{C}_{j\sigma\mu}(D_y,\ \theta)v,\ \mathcal{T}_{jq\nu}(D_y,\ \theta)v\}.$$

Here $\mathcal{C}_{j\sigma\mu}(D_y,\ \theta)v$ and $\mathcal{T}_{jq\nu}(D_y,\ \theta)v$ are given by (5.9) and (5.10) correspondingly; $w_j(y) \equiv \mathcal{Q}_j(D_y,\ \theta)v_{jt}(y)$ for $y \in K_{jt}$.

Repeating the proof of Lemma 5.1, from lemma 3.1′ [7], we get the following statement.

Lemma 5.2. *For any* $g_{j\sigma\mu} \in E_a^{l+2m-m'_{j\sigma\mu}-1/2}(\gamma_{j\sigma}),\ h_{jq\nu} \in E_a^{l+2m-\nu+1/2}(\gamma_{jq})$ $(j = 1,\ \dots,\ N;\ \sigma = 1,\ R_j+1;\ \mu = 1,\ \dots,\ m;\ q = 2,\ \dots,\ R_j;\ \nu = 1,\ \dots,\ 2m)$ *there exists a vector–function* $v \in \mathcal{E}_a^{l+2m,\,N}(\Omega)$ *such that*

$$\mathcal{C}_{j\sigma\mu}(D_y,\ \theta)v = g_{j\sigma\mu}(y)\ (y \in \gamma_{j\sigma}), \quad \mathcal{T}_{jq\nu}(D_y,\ \theta)v = h_{jq\nu}(y)\ (y \in \gamma_{jq}),$$

$$\|v\|_{\mathcal{E}_a^{l+2m,\,N}(\Omega)} \le c\sum_j \Big\{ \sum\nolimits_{\sigma,\,\mu} \|g_{j\sigma\mu}\|_{E_a^{l+2m-m'_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} + \\ + \sum\nolimits_{q,\,\nu} \|h_{jq\nu}\|_{E_a^{l+2m-\nu+1/2}(\gamma_{jq})} \Big\},$$

where $c > 0$ *is independent of* $g_{j\sigma\mu},\ h_{jq\nu},\ \theta$.

6 Solvability of nonlocal transmission problems in plane angles

The results of this section are analogous to those of §2. We shall need these results for obtaining a priori estimates of solutions to nonlocal transmission problems in dihedral angles in §7.

1 Reduction of nonlocal problems in plane angles to nonlocal problems on arcs.

Consider the nonlocal transmission problem for a vector–function $V = (V_1,\ \dots,\ V_N) \in \mathcal{H}_a^{l+2m,\,N}(K)$

$$\mathcal{Q}_j(D_y,\ 0)V_{jt} = f_{jt}(y) \quad (y \in K_{jt};\ t = 1,\ \dots,\ R_j), \tag{6.1}$$

$$\mathcal{C}_{j1\mu}(D_y,\ 0)V = C_{j1\mu}(D_y,\ 0)V_{j1}(y)|_{\gamma_{j1}} = g_{j1\mu}(y) \quad (y \in \gamma_{j1}),$$
$$\mathcal{C}_{j,R_j+1,\mu}(D_y,\ 0)V = C_{j,R_j+1,\mu}(D_y,\ 0)V_{jR_j}(y)|_{\gamma_{j,R_j+1}} = \tag{6.2}$$
$$= g_{j,R_j+1,\mu}(y) \quad (y \in \gamma_{j,R_j+1}),$$

$$\mathcal{T}_{jq\nu}(D_y,\ 0)V = T_{jq\nu}(D_y,\ 0)V_{j,q-1}(y)|_{\gamma_{jq}} - T_{jq\nu}(D_y,\ 0)V_{jq}(y)|_{\gamma_{jq}} +$$
$$+ \sum_{k,\sigma,s} (T_{jq\nu k\sigma s}(D_y,\ 0)V_k)(\mathcal{G}'_{jqk\sigma s}y)|_{\gamma_{jq}} = h_{jq\nu}(y) \quad (y \in \gamma_{jq}) \tag{6.3}$$

$$(j = 1,\ \dots,\ N;\ \mu = 1,\ \dots,\ m;\ q = 2,\ \dots,\ R_j;\ \nu = 1,\ \dots,\ 2m),$$

where $f = \{f_j,\ g_{j\sigma\mu},\ h_{jq\nu}\} \in \mathcal{H}_a^{l,\,N}(K,\ \gamma)$.

Put formally $D_z = 0$ and write the differential operators in the polar coordinates: $\mathcal{Q}_j(D_y,\ 0) = r^{-2m}\tilde{\mathcal{Q}}_j(\varphi,\ D_\varphi,\ rD_r)$, $C_{j\sigma\mu}(D_y,\ 0) = r^{-m'_{j\sigma\mu}}\tilde{C}_{j\sigma\mu}(\varphi,\ D_\varphi,\ rD_r)$, $T_{jq\nu}(D_y,\ 0) = r^{-\nu+1}\tilde{T}_{jq\nu}(\varphi,\ D_\varphi,\ rD_r)$, $T_{jq\nu k\sigma s}(D_y,\ 0) = r^{-\nu+1}\tilde{T}_{jq\nu k\sigma s}(\varphi,\ D_\varphi,\ rD_r)$.

Put $\tau = \ln r$ and do the Fourier transform with respect to τ; then from (6.1)–(6.3), we get

$$\tilde{\mathcal{Q}}_j(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_{jt}(\varphi,\ \lambda) = \tilde{F}_{jt}(\varphi,\ \lambda) \quad (\varphi \in (b_{jt},\ b_{j,t+1});\ t = 1,\ \dots,\ R_j), \tag{6.4}$$

$$\tilde{\mathcal{C}}_{j1\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}(\varphi,\ \lambda) = \tilde{C}_{j1\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_{j1}(\varphi,\ \lambda)|_{\varphi=b_{j1}} =$$
$$= \tilde{G}_{j1\mu}(\lambda),$$
$$\tilde{\mathcal{C}}_{j,R_j+1,\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}(\varphi,\ \lambda) = \tilde{C}_{j,R_j+1,\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_{jR_j}(\varphi,\ \lambda)|_{\varphi=b_{j,R_j+1}} =$$
$$= \tilde{G}_{j,R_j+1,\mu}(\lambda), \tag{6.5}$$

$$\tilde{\mathcal{T}}_{jq\nu}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}(\varphi,\ \lambda) = \tilde{T}_{jq\nu}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_{j,q-1}(\varphi,\ \lambda)|_{\varphi=b_{jq}} -$$
$$-\tilde{T}_{jq\nu}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_{jq}(\varphi,\ \lambda)|_{\varphi=b_{jq}} +$$
$$+ \sum_{k,\sigma,s} e^{(i\lambda-(\nu-1))\ln\chi'_{jqk\sigma s}}\tilde{T}_{jq\nu k\sigma s}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_k(\varphi + \varphi'_{jqk\sigma},\ \lambda)|_{\varphi=b_{jq}} = \tilde{H}_{jq\nu}(\lambda) \tag{6.6}$$

$$(j = 1,\ \dots,\ N;\ \mu = 1,\ \dots,\ m;\ q = 2,\ \dots,\ R_j;\ \nu = 1,\ \dots,\ 2m).$$

Here $F_{jt}(\varphi,\ \tau) = e^{2m\tau}f_{jt}(\varphi,\ \tau)$, $G_{j\sigma\mu}(\tau) = e^{m'_{j\sigma\mu}\tau}g_{j\sigma\mu}(\tau)$; $H_{jq\nu}(\tau) = e^{(\nu-1)\tau}h_{jq\nu}(\tau)$; $\tilde{V}_{jt}$, $\tilde{F}_{jt}$, $\tilde{G}_{j\sigma\mu}$, and $\tilde{H}_{jq\nu}$ are the Fourier transforms of V_{jt}, F_{jt}, $G_{j\sigma\mu}$, and $H_{jq\nu}$ correspondingly.

This problem is a system of $R_1 + \dots + R_N$ ordinary differential equations (6.4) for the functions $\tilde{V}_{jt} \in W^{l+2m}(b_{jt},\ b_{j,t+1})$ with boundary conditions (6.5) and nonlocal transmission conditions (6.6) connecting jumps of the functions $\tilde{V}_j$ and their derivatives at the points of the intervals $(b_{j1},\ b_{j,R_j+1})$ with values of the functions $\tilde{V}_{k1}$ and $\tilde{V}_{k,R_k+1}$ and their derivatives at the points $\varphi = b_{k1}$ and $\varphi = b_{k,R_k+1}$ correspondingly.

Notice that problem (6.4)–(6.6) is formally adjoint to problem (2.3), (2.4) with respect to Green formula (4.16).

2 Solvability of nonlocal problems with parameter λ on arcs.

Consider the space $\mathcal{W}^l(b_{j1},\ b_{j,R_j+1}) = \bigoplus\limits_{t=1}^{R_j} W^l(b_{jt},\ b_{j,t+1})$ with the norm $\|\tilde V_j\|_{\mathcal{W}^l(b_{j1},\ b_{j,R_j+1})} = \left(\sum\limits_{t=1}^{R_j} \|\tilde V_{jt}\|^2_{W^l(b_{jt},\ b_{j,t+1})}\right)^{1/2}$. Introduce the spaces of vector–functions

$$\mathcal{W}^{l+2m,\,N}(b_1,\ b_2) = \prod_{j=1}^{N} \mathcal{W}^{l+2m}(b_{j1},\ b_{j,R_j+1}),$$

$$\mathcal{W}^{l,\,N}[b_1,\ b_2] = \prod_{j=1}^{N} \mathcal{W}^{l}[b_{j1},\ b_{j,R_j+1}],$$

$$\mathcal{W}^l[b_{j1},\ b_{j,R_j+1}] = \mathcal{W}^l(b_{j1},\ b_{j,R_j+1}) \times \mathbb{C}^m \times \mathbb{C}^m \times \prod_{q=2}^{R_j} \mathbb{C}^{2m}.$$

Introduce the equivalent norms depending on the parameter λ ($|\lambda| \geq 1$) in the Hilbert spaces $\mathcal{W}^l(b_{j1},\ b_{j,R_j+1})$ and $\mathcal{W}^l[b_{j1},\ b_{j,R_j+1}]$:

$$|||\tilde V_j|||_{\mathcal{W}^l(b_{j1},\ b_{j,R_j+1})} = \left(\|\tilde V_j\|^2_{\mathcal{W}^l(b_{j1},\ b_{j,R_j+1})} + |\lambda|^{2l}\, \|\tilde V_j\|^2_{L_2(b_{j1},\ b_{j,R_j+1})}\right)^{1/2},$$

$$|||\{\tilde F_j,\ \tilde G_{j\sigma\mu},\ \tilde H_{jq\nu}\}|||_{\mathcal{W}^l[b_{j1},\ b_{j,R_j+1}]} = \Big(|||\tilde F_j|||^2_{\mathcal{W}^l(b_{j1},\ b_{j,R_j+1})} +$$
$$+ \sum_{\sigma,\,\mu} (1 + |\lambda|^{2(l+2m-m'_{j\sigma\mu}-1/2)})|\tilde G_{j\sigma\mu}|^2 + \sum_{q,\,\nu} (1 + |\lambda|^{2(l+2m-\nu+1/2)})|\tilde H_{jq\nu}|^2\Big)^{1/2},$$

where $\tilde V_j \in \mathcal{W}^l(b_{j1},\ b_{j,R_j+1})$, $\{\tilde F_j,\ \tilde G_{j\sigma\mu},\ \tilde H_{jq\nu}\} \in \mathcal{W}^l[b_{j1},\ b_{j,R_j+1}]$. And therefore we have

$$|||\tilde V|||_{W^{l+2m,\,N}(b_1,\ b_2)} = \Big(\sum_j |||\tilde V_j|||^2_{\mathcal{W}^{l+2m}(b_{j1},\ b_{j,R_j+1})}\Big)^{1/2},$$

$$|||\tilde \Phi|||_{\mathcal{W}^{l,\,N}[b_1,\ b_2]} = \Big(\sum_j |||\tilde \Phi_j|||^2_{\mathcal{W}^l[b_{j1},\ b_{j,R_j+1}]}\Big)^{1/2},$$

where $\tilde{V} = (\tilde{V}_1, \ \ldots, \ \tilde{V}_N) \in \mathcal{W}^{l+2m,\,N}(b_1, \ b_2)$, $\tilde{\Phi} = (\tilde{\Phi}_1, \ \ldots, \ \tilde{\Phi}_N) \in \mathcal{W}^{l,\,N}[b_1, \ b_2]$.

Consider the operator–valued function $\tilde{\mathcal{M}}(\lambda) : \mathcal{W}^{l+2m,\,N}(b_1, \ b_2) \to \mathcal{W}^{l,\,N}[b_1, \ b_2]$ corresponding to problem (6.4)–(6.6) and given by

$$\tilde{\mathcal{M}}(\lambda)\tilde{V} = \{\tilde{W}_j, \ \tilde{\mathcal{C}}_{j\sigma\mu}(\varphi, \ D_\varphi, \ \lambda)\tilde{V}, \ \tilde{\mathcal{T}}_{jq\nu}(\varphi, \ D_\varphi, \ \lambda)\tilde{V}\}.$$

Here $\tilde{\mathcal{C}}_{j\sigma\mu}(\varphi, \ D_\varphi, \ \lambda)\tilde{V}$ and $\tilde{\mathcal{T}}_{jq\nu}(\varphi, \ D_\varphi, \ \lambda)\tilde{V}$ are given by (6.5) and (6.6) correspondingly; $\tilde{W}_j(\varphi) = \tilde{\mathcal{Q}}_j(\varphi, \ D_\varphi, \ \lambda)\tilde{V}_{jt}(\varphi)$ for $\varphi \in (b_{jt}, \ b_{j,t+1})$.

Lemma 6.1. *For all* $\lambda \in \mathbb{C}$, *the operator* $\tilde{\mathcal{M}}(\lambda) : \mathcal{W}^{l+2m,\,N}(b_1, \ b_2) \to \mathcal{W}^{l,\,N}[b_1, \ b_2]$ *is Fredholm,* $\operatorname{ind}\tilde{\mathcal{M}}(\lambda) = 0$; *for any* $h \in \mathbb{R}$, *there exists a* $q_0 > 0$ *such that for* $\lambda \in J_{h,\,q_0} = \{\lambda \in \mathbb{C} : \ \operatorname{Im}\lambda = h, \ |\operatorname{Re}\lambda| \geq q_0\}$, *the operator* $\tilde{\mathcal{M}}(\lambda)$ *has the bounded inverse* $\tilde{\mathcal{M}}^{-1}(\lambda) : \mathcal{W}^{l,\,N}[b_1, \ b_2] \to \mathcal{W}^{l+2m,\,N}(b_1, \ b_2)$ *and*

$$|||\tilde{\mathcal{M}}^{-1}(\lambda)\tilde{\Phi}|||_{\mathcal{W}^{l+2m,\,N}(b_1, \ b_2)} \leq c|||\tilde{\Phi}|||_{\mathcal{W}^{l,\,N}[b_1, \ b_2]} \qquad (6.7)$$

for all $\tilde{\Phi} \in \mathcal{W}^{l,\,N}[b_1, \ b_2]$, *where* $c > 0$ *is independent of* λ *and* $\tilde{\Phi}$; *the operator–valued function* $\tilde{\mathcal{M}}^{-1}(\lambda) : \mathcal{W}^{l,\,N}[b_1, \ b_2] \to \mathcal{W}^{l+2m,\,N}(b_1, \ b_2)$ *is finitely meromorphic.*

Proof. If

$$\tilde{\mathcal{T}}_{jq\nu}(\varphi, \ D_\varphi, \ \lambda)\tilde{V}(\varphi, \ \lambda) = \tilde{T}_{jq\nu}(\varphi, \ D_\varphi, \ \lambda)\tilde{V}_{j,q-1}(\varphi, \ \lambda)|_{\varphi=b_{jq}} - \\ -\tilde{T}_{jq\nu}(\varphi, \ D_\varphi, \ \lambda)\tilde{V}_{jq}(\varphi, \ \lambda)|_{\varphi=b_{jq}}$$

(i.e., if the operators $T_{jq\nu k\sigma s}(\varphi, \ D_\varphi, \ rD_r)$ corresponding to the nonlocal terms are absent), then we denote by $\tilde{\mathcal{M}}_0(\lambda)$ the operator $\tilde{\mathcal{M}}(\lambda)$. Following the scheme developed by M.S. Agranovich and M.I. Vishik in [13] (see also [10, §5]), one can show that there exist $0 < \varepsilon_1 < \pi/2$ and $q_1 > 0$ such that for

$$\lambda \in Q_{\varepsilon_1, q_1} = \{\lambda : |\lambda| \geq q_1, \ |\arg\lambda| \leq \varepsilon_1\} \cup \{\lambda : |\lambda| \geq q_1, \ |\arg\lambda - \pi| \leq \varepsilon_1\},$$

there exists the bounded inverse operator $\tilde{\mathcal{M}}_0^{-1}(\lambda)$; moreover, for all $\tilde{\Phi} \in W^{l,\,N}[b_1, \ b_2]$,

$$|||\tilde{\mathcal{M}}_0^{-1}(\lambda)\tilde{\Phi}|||_{\mathcal{W}^{l+2m,\,N}(b_1, \ b_2)} \leq k_1|||\tilde{\Phi}|||_{\mathcal{W}^{l,\,N}[b_1, \ b_2]}. \qquad (6.8)$$

Here $k_1 > 0$ is independent of λ and $\tilde{\Phi}$.

Consider the operator $\tilde{\mathcal{M}}_t(\lambda) = \tilde{\mathcal{M}}_0(\lambda) + t(\tilde{\mathcal{M}}(\lambda) - \tilde{\mathcal{M}}_0(\lambda))$, $0 \le t \le 1$. We shall prove that for any $h \in \mathbb{R}$, there exists a $q_0 > 0$ such that if $\lambda \in J_{h,q_0}$ and $0 \le t \le 1$, then we have

$$k_2|||\tilde{\mathcal{M}}_t(\lambda)\tilde{V}|||_{\mathcal{W}^{l,\,N}[b_1,\ b_2]} \le |||\tilde{V}|||_{\mathcal{W}^{l+2m,\,N}(b_1,\ b_2)} \le k_3|||\tilde{\mathcal{M}}_t(\lambda)\tilde{V}|||_{\mathcal{W}^{l,\,N}[b_1,\ b_2]} \tag{6.9}$$

for all $\tilde{V} \in \mathcal{W}^{l+2m,\,N}(b_1,\ b_2)$. Here $k_2,\ k_3 > 0$ are independent of $\lambda,\ t$ and V.

Denote $\tilde{\mathcal{M}}_t(\lambda)\tilde{V} = \tilde{\Phi}$; then we have

$$\tilde{\mathcal{M}}_0(\lambda)\tilde{V} = \tilde{\Phi} + \tilde{\Psi},$$

where

$$\tilde{\Psi} = \{0,\ 0,\ -t\sum_{k,\,\sigma,\,s} e^{(i\lambda-(\nu-1))\ln\chi'_{jqk\sigma s}}\tilde{T}_{jq\nu k\sigma s}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_k(\varphi+\varphi'_{jqk\sigma},\ \lambda)|_{\varphi=b_{jq}}\}.$$

By virtue of (6.8), we have

$$|||\tilde{V}|||_{\mathcal{W}^{l+2m,\,N}(b_1,\ b_2)} \le k_1|||\tilde{\Phi} + \tilde{\Psi}|||_{\mathcal{W}^{l,\,N}[b_1,\ b_2]}. \tag{6.10}$$

Take $\varepsilon > 0$ from formula (3.3) and a $q_0 \ge q_1$ such that $J_{h,q_0} \subset Q_{\varepsilon_1,q_1}$. Then using inequalities (1.3), (1.4), we get

$$\begin{gathered} I_{jq\nu k1s} = (1+|\lambda|^{l+2m-\nu+1/2})\Big|e^{(i\lambda-(\nu-1))\ln\chi'_{jqk1s}}\times \\ \times\tilde{T}_{jq\nu k1s}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_k(\varphi+\varphi'_{jqk1})|_{\varphi=b_{jq}}\Big| \le \\ k_4|\lambda|^{l+2m-\nu}\big\{\|\tilde{T}_{jq\nu k1s}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_{k1}\|_{W^1(b_{k1},\ b_{k1}+\varepsilon/2)}+ \\ |\lambda|\,\|\tilde{T}_{jq\nu k1s}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_{k1}\|_{L^2(b_{k1},\ b_{k1}+\varepsilon/2)}\big\} \le k_5|||\tilde{V}_{k1}|||_{W^{l+2m}(b_{k1},\ b_{k1}+\varepsilon/2)}. \end{gathered} \tag{6.11}$$

If ε_1 is sufficiently small and q_1 is sufficiently large, then from inequality (6.11), theorem 4.1 [13, Chapter 1, §4], Leibniz' formula, and interpolation inequality (1.3), we obtain

$$\begin{gathered} I_{jq\nu k1s} \le k_5|||\zeta_{k1}\tilde{V}_{k1}|||_{W^{l+2m}(b_{k1},\ b_{k1}+\varepsilon/2)} \le k_6\big(|||\tilde{\mathcal{Q}}_k(\zeta_{k1}\tilde{V}_{k1})|||_{W^l(b_{k1},\ b_{k2})}+ \\ +\sum_{\mu=1}^{m}(1+|\lambda|^{l+2m-m'_{k1\mu}-1/2})\Big|\tilde{C}_{k1\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_{k1}(\varphi)|_{\varphi=b_{k1}}\Big|\big) \le \\ \le k_7\big(|||\tilde{\mathcal{Q}}_k\tilde{V}_{k1}|||_{W^l(b_{k1},\ b_{k2})} + |\lambda|^{-1}|||\tilde{V}_{k1}|||_{W^{l+2m}(b_{k1},\ b_{k2})}+ \\ +\sum_{\mu=1}^{m}(1+|\lambda|^{l+2m-m'_{k1\mu}-1/2})|\tilde{C}_{k1\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde{V}_{k1}(\varphi)|_{\varphi=b_{k1}}|\big). \end{gathered} \tag{6.12}$$

Similarly to (6.11), (6.12), one can estimate

$$I_{jq\nu k,R_k+1,s}=(1+|\lambda|^{l+2m-\nu+1/2})\times$$
$$\times\Big|e^{(i\lambda-(\nu-1))\ln\chi'_{jqk,R_k+1,s}}\tilde T_{jq\nu k,R_k+1,s}(\varphi,\ D_\varphi,\ \lambda)\tilde V_k(\varphi+\varphi'_{jqk,R_k+1})|_{\varphi=b_{jq}}\Big|:$$

$$\begin{gathered}I_{jq\nu k,R_k+1,s}\le k_8\big(|||\tilde{\mathcal Q}_k\tilde V_{kR_k}|||_{W^l(b_{kR_k},\ b_{k,R_k+1})}+\\ +|\lambda|^{-1}|||\tilde V_{kR_k}|||_{W^{l+2m}(b_{kR_k},\ b_{kR_k+1})}+\\ +\sum_{\mu=1}^{m}(1+|\lambda|^{l+2m-m'_{k,R_k+1,\mu}-1/2})|\tilde C_{k,R_k+1,\mu}(\varphi,\ D_\varphi,\ \lambda)\tilde V_{kR_k}(\varphi)|_{\varphi=b_{k,R_k+1}}|\big).\end{gathered}\tag{6.13}$$

Now if q_0 is sufficiently large, then (6.10), (6.12), and (6.13) imply right-hand side of inequality (6.9). Left-hand side of inequality (6.9) is obvious. Using a standard method of continuation with respect to parameter t (see the proof of theorem 7.1 [14, Chapter 2, §7]), inequality (6.9) and existence of a bounded inverse operator $\tilde{\mathcal M}_0^{-1}(\lambda)$ for $\lambda\in Q_{\varepsilon_1,q_1}$, one can easily see that for $\lambda\in J_{h,q_0}$, the operator $\tilde{\mathcal M}(\lambda)$ also has a bounded inverse and (6.7) holds.

Let us prove that the operator $\tilde{\mathcal M}(\lambda)$ is Fredholm. For $\lambda_0\in Q_{\varepsilon_1,q_1}$, we have

$$\tilde{\mathcal M}(\lambda)\tilde{\mathcal M}_0^{-1}(\lambda_0)=I+(\tilde{\mathcal M}(\lambda)-\tilde{\mathcal M}_0(\lambda_0))\tilde{\mathcal M}_0^{-1}(\lambda_0),$$

where I is the identity operator in $\mathcal W^{l,\,N}[b_1,\ b_2]$. Since the operators $\tilde{\mathcal Q}_j(\varphi,\ D_\varphi,\ \lambda)$ contain the parameter λ only in junior terms, the operator

$$\tilde{\mathcal M}(\lambda)-\tilde{\mathcal M}_0(\lambda_0):\mathcal W^{l+2m,\,N}(b_1,\ b_2)\to\mathcal W^{l+1,\,N}[b_1,\ b_2]$$

is bounded for every fixed $\lambda\in\mathbb C$. Hence from the compactness of the imbedding operator of $W^{l+1}(b_{jt},\ b_{j,t+1})$ into $W^l(b_{jt},\ b_{j,t+1})$, it follows that the operator

$$(\tilde{\mathcal M}(\lambda)-\tilde{\mathcal M}_0(\lambda_0))\tilde{\mathcal M}_0^{-1}(\lambda_0):\mathcal W^{l,\,N}[b_1,\ b_2]\to\mathcal W^{l,\,N}[b_1,\ b_2]$$

is compact. Thus by theorem 15.1 [15, §15], the operator $\tilde{\mathcal M}(\lambda)$ is Fredholm and $\operatorname{ind}\tilde{\mathcal M}(\lambda)=0$ for all $\lambda\in\mathbb C$.

From this, from existence of the bounded inverse operator $\tilde{\mathcal M}^{-1}(\lambda)$ for $\lambda\in J_{h,q_0}$, and from theorem 1 [16], it follows that the operator–valued function $\tilde{\mathcal M}^{-1}(\lambda)$ is finitely meromorphic. □

Repeating the proof of lemma 2.2 [4, §2], from (6.10)–(6.13), we obtain the following statement.

Lemma 6.2. *For any $0 < \varepsilon' < 1/d'$, there exists a $q > 1$ such that the set $\{\lambda \in \mathbb{C}:\ |\operatorname{Im}\lambda| \le \varepsilon' \ln|\operatorname{Re}\lambda|,\ |\operatorname{Re}\lambda| \ge q\}$ contains no poles of the operator–valued function $\tilde{\mathcal{M}}^{-1}(\lambda)$, where $d' = \max|\ln \chi'_{jqk\sigma s}|$; for every pole λ_0 of the operator–valued function $\tilde{\mathcal{M}}^{-1}(\lambda)$, there exists a $\delta > 0$ such that the set $\{\lambda \in \mathbb{C}:\ 0 < |\operatorname{Im}\lambda - \operatorname{Im}\lambda_0| < \delta\}$ contains no poles of the operator–valued function $\tilde{\mathcal{M}}^{-1}(\lambda)$.*

3 One–valued solvability of nonlocal problems in plane angles.

Replacing in the proof of theorem 2.1 [4, §2] Sobolev spaces $W^l(\cdot)$ by $\mathcal{W}^l(\cdot)$ and weighted spaces $H^l_a(\cdot)$ by $\mathcal{H}^l_a(\cdot)$, from Lemma 6.1, we obtain the following result.

Theorem 6.1. *Suppose the line $\operatorname{Im}\,\lambda = a+1-l-2m$ contains no poles of the operator–valued function $\tilde{\mathcal{M}}^{-1}(\lambda)$; then nonlocal transmission problem (6.1)–(6.3) has a unique solution $V \in \mathcal{H}^{l+2m,\,N}_a(K)$ for every right-hand side $f \in \mathcal{H}^{l,\,N}_a(K,\ \gamma)$ and*

$$\|V\|_{\mathcal{H}^{l+2m,\,N}_a(K)} \le c\|f\|_{\mathcal{H}^{l,\,N}_a(K,\ \gamma)},$$

where $c > 0$ does not depend on f.

7 A priori estimates of solutions for nonlocal transmission problems

In this section, we prove a priori estimates for solutions to nonlocal transmission problems analogous to those of §3.

1 A priori estimates in dihedral angles.

Denote $d_1' = \min\{1,\ \chi'_{jqk\sigma s}\}/2$, $d_2' = 2\max\{1,\ \chi'_{jqk\sigma s}\}$, $\Omega_j^p = \Omega_j \cap \{r_1(d_1')^{6-p} < r < r_2(d_2')^{6-p},\ |z| < 2^{-p-1}\}$, $\Omega^p_{jt} = \Omega_{jt} \cap \{r_1(d_1')^{6-p} < r < r_2(d_2')^{6-p},\ |z| < 2^{-p-1}\}$, where $j = 1,\ \dots,\ N;\ t = 1,\ \dots,\ R_j;\ p = 0,\ \dots,\ 6;\ 0 < r_1 < r_2$.

Introduce the space $\mathcal{W}^l(\Omega_j^p) = \bigoplus\limits_{t=1}^{R_j} W^l(\Omega^p_{jt})$ with the norm $\|V_j\|_{\mathcal{W}^l(\Omega^p_j)} = \left(\sum\limits_{t=1}^{R_j} \|V_{jt}\|^2_{W^l(\Omega^p_{jt})}\right)^{1/2}$.

Lemma 7.1. *Suppose* $V_j \in \mathcal{W}^{2m}(\Omega_j^0)$,

$$\begin{gathered} \mathcal{Q}_j(D_y,\ D_z)V_{jt} \in W^l(\Omega_{jt}^0), \\ \mathcal{C}_{j\sigma\mu}(D_y,\ D_z)V \in W^{l+2m-m'_{j\sigma\mu}-1/2}(\Gamma_{j\sigma} \cap \bar\Omega_j^0), \\ \mathcal{T}_{jq\nu}(D_y,\ D_z)V \in W^{l+2m-\nu+1/2}(\Gamma_{jq} \cap \bar\Omega_j^0) \end{gathered} \tag{7.1}$$

$$\begin{gathered} (j=1,\ \dots,\ N;\ \sigma=1,\ R_j+1;\ \mu=1,\ \dots,\ m; \\ q=2,\ \dots,\ R_j;\ \nu=1,\ \dots,\ 2m); \end{gathered}$$

then we have $V \in \prod\limits_j \mathcal{W}^{l+2m}(\Omega_j^3)$ *and for* $|\lambda| \geq 1$,

$$\begin{gathered} \sum_j \|V_j\|_{\mathcal{W}^{l+2m}(\Omega_j^6)} \leq c \sum_j \Big\{ \sum_t \|\mathcal{Q}_j(D_y,\ D_z)V_{jt}\|_{W^l(\Omega_{jt}^3)} + \\ + \sum_{\sigma,\,\mu} \|\mathcal{C}_{j\sigma\mu}(D_y,\ D_z)V\|_{W^{l+2m-m'_{j\sigma\mu}-1/2}(\Gamma_{j\sigma}\cap\bar\Omega_j^3)} + \\ + \sum_{q,\,\nu} \|\mathcal{T}_{jq\nu}(D_y,\ D_z)V\|_{W^{l+2m-\nu+1/2}(\Gamma_{jq}\cap\bar\Omega_j^3)} + \\ + |\lambda|^{-1}\|V_j\|_{\mathcal{W}^{l+2m}(\Omega_j^3)} + |\lambda|^{l+2m-1}\|V_j\|_{L_2(\Omega_j^3)} \Big\}, \end{gathered} \tag{7.2}$$

where $c > 0$ *is independent of* λ *and* V.

Proof. Since the functions ζ_{jq} $(q = 1,\ \dots,\ R_j+1)$ given by (3.4) are the multiplicators in the spaces $W^l(\Omega_{jt}^p)$ $(t = 1,\ \dots,\ R_j)$, we have $\zeta_{j\sigma}V_j \in W^{2m}(\Omega_j^0)$ $(\sigma = 1,\ R_j+1)$. Apply theorem 5.1 [12, Chapter 2, §5.1] to the functions $\zeta_{j\sigma}V_j$ and to the operator $\{\mathcal{Q}_j(D_y,\ D_z),\ \mathcal{C}_{j\sigma\mu}(D_y,\ D_z)\}$; then from (7.1) and Leibniz' formula, we get

$$\zeta_{j\sigma}V_j \in\ W^{l+2m}(\Omega_j^1). \tag{7.3}$$

Denote $W_{jq\nu} = \sum\limits_{k,\,\sigma,\,s} (T_{jq\nu k\sigma s}(D_y,\ D_z)(\zeta_{k\sigma}V_k))(\mathcal{G}'_{jqk\sigma s}y,\ z)$. Clearly,

$$W_{jq\nu}|_{\Gamma_{jq}\cap\bar\Omega_j^2} = \sum_{k,\ \sigma,\,s} (T_{jq\nu k\sigma s}(D_y,\ D_z)V_k))(\mathcal{G}'_{jqk\sigma s}y,\ z)|_{\Gamma_{jq}\cap\bar\Omega_j^2}. \tag{7.4}$$

From equality (7.4) and relations (7.1), (7.3), it follows that

$$\begin{gathered} T_{jq\nu}(D_y,\ D_z)V_{j,q-1}|_{\Gamma_{jq}\cap\bar\Omega_j^2} - T_{jq\nu}(D_y,\ D_z)V_{jq}|_{\Gamma_{jq}\cap\bar\Omega_j^2} = \\ = \mathcal{T}_{jq\nu}(D_y,\ D_z)V - W_{jq\nu}|_{\Gamma_{jq}\cap\bar\Omega_j^2} \in W^{l+2m-\nu+1/2}(\Gamma_{jq} \cap \bar\Omega_j^2). \end{gathered} \tag{7.5}$$

Now (7.1), (7.5), and theorem 1 [9, §2] imply that $V_j \in \mathcal{W}^{l+2m}(\Omega_j^3)$ and

$$\begin{gathered}\sum_j \|V_j\|_{\mathcal{W}^{l+2m}(\Omega_j^6)} \le k_1 \sum_j \Big\{ \sum_t \|\mathcal{Q}_j(D_y,\ D_z)V_{jt}\|_{W^l(\Omega_{jt}^5)} + \\ + \sum_{\sigma,\,\mu} \|\mathcal{C}_{j\sigma\mu}(D_y,\ D_z)V\|_{W^{l+2m-m'_{j\sigma\mu}-1/2}(\Gamma_{j\sigma}\cap\bar{\Omega}_j^5)} + \\ + \sum_{q,\,\nu} \|T_{jq\nu}(D_y,\ D_z)V_{j,q-1}|_{\Gamma_{jq}\cap\bar{\Omega}_j^5} - \\ - T_{jq\nu}(D_y,\ D_z)V_{jq}|_{\Gamma_{jq}\cap\bar{\Omega}_j^5}\|_{W^{l+2m-\nu+1/2}(\Gamma_{jq}\cap\bar{\Omega}_j^5)} + \|V_j\|_{L_2(\Omega_j^5)} \Big\}.\end{gathered} \tag{7.6}$$

Again using theorem 5.1 [12, Chapter 2, §5.1], Leibniz' formula, and inequality (1.3), we get

$$\begin{gathered}\|W_{jq\nu}|_{\Gamma_{jq}\cap\bar{\Omega}_j^5}\|_{W^{l+2m-\nu+1/2}(\Gamma_{jq}\cap\bar{\Omega}_j^5)} \le k_2 \sum_{k,\,\sigma} \|\zeta_{k\sigma}V_k\|_{W^{l+2m}(\Omega_k^4)} \le \\ \le k_3 \sum_k \Big\{ \sum_t \|\mathcal{Q}_k(D_y,\ D_z)V_{kt}\|_{W^l(\Omega_{kt}^3)} + \\ + \sum_{\sigma,\,\mu} \|\mathcal{C}_{k\sigma\mu}(D_y,\ D_z)V\|_{W^{l+2m-m'_{k\sigma\mu}-1/2}(\Gamma_{k\sigma}\cap\bar{\Omega}_j^3)} + \\ + |\lambda|^{-1}\|V_k\|_{\mathcal{W}^{l+2m}(\Omega_k^3)} + |\lambda|^{l+2m-1}\|V_k\|_{L_2(\Omega_k^3)} \Big\}.\end{gathered} \tag{7.7}$$

From (7.6), (7.4), and (7.7), it follows inequality (7.2). □

Denote $\mathcal{W}^l_{\text{loc}}(\bar{\Omega}_j \backslash M) = \bigoplus_{t=1}^{R_j} W^l_{\text{loc}}(\bar{\Omega}_{jt} \backslash M)$.

Theorem 7.1. *Let $V \in \prod_j \mathcal{W}^{2m}_{\text{loc}}(\bar{\Omega}_j \backslash M)$ be a solution for nonlocal transmission problem (5.1)–(5.3) such that $V \in H^{0,\,N}_{a-l-2m}(\Omega)$ and $f \in \mathcal{H}^{l,\,N}_a(\Omega,\ \Gamma)$; then $V \in \mathcal{H}^{l+2m,\,N}_a(\Omega)$ and*

$$\|V\|_{\mathcal{H}^{l+2m,\,N}_a(\Omega)} \le c\big(\|f\|_{\mathcal{H}^{l,\,N}_a(\Omega,\ \Gamma)} + \|V\|_{H^{0,\,N}_{a-l-2m}(\Omega)}\big), \tag{7.8}$$

where $c > 0$ is independent of V.

Proof. From Lemma 7.1, it follows that $V \in \prod_j \mathcal{W}^{l+2m}_{\text{loc}}(\bar{\Omega}_j \backslash M)$. Now repeating the proof of lemma 3.2 [4, §3] and replacing there $W^l(\cdot)$ by $\mathcal{W}^l(\cdot)$ and weighted spaces $H^l_a(\cdot)$ by $\mathcal{H}^l_a(\cdot)$, from Lemmas 5.1 and 7.1, we derive that $V \in \mathcal{H}^{l+2m,\,N}_a(\Omega)$ and a priori estimate (7.8) holds. □

2 A priori estimates in plane angles.

Put $K_j^{ps}=K_j\cap\{r_1(d_1')^{6-p}\cdot 2^s<r<r_2(d_2')^{6-p}\cdot 2^s\}$, $K_{jt}^{ps}=K_{jt}\cap\{r_1(d_1')^{6-p}\cdot 2^s<r<r_2(d_2')^{6-p}\cdot 2^s\}$, where $0<r_1<r_2$; $s\geq 1$; $j=1,\ \dots,\ N$; $p=0,\ \dots,\ 6$.

Introduce the space $\mathcal{W}^l(K_j^{ps})=\bigoplus\limits_{t=1}^{R_j}W^l(K_{jt}^{ps})$ with the norm $\|v_j\|_{\mathcal{W}^l(K_j^{ps})}=\left(\sum\limits_{t=1}^{R_j}\|v_{jt}\|^2_{W^l(\Omega_{jt}^{ps})}\right)^{1/2}$.

Lemma 7.2. *Suppose $s\geq 1,\ \theta\in S^{n-3}$. Assume that $v_j\in\mathcal{W}^{2m}(K_j^{0s})$,*

$$\mathcal{Q}_j(D_y,\ \theta)v_{jt}\in W^l(K_{jt}^{0s}),$$

$$\mathcal{C}_{j\sigma\mu}(D_y,\ \theta)v=0\quad(y\in\gamma_{j\sigma}\cap\bar K_j^{0s}),\ \mathcal{T}_{jq\nu}(D_y,\ \theta)v=0\quad(y\in\gamma_{jq}\cap\bar K_j^{0s})$$

$$(j=1,\ \dots,\ N,\ \sigma=1,\ R_j+1,\ \mu=1,\ \dots,\ m,\ q=2,\ \dots,\ R_j,\ \nu=1,\ \dots,\ 2m);$$

then $v\in\prod\limits_j\mathcal{W}^{l+2m}(K_j^{3s})$ and for all $|\lambda|\geq 1$,

$$\sum_j 2^{sa}\|v_j\|_{\mathcal{W}^{l+2m}(K_j^{6s})}\leq c\sum_j\big\{2^{sa}\sum_t\|\mathcal{Q}_j(D_y,\ \theta)v_{jt}\|_{W^l(K_{jt}^{3s})}+ +|\lambda|^{-1}2^{sa}\|v_j\|_{\mathcal{W}^{l+2m}(K_j^{3s})}+|\lambda|^{l+2m-1}2^{s(a-l-2m)}\|v_j\|_{L_2(K_j^{3s})}\big\}, \tag{7.9}$$

where $c>0$ is independent of v, $\theta,\ \lambda$, and s.

Proof. Repeating the proof of Lemma 7.1 and replacing Ω_j^p by K_j^{ps} and D_z by θ, we get $v\in\prod\limits_j\mathcal{W}^{l+2m}(K_j^{3s})$. Now repeating the proof of lemma 3.3 [4, §3] and replacing there $W^l(\cdot)$ by $\mathcal{W}^l(\cdot)$ and $H_a^l(\cdot)$ by $\mathcal{H}_a^l(\cdot)$, from a priori estimate (7.2), we derive estimate (7.9). □

Theorem 7.2. *Let $v\in\prod\limits_j\mathcal{W}^{2m}_{\rm loc}(\bar K_j\backslash\{0\})$ be a solution for problem (5.8)–(5.10) such that $v\in E^{0,\,N}_{a-l-2m}(K)$ and $f\in\mathcal{E}_a^{l,\,N}(K,\ \gamma)$; then $v\in\mathcal{E}_a^{l+2m,\,N}(K)$ and*

$$\|v\|_{\mathcal{E}_a^{l+2m,\,N}(K)}\leq c\big(\|f\|_{\mathcal{E}_a^{l,\,N}(K,\ \gamma)}+\|v\|_{E^{0,\,N}_{a-l-2m}(K)}\big), \tag{7.10}$$

where $c>0$ is independent of v and $\theta\in S^{n-3}$.

Proof. The proof is analogous to the proof of Theorem 3.2, where one must replace $W^l(\cdot)$, $H_a^l(\cdot)$, $E_a^l(\cdot)$ by $\mathcal{W}^l(\cdot)$, $\mathcal{H}_a^l(\cdot)$, $\mathcal{E}_a^l(\cdot)$; Lemmas 1.5, 3.1, 3.2 by Lemmas 5.2, 7.1, 7.2 correspondingly; Theorem 3.1 by Theorem 7.1. $\square$

From Theorem 6.1 and Lemma 7.2, we obtain the following result (see theorem 3.1 [4, §3] with $E_a^l(\cdot)$ replaced by $\mathcal{E}_a^l(\cdot)$).

Theorem 7.3. *Suppose the line* $\operatorname{Im}\lambda = a+1-l-2m$ *contains no poles of the operator–valued function* $\tilde{\mathcal{M}}^{-1}(\lambda)$*; then for all solutions* $v \in \mathcal{E}_a^{l+2m,\,N}(K)$ *to nonlocal transmission problem (5.8)–(5.10) and all* $\theta \in S^{n-3}$*, we have*

$$\|v\|_{\mathcal{E}_a^{l+2m,\,N}(K)} \le c\big(\|f\|_{\mathcal{E}_a^{l,\,N}(K,\ \gamma)} + \sum_j \|v_j\|_{L_2(K_j\cap S')}\big), \tag{7.11}$$

where $S' = \{y \in \mathbb{R}^2 :\ 0 < R_1' < r < R_2'\}$; $c > 0$ *is independent of* θ *and* v.

If for any $\theta \in S^{n-3}$*, estimate (7.11) holds for all solutions to nonlocal transmission problem (5.8)–(5.10), then the line* $\operatorname{Im}\lambda = a+1-l-2m$ *contains no poles of the operator–valued function* $\tilde{\mathcal{M}}^{-1}(\lambda)$.

Theorem 7.3 implies that kernel of $\mathcal{M}(\theta)$ is of finite dimension and range of $\mathcal{L}(\theta)$ is closed.

8 Adjoint nonlocal problems

In this section, we study operators that are adjoint to the operators of the nonlocal boundary value problems with parameter $\theta \in S^{n-3}$.

1 Operators $\mathcal{L}(\theta)^*$.

Let $\mathcal{L}(\theta) = \{\mathcal{P}_j(D_y,\ \theta),\ \mathcal{B}_{j\sigma\mu}(D_y,\ \theta)\} : E_a^{2m,\,N}(K) \to E_a^{0,\,N}(K,\ \gamma)$ be the operator corresponding to problem (1.6), (1.7). Consider the adjoint operator $\mathcal{L}(\theta)^* : (E_a^{0,\,N}(K,\ \gamma))^* \to (E_a^{2m,\,N}(K))^*$, where

$$(E_a^{0,\,N}(K,\ \gamma))^* = \prod_{j=1}^{N}\big\{E_{-a}^0(K_j) \times \prod_{\sigma=1,\,R_j+1}\prod_{\mu=1}^{m}(E_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}))^*\big\},$$

$$(E_a^{2m,\,N}(K))^* = \prod_{j=1}^{N}(E_a^{2m}(K_j))^*.$$

$\mathcal{L}(\theta)^*$ takes $f = \{f_j,\ g_{j\sigma\mu}\} \in (E_a^{0,\,N}(K,\ \gamma))^*$ to $\mathcal{L}(\theta)^* f$ by the rule

$$< u,\ \mathcal{L}(\theta)^* f >= \sum_j \big\{ < \mathcal{P}_j(D_y,\ \theta) u_j,\ f_j >_{K_j} + \\ + \sum_{\sigma,\mu} < \mathcal{B}_{j\sigma\mu}(D_y,\ \theta) u,\ g_{j\sigma\mu} >_{\gamma_{j\sigma}} \big\}$$

for all $u \in E_a^{2m,\,N}(K)$. Here $< \cdot,\ \cdot >$, $< \cdot,\ \cdot >_{K_j}$, $< \cdot,\ \cdot >_{\gamma_{j\sigma}}$ are the sesquilinear forms on the corresponding dual pairs of the spaces.

Introduce the space $\mathcal{W}^l(K_j) = \bigoplus_{t=1}^{R_j} W^l(K_{jt})$ with the norm $\|v_j\|_{\mathcal{W}^l(K_j)} = \left(\sum_{t=1}^{R_j} \|v_{jt}\|^2_{W^l(K_{jt})} \right)^{1/2}$. Further (see Theorem 8.1), we shall see that if the j-th component of $\mathcal{L}(\theta)^* f$ is smooth in K_j ($j = 1,\ \dots,\ N$), then f_j is smooth only in K_{jt} and, generally, may have discontinuity on γ_{jq} ($q = 2,\ \cdots,\ R_j$). This happens because of nonlocal terms with supports on γ_{jq} in the operator $\mathcal{L}(\theta)$ and therefore in the operator $\mathcal{L}(\theta)^*$. Hence it is natural to consider spaces $\mathcal{W}^l(\cdot)$ (but not $W^l(\cdot)$) when studying smoothness of f.

Consider the functions $\psi_p \in C_0^\infty(\mathbb{R}^1)$ such that

$$\psi_p(r) = 1 \text{ for } r_1 d_1^{3-p} < r < r_2 d_2^{3-p}, \\ \psi_p(r) = 0 \text{ for } r < \frac{2}{3} r_1 d_1^{3-p} \text{ and } r > \frac{3}{2} r_2 d_2^{3-p},$$

where $0 < r_1 < r_2$; $p = 0,\ \dots,\ 3$. Put $\hat\gamma_{jq} = \{y:\ \varphi = b_{jq} \text{ or } \varphi = b_{jq} + \pi\}$ ($j = 1,\ \dots,\ N;\ q = 1,\ \dots,\ R_j + 1$). Clearly, $\gamma_{jq} \subset \hat\gamma_{jq}$.

Theorem 8.1. *Suppose* $f = \{f_j,\ g_{j\sigma\mu}\} \in (E_a^{0,\,N}(K,\ \gamma))^*$, $\mathcal{L}(\theta)^* f \in (E_a^{2m,\,N}(K))^*$,

$$\psi_0 \mathcal{L}(\theta)^* f \in \begin{cases} \prod_j W_{\bar K_j}^{-2m+l}(\mathbb{R}^n)^5 & \textit{for } l < 2m, \\ \prod_j \mathcal{W}^{-2m+l}(K_j) & \textit{for } l \ge 2m; \end{cases}$$

then $\psi_3 f \in \prod_j \big\{ \mathcal{W}^l(K_j) \times \prod_{\sigma,\mu} W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma}) \big\}$ *and*

$$\|\psi_3 f\|_{\prod_j \{\mathcal{W}^l(K_j) \times \prod_{\sigma,\mu} W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma})\}} \le \\ c_l \big(\|\psi_0 \mathcal{L}(\theta)^* f\|_{-2m+l} + \|\psi_0 f\|_{\prod_j \big\{ W_{\bar K_j}^{-1}(\mathbb{R}^n) \times \prod_{\sigma,\mu} W^{-2m-1+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma}) \big\}} \big), \tag{8.1}$$

where

$$\|\cdot\|_{-2m+l}=\begin{cases}\|\cdot\|_{\prod\limits_j W^{-2m+l}_{\bar K_j}(\mathbb{R}^n)} \text{ for } l<2m,\\ \|\cdot\|_{\prod\limits_j \mathcal{W}^{-2m+l}(K_j)} \text{ for } l\ge 2m,\end{cases}$$

$c_l>0$ depends on $l\ge 0$ and does not depend on f.

Proof. 1) For any $g\in(E_a^{l-1/2}(\gamma_{jq}))^*$ and ψ_p, denote by $\psi g\otimes\delta(\gamma_{jq})$ the distribution from $W^{-l}_{\bar K_j}(\mathbb{R}^n)$ given by

$$<u_j,\ \psi g\otimes\delta(\gamma_{jq})>_{K_j}=<\psi u_j|_{\gamma_{jq}},\ g>_{\gamma_{jq}}\ \text{ for all } u_j\in W^l(K_j),$$

$j=1,\ \dots,\ N;\ q=1,\ \dots,\ R_j+1.$

Introduce the auxiliary operator

$$\mathcal{L}_{\mathcal{G}}(\theta)^*:\prod_{j=1}^N\big\{E^0_{-a}(K_j)\times\prod_{\sigma=1,\,R_j+1}\prod_{\mu=1}^m\big(E_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})\big)^*\times$$
$$\times\prod_{k=1}^N\prod_{q=2}^{R_k}\prod_{s=1}^{S_{j\sigma kq}}\big(E_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{kq})\big)^*\big)\big\}\to(E_a^{2m,\,N}(K))^*$$

that takes $f'=\{f_j,\ g_{j\sigma\mu},\ g'_{j\sigma\mu kqs}\}\in\prod\limits_j\big\{E^0_{-a}(K_j)\times\prod\limits_{\sigma,\mu}\big(E_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})\big)^*\times$ $\prod\limits_{k,q,s}\big(E_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{kq})\big)^*\big)\big\}$ to $\mathcal{L}_{\mathcal{G}}(\theta)^*f'$ by the rule

$$<u,\ \mathcal{L}_{\mathcal{G}}(\theta)^*f'>=\sum_j\big\{<\mathcal{P}_j(D_y,\ \theta)u_j,\ f_j>_{K_j}+$$
$$+\sum_{\sigma,\mu}\big(<B_{j\sigma\mu}(D_y,\ \theta)u_j|_{\gamma_{j\sigma}},\ g_{j\sigma\mu}>_{\gamma_{j\sigma}}+$$
$$\sum_{k,q,s}<B_{j\sigma\mu kqs}(D_y,\ \theta)u_k|_{\gamma_{kq}},\ g'_{j\sigma\mu kqs}>_{\gamma_{kq}}\big)\big\}\quad\text{for all } u\in E_a^{2m,\,N}(K).$$

Now for every $g_{j\sigma\mu}\in(E_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}))^*$ and ψ_p we introduce the distributions $g^{\mathcal{G}}_{j\sigma\mu kqs}\in(E_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{kq}))^*$ and $\psi_p g^{\mathcal{G}}_{j\sigma\mu kqs}\in W^{-2m+m_{j\sigma\mu}+1/2}(\hat\gamma_{kq})$ given by

$$<u_{\gamma_{kq}},\ g^{\mathcal{G}}_{j\sigma\mu kqs}>_{\gamma_{kq}}=<u_{\gamma_{kq}}(\mathcal{G}_{j\sigma kqs}\cdot),\ g_{j\sigma\mu}>_{\gamma_{j\sigma}}$$
$$\text{for all } u_{\gamma_{kq}}\in E_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{kq})$$

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[5] $W^{-l}_{\bar K_j}(\mathbb{R}^n)$ $(l>0)$ is the space that is adjoint to $W^l(K_j)$. One can identify the space $W^{-l}_{\bar K_j}(\mathbb{R}^n)$ with the subspace of the space $W^{-l}(\mathbb{R}^n)$ consisting of distributions with supports from $\bar K_j$ (see remark 12.4 [12, Chapter 1, §12.6]).

and

$$< W_{\gamma_{kq}},\ \psi_p g^{\mathcal G}_{j\sigma\mu kqs} >_{\hat\gamma_{kq}} = < (\psi_p W_{\gamma_{kq}})(\mathcal G_{j\sigma kqs}\cdot),\ g_{j\sigma\mu} >_{\gamma_{j\sigma}}$$
$$\text{for all } W_{\gamma_{kq}} \in W^{2m-m_{j\sigma\mu}-1/2}(\hat\gamma_{kq}).$$

From this, it follows in particular that $\psi_p g^{\mathcal G}_{j\sigma\mu kqs} \in W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{kq})$ iff $\psi_p(\mathcal G_{j\sigma kqs}\cdot)g_{j\sigma\mu} \in W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma})$; moreover, there are constants $k_1,\ k_2 > 0$ (depending on l) such that

$$k_1\|\psi_p(\mathcal G_{j\sigma kqs}\cdot)g_{j\sigma\mu}\|_{W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma})} \le \|\psi_p g^{\mathcal G}_{j\sigma\mu kqs}\|_{W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{kq})} \le$$
$$\le k_2\|\psi_p(\mathcal G_{j\sigma kqs}\cdot)g_{j\sigma\mu}\|_{W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma})}. \tag{8.2}$$

Put $f^{\mathcal G} = \{f_j,\ g_{j\sigma\mu},\ g^{\mathcal G}_{j\sigma\mu kqs}\}$. From the definitions of the operators $\mathcal L(\theta)^*$ and $\mathcal L_{\mathcal G}(\theta)^*$, it follows that

$$\mathcal L_{\mathcal G}(\theta)^* f^{\mathcal G} = \mathcal L(\theta)^* f. \tag{8.3}$$

Denote $\Xi f = \{\Xi_j f_j,\ \Xi_j g_{j\sigma\mu}\}$, $\Xi f^{\mathcal G} = \{\Xi_j f_j,\ \Xi_j g_{j\sigma\mu},\ \Xi_k g^{\mathcal G}_{j\sigma\mu kqs}\}$, where $\Xi = (\Xi_1,\ \dots,\ \Xi_N)$, $\Xi_j = \Xi_j(\varphi)$ are arbitrary infinitely differentiable on $[b_{j1},\ b_{j,R_j+1}]$ functions. Notice that in the formula $\Xi f^{\mathcal G} = \{\Xi_j f_j,\ \Xi_j g_{j\sigma\mu},\ \Xi_k g^{\mathcal G}_{j\sigma\mu kqs}\}$, a distribution $g^{\mathcal G}_{j\sigma\mu kqs}$ is multiplied by Ξ_k, but not by Ξ_j. This will be important further.

2) Let ζ_{jq} be the functions given by formula (3.4). We also consider the functions

$$\hat\zeta_{jq} \in C^\infty(\mathbb R),\ \hat\zeta_{jq}(\varphi) = 1 \text{ for } |b_{jq}-\varphi| < 3\varepsilon/2,\ \hat\zeta_{jq}(\varphi) = 0 \text{ for } |b_{jq}-\varphi| > 2\varepsilon; \tag{8.4}$$

$$\bar\zeta_{jq} \in C^\infty(\mathbb R),\ \bar\zeta_{jq}(\varphi) = 1 \text{ for } |b_{jq}-\varphi| < \varepsilon/8,\ \bar\zeta_{jq}(\varphi) = 0 \text{ for } |b_{jq}-\varphi| > \varepsilon/4 \tag{8.5}$$

$(j = 1,\ \dots,\ N;\ q = 1,\ \dots,\ R_j+1)$, where ε is given by formula (3.3).

Introduce the N-dimensional vector–function

$$\Xi^{j'\sigma'} = (0,\ \dots,\ \zeta_{j'\sigma'},\ \dots,\ 0).$$

Here "zeroes" are everywhere, except the j'-th position, $j' = 1,\ \dots,\ N$; $\sigma' = 1,\ R_{j'}+1$. If $j \ne j'$, then we have $\Xi^{j'\sigma'}_j = 0$. If $j = j'$, then we see that the support of $\Xi^{j'\sigma'}_{j'} = \zeta_{j'\sigma'}$ does not intersect with $\gamma_{j'q}$, but the support of $g^{\mathcal G}_{j\sigma\mu j'qs}$ is contained in $\gamma_{j'q}$ $(q = 2,\ \dots,\ R_{j'})$; therefore, $\zeta_{j'\sigma'} g^{\mathcal G}_{j\sigma\mu j'qs} = 0$. Thus we have

$$\mathcal L_{\mathcal G}(\theta)^*(\psi_p \Xi^{j'\sigma'} f^{\mathcal G}) = (0,\ \dots,\ \mathcal Q_{j'}(D_y,\ \theta)(\psi_p\zeta_{j'\sigma'}f_{j'})+$$
$$+\sum_{\mu=1}^{m} B^*_{j'\sigma'\mu}(D_y,\ \theta)(\psi_p\zeta_{j'\sigma'}g_{j'\sigma'\mu}\otimes\delta(\gamma_{j'\sigma'})),\ \dots,\ 0)$$

($p = 0, \dots, 3$). Here "zeroes" are everywhere, except the j'-th position, $\mathcal{Q}_{j'}(D_y,\ \theta)$ and $B^*_{j'\sigma'\mu}(D_y,\ \theta)$ are formally adjoint to $\mathcal{P}_{j'}(D_y,\ \theta)$ and $B_{j'\sigma'\mu}(D_y,\ \theta)$ correspondingly.

Notice that the operator

$$\mathcal{Q}_{j'}(D_y,\ \theta)(\psi_p\zeta_{j'\sigma'}f_{j'}) + \sum_{\mu=1}^{m} B^*_{j'\sigma'\mu}(D_y,\ \theta)(\psi_p\zeta_{j'\sigma'}g_{j'\sigma'\mu}\otimes\delta(\gamma_{j'\sigma'}))$$

can be identified with the adjoint to the operator

$$\{\mathcal{P}_{j'}(D_y,\ \theta)u_{j'},\ B_{j'\sigma'\mu}(D_y,\ \theta)u_{j'}|_{\hat\gamma_{j'\sigma'}}\}_{\mu=1}^m.$$

Therefore we can use theorem 4.3 [12, Chapter 2, §4.5][6]. Thus from relation (8.3) and Leibniz' formula, it follows that

$$\psi_1\Xi^{j'\sigma'}f^{\mathcal{G}} \in \prod_j\big\{W^l(K_j)\times\prod_{\sigma,\mu}\big(W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma})\times \\ \times\prod_{k,q,s}W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{kq})\big)\big\}$$

and

$$\|\psi_1\Xi^{j'\sigma'}f^{\mathcal{G}}\|_{\prod\limits_j\big\{W^l(K_j)\times\prod\limits_{\sigma,\mu}\big(W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma})\times\prod\limits_{k,q,s}W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{kq})\big)\big\}} \le \\ \le k_3\big(\|\psi_0\mathcal{L}(\theta)^*f\|_{-2m+l} + \|\psi_0\hat\zeta_{j'\sigma'}f_{j'}\|_{W^{-1}_{\bar K_{j'}}(\mathbb{R}^n)} + \\ + \sum_{\mu=1}^m\|\psi_0 g_{j'\sigma'\mu}\|_{W^{-2m-1+m_{j'\sigma'\mu}+1/2}(\hat\gamma_{j'\sigma'})}\big). \tag{8.6}$$

From (8.6) and (8.2), it follows in particular that $\psi_2 g^{\mathcal{G}}_{j'\sigma'\mu kqs} \in W^{-2m+l+m_{j'\sigma'\mu}+1/2}(\hat\gamma_{kq})$ and

$$\|\psi_2 g^{\mathcal{G}}_{j'\sigma'\mu kqs}\|_{W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{kq})} \le k_4\big(\|\psi_0\mathcal{L}(\theta)^*f\|_{-2m+l}+ \\ +\|\psi_0\hat\zeta_{j'\sigma'}f_{j'}\|_{W^{-1}_{\bar K_{j'}}(\mathbb{R}^n)} + \sum_{\mu=1}^m\|\psi_0 g_{j'\sigma'\mu}\|_{W^{-2m-1+m_{j'\sigma'\mu}+1/2}(\hat\gamma_{j'\sigma'})}\big). \tag{8.7}$$

3) Put $\Xi^{k'q'} = (0,\ \dots,\ \zeta_{k'q'},\ \dots,\ 0)$. Here "zeroes" are everywhere, except the k'-th position, $k' = 1,\ \dots,\ N$; $q' = 2,\ \dots,\ R_{k'}$. If $k \ne k'$, then we have

[6]Theorem 4.3 [12, Chapter 2, §4.5] deals with operators having variable coefficients; therefore some additional restrictions are imposed on supports of considered functions. It is easy to see that these restrictions may be omitted if the coefficients are constant.

$\Xi_k^{k'q'} = 0$. If $k = k'$, then we see that the support of $\Xi_{k'}^{k'q'} = \zeta_{k'q'}$ does not intersect with the supports of $g_{k'\sigma\mu}$ and $g^{\mathcal{G}}_{j\sigma\mu k'qs}$ for $q \ne q'$; therefore, $\zeta_{k'q'}g_{k'\sigma\mu} = 0$ and $\zeta_{k'q'}g^{\mathcal{G}}_{j\sigma\mu k'qs} = 0$ for $q \ne q'$. Thus we have

$$\mathcal{L}_{\mathcal{G}}(\theta)^*(\psi_p\Xi^{k'q'}f^{\mathcal{G}}) = (0,\ \dots,\ \mathcal{Q}_{k'}(D_y,\ \theta)(\psi_p\zeta_{k'q'}f_{k'})+ \\ + \sum_{j,\sigma,\mu,s} B^*_{j\sigma\mu k'q's}(D_y,\ \theta)(\psi_p\zeta_{k'q'}g^{\mathcal{G}}_{j\sigma\mu k'q's}\otimes\delta(\gamma_{k'q'})),\ \dots,\ 0)$$

($p = 0,\ \dots,\ 3$), where "zeroes" are everywhere, except the k'-th position, $B^*_{j\sigma\mu k'q's}(D_y,\ \theta)$ is formally adjoint to $B_{j\sigma\mu k'q's}(D_y,\ \theta)$.

Notice that the operator

$$\mathcal{Q}_{k'}(D_y,\ \theta)(\psi_p\zeta_{k'q'}f_{k'}) + \sum_{j,\sigma,\mu,s} B^*_{j\sigma\mu k'q's}(D_y,\ \theta)(\psi_p\zeta_{k'q'}g^{\mathcal{G}}_{j\sigma\mu k'q's}\otimes\delta(\gamma_{k'q'}))$$

can be identified with the adjoint to the operator of the problem

$$\mathcal{P}_{k'}(D_y,\ \theta)u_{k'} = \hat{f}_{k'}(y)\ (y\in\mathbb{R}^2),$$
$$B_{j\sigma\mu k'q's}(D_y,\ \theta)u_{k'}|_{\hat{\gamma}_{k'q'}} = \hat{g}_{j\sigma\mu s}(y)\ (y\in\hat{\gamma}_{k'q'})$$
$$(j = 1,\ \dots,\ N;\ \sigma = 1,\ R_j+1;\ \mu = 1,\ \dots,\ m;\ s = 1,\ \dots,\ S_{j\sigma k'q'}).$$

This problem differs from the problem studied in Appendix A only in junior terms.

In 1), we showed that $\psi_2 g^{\mathcal{G}}_{j\sigma\mu k'q's} \in W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat{\gamma}_{k'q'})$; hence we can apply theorem A.1. Thus from relation (8.3) and Leibniz' formula, we obtain

$$\psi_3\Xi^{k'q'}f^{\mathcal{G}} \in \prod_j\big\{\mathcal{W}^l(K_j)\times\prod_{\sigma,\mu}\big(W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat{\gamma}_{j\sigma})\times \\ \times\prod_{k,q,s}W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat{\gamma}_{kq})\big)\big\}$$

and

$$\|\psi_3\Xi^{k'q'}f^{\mathcal{G}}\|_{\prod\limits_j\{\mathcal{W}^l(K_j)\times\prod\limits_{\sigma,\mu}(W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat{\gamma}_{j\sigma})\times\prod\limits_{k,q,s}W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat{\gamma}_{kq}))\}} \le \\ \le k_5\big(\|\psi_2\mathcal{L}(\theta)^*f\|_{-2m+l} + \|\psi_2\hat{\zeta}_{k'q'}f_{k'}\|_{W^{-1}_{\bar{K}_{k'}}(\mathbb{R}^n)} + \\ + \sum_{j,\sigma,\mu,s}\|\psi_2 g^{\mathcal{G}}_{j\sigma\mu k'q's}\|_{W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat{\gamma}_{k'q'})}\big). \tag{8.8}$$

Notice that the space $\mathcal{W}^l(\cdot)$ appeared just here. As we noted earlier, this is connected with the nonlocal terms $g^{\mathcal{G}}_{j\sigma\mu k'q's}$, which have supports on $\gamma_{k'q'}$ ($q' = 2,\ \dots,\ R_{k'}$).

From inequalities (8.8) and (8.7), we get

$$\|\psi_3\Xi^{k'q'}f^{\mathcal{G}}\|_{\prod\limits_j\left\{\mathcal{W}^l(K_j)\times\prod\limits_{\sigma,\mu}\left(W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma})\times\prod\limits_{k,q,s}W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{kq})\right)\right\}}\le$$
$$\le k_6\Big(\|\psi_0\mathcal{L}(\theta)^*f\|_{-2m+l}+\sum_{j=1}^{N}\sum_{\sigma=1,\,R_j+1}\{\|\psi_0\hat\zeta_{j\sigma}f_j\|_{W^{-1}_{\bar K_j}(\mathbb{R}^n)}+$$
$$+\sum_{\mu=1}^{m}\|\psi_0 g_{j\sigma\mu}\|_{W^{-2m-1+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma})}\}\Big). \tag{8.9}$$

4) Finally, we put $\zeta_{i0}=1-\sum\limits_{q=1}^{R_i+1}\zeta_{iq}$, $\Xi^{i0}=(0,\ \dots,\ \zeta_{i0},\ \dots,\ 0)$. Here "zeroes" are everywhere, except the i-th position, $i=1,\ \dots,\ N$.

Since the support of ζ_{i0} does not intersect with γ_{iq} ($q=1,\ \dots,R_i+1$), we have

$$\mathcal{L}_{\mathcal{G}}(\theta)^*(\psi_p\Xi^{i0}f^{\mathcal{G}})=(0,\ \dots,\ \mathcal{Q}_i(D_y,\ \theta)(\psi_p\zeta_{i0}f_i),\ \dots,\ 0)$$

($p=0,\ \dots,\ 3$). Here "zeroes" are everywhere, except the i-th position.

The operator $\mathcal{Q}_i(D_y,\ \theta)(\psi_p\zeta_{j'\sigma'}f_{j'})$ can be identified with the adjoint one to the operator of the problem

$$\mathcal{P}_i(D_y,\ \theta)u_i=\hat f_i(x)\ (y\in\mathbb{R}^2).$$

Therefore applying theorem 3.1 [12, Chapter 2, §3.2], from (8.3) and Leibniz' formula, we get

$$\psi_1\Xi^{i0}f^{\mathcal{G}}\in\prod_j\Big\{W^l(K_j)\times\prod_{\sigma,\mu}\big(W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma})\times$$
$$\times\prod_{k,q,s}W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{kq})\big)\Big\}$$

and

$$\|\psi_1\Xi^{i0}f^{\mathcal{G}}\|_{\prod\limits_j\left\{W^l(K_j)\times\prod\limits_{\sigma,\mu}\left(W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{j\sigma})\times\prod\limits_{k,q,s}W^{-2m+l+m_{j\sigma\mu}+1/2}(\hat\gamma_{kq})\right)\right\}}\le$$
$$\le k_7\big(\|\psi_0\mathcal{L}(\theta)^*f\|_{-2m+l}+\|\psi_0\bar\zeta_{i0}f_i\|_{W^{-1}_{\bar K_i}(\mathbb{R}^n)}\big), \tag{8.10}$$

where $\zeta_{i0}=1-\sum\limits_{q=1}^{R_i+1}\bar\zeta_{iq}$.

Now a priori estimate (8.1) follows from inequalities (8.6), (8.9), and (8.10). □

2 Connection between kernel of $\mathcal{L}(\theta)^*$ and kernel of $\mathcal{M}(\theta)$.

Lemma 8.1. *The kernel* $\ker(\mathcal{L}(\theta)^*)$ *of the operator* $\mathcal{L}(\theta)^*$ *coincides with the set* $\{v_j,\ F_{j\sigma\mu}(D_y,\ \theta)v|_{\gamma_{j\sigma}}\}$, *where* $v_j \in \mathcal{E}^{2m}_{-a+2m}(K_j)$, $v_{jt} \in C^\infty(\bar{K}_{jt}\backslash\{0\})$ $(j = 1, \dots,\ N;\ t = 1,\ \dots,\ R_j)$, *and* v *is a solution to problem (5.8)–(5.10) for* $\{f_j,\ g_{j\sigma\mu},\ h_{jq\nu}\} = 0$.

Proof. 1) In this proof, we shall omit the arguments $(D_y,\ \theta)$ in differential operators; so we shall write $\mathcal{P}_j$ instead of $\mathcal{P}_j(D_y,\ \theta)$ and so on.

Suppose $v_j \in \mathcal{E}^{2m}_{-a+2m}(K_j)$, $v_{jt} \in C^\infty(\bar{K}_{jt}\backslash\{0\})$ and v is a solution to problem (5.8)–(5.10) for $\{f_j,\ g_{j\sigma\mu},\ h_{jq\nu}\} = 0$. Then for any functions $u_j \in C^\infty_0(\bar{K}_j\backslash\{0\})$, by virtue of Theorem 4.1, we have

$$\sum_j\Big\{\sum_t(\mathcal{P}_j u_j,\ v_{jt})_{K_{jt}} + \sum_{\sigma,\mu}(\mathcal{B}_{j\sigma\mu}u,\ F_{j\sigma\mu}v_j|_{\gamma_{j\sigma}})_{\gamma_{j\sigma}} = 0. \tag{8.11}$$

Since the imbedding operator of $\mathcal{E}^{2m}_{-a+2m}(K_j)$ into $E^0_{-a}(K_j)$ is bounded, we have $v_j \in \mathcal{E}^0_{-a}(K_j)$. Besides, the operator $F_{j\sigma\mu}(D_y,\ \theta)$ is of order $2m - 1 - m_{j\sigma\mu}$; hence, from the Schwarz inequality and Theorem B.2, for all $u_{\gamma_{j\sigma}} \in E^{2m-m_{j\sigma\mu}-1/2}_a(\gamma_{j\sigma})$, we obtain

$$\begin{aligned}|(u_{\gamma_{j\sigma}},\ F_{j\sigma\mu}v_j|_{\gamma_{j\sigma}})_{\gamma_{j\sigma}}|^2 \le \int\limits_{\gamma_{j\sigma}} r^{2(a-(2m-m_{j\sigma\mu}-1/2))}|u_{\gamma_{j\sigma}}|^2\, d\gamma\times\\ \times\int\limits_{\gamma_{j\sigma}} r^{2(-a+2m-(m_{j\sigma\mu}+1/2))}|F_{j\sigma\mu}v_j|_{\gamma_{j\sigma}}|^2\, d\gamma \le\\ \le k_1\|u_{\gamma_{j\sigma}}\|^2_{E^{2m-m_{j\sigma\mu}-1/2}_a(\gamma_{j\sigma})}\cdot\|F_{j\sigma\mu}v_j|_{\gamma_{j\sigma}}\|^2_{E^{m_{j\sigma\mu}+1/2}_{-a+2m}(\gamma_{j\sigma})}\end{aligned}$$

Therefore, $F_{j\sigma\mu}v_j|_{\gamma_{j\sigma}} \in (E^{2m-m_{j\sigma\mu}-1/2}_a(\gamma_{j\sigma}))^*$.

Thus, $\{v_j,\ F_{j\sigma\mu}(D_y,\ \theta)v|_{\gamma_{j\sigma}}\} \in \prod\limits_j\big\{E^0_{-a}(K_j)\times\prod\limits_{\sigma,\mu}(E^{2m-m_{j\sigma\mu}-1/2}_a(\gamma_{j\sigma}))^*\big\}$ and from the definition of the operator $\mathcal{L}(\theta)^*$ and identity (8.11), we get

$$< u,\ \mathcal{L}(\theta)^*\{v_j,\ F_{j\sigma\mu}(D_y,\ \theta)v|_{\gamma_{j\sigma}}\} >= 0 \text{ for all } u \in \prod_j C^\infty_0(\bar{K}_j\backslash\{0\}).$$

But $\prod\limits_j C^\infty_0(\bar{K}_j\backslash\{0\})$ is dense in $E^{2m,N}_a(K)$; hence, $\{v_j,\ F_{j\sigma\mu}(D_y,\ \theta)v|_{\gamma_{j\sigma}}\} \in \ker(\mathcal{L}(\theta)^*)$.

2) Now suppose $\{v_j,\ \psi_{j\sigma\mu}\} \in \ker(\mathcal{L}(\theta)^*)$. From Theorem 8.1, it follows that $v_{jt} \in C^\infty(\bar{K}_{jt}\backslash\{0\})$, $\psi_{j\sigma\mu} \in C^\infty(\gamma_{j\sigma})$. Then from the definition of the

operator $\mathcal{L}(\theta)^*$, it follows that

$$\sum_j \{(\mathcal{P}_j u_j,\ v_j)_{K_j} = -\sum_{j,\sigma,\mu} (\mathcal{B}_{j\sigma\mu} u,\ \psi_{j\sigma\mu})_{\gamma_{j\sigma}},\ \text{for all } u_j \in C_0^\infty(\bar{K}_j\backslash\{0\}).$$

The last identity and Green formula (4.15) imply

$$\begin{aligned} &\sum_j \{\sum_{\sigma,\mu} (\mathcal{B}_{j\sigma\mu} u,\ F_{j\sigma\mu} v_j|_{\gamma_{j\sigma}} - \psi_{j\sigma\mu})_{\gamma_{j\sigma}} + \sum_{q,\mu} (B_{jq\mu} u_j|_{\gamma_{jq}},\ \mathcal{T}_{jq\mu} v)_{\gamma_{jq}}\} = \\ &\quad = \sum_j \{\sum_t (u_j,\ \mathcal{Q}_j v_{jt})_{K_{jt}} + \sum_{\sigma,\mu} (B'_{j\sigma\mu} u_j|_{\gamma_{j\sigma}},\ C_{j\sigma\mu} v_j|_{\gamma_{j\sigma}})_{\gamma_{j\sigma}} + \\ &\qquad + \sum_{q,\mu} (B'_{jq\mu} u_j|_{\gamma_{jq}},\ \mathcal{T}_{jq,m+\mu} v)_{\gamma_{jq}}\}. \end{aligned} \tag{8.12}$$

Putting $\operatorname{supp} u_j \in C_0^\infty(K_{jt})$, from (8.12), we obtain $\mathcal{Q}_j v_{jt} = 0$, $j = 1,\ \dots,\ N$; $t = 1,\ \dots,\ R_j$.

By Theorem 4.1, the system $\{B_{j\sigma\mu},\ B'_{j\sigma\mu}\}_{\mu=1}^m$ is a Dirichlet system on $\gamma_{j\sigma}$ ($j = 1,\ \dots,\ N;\ \sigma = 1,\ R_j + 1$) of order $2m$. Therefore, for any system of functions $\{\Theta_{j\sigma\nu}\}_{\nu=1}^{2m} \subset C_0^\infty(\gamma_{j\sigma})$ there exist functions $u_j \in C_0^\infty(\bar{K}_j\backslash\{0\})$ such that

$$\begin{gathered} B_{j\sigma\mu} u_j|_{\gamma_{j\sigma}} = \Theta_{j\sigma\mu},\ B'_{j\sigma\mu} u_j|_{\gamma_{j\sigma}} = \Theta_{j\sigma,\mu+m},\ \mu = 1,\ \dots,\ m, \\ u_j = 0 \text{ in a neighbourhood of } \gamma_{jq}\ (j = 1,\ \dots,\ N;\ q = 2,\ \dots,\ R_j) \end{gathered}$$

(see lemma 2.2 [12, Chapter 2, §2.3]). Therefore, taking into account that $\mathcal{Q}_j v_{jt} = 0$, from (8.12), we obtain $F_{j\sigma\mu} v_j|_{\gamma_{j\sigma}} - \psi_{j\sigma\mu} = 0$ and $C_{j\sigma\mu} v_j|_{\gamma_{j\sigma}} = 0$.

Similarly, since $\{B_{jq\mu},\ B'_{jq\mu}\}_{\mu=1}^m$ is a Dirichlet system on γ_{jq} ($j = 1,\ \dots,\ N;\ q = 2,\ \dots,\ R_j$) of order $2m$, we get $\mathcal{T}_{jq\nu} v = 0$.

Finally, we know that $v_j \in E_{-a}^0(K_j)$ by assumption and we showed that $v_{jt} \in C^\infty(\bar{K}_{jt}\backslash\{0\})$; therefore, from Theorem 7.1, it follows that $v_j \in \mathcal{E}_{-a+2m}^{2m}(K_j)$. □

9 Solvability of nonlocal boundary value problems

In this section, we study solvability of nonlocal boundary value problems. In subsection 1, we establish necessary and sufficient conditions for Fredholm solvability of the nonlocal boundary value problems with parameter θ in plane angles. In subsection 2, we study necessary conditions for Fredholm solvability and sufficient conditions for one–valued solvability of nonlocal boundary value problems in dihedral angles.

1 Fredholm solvability of nonlocal boundary value problems with parameter θ.

Theorem 9.1. *Put $a = b + l$. Suppose the line $\operatorname{Im}\lambda = b + 1 - 2m$ contains no poles of the operator–valued function $\tilde{\mathcal{L}}^{-1}(\lambda)$; then the operator*

$$\mathcal{L}(\theta) = \{\mathcal{P}_j(D_y,\ \theta),\ \mathcal{B}_{j\sigma\mu}(D_y,\ \theta)\} : E_a^{l+2m,\,N}(K) \to E_a^{l,\,N}(K,\ \gamma)$$

is Fredholm for all $\theta \in S^{n-3}$.

If there is a $\theta \in S^{n-3}$ such that the operator $\mathcal{L}(\theta)$ is Fredholm, then the line $\operatorname{Im}\lambda = b + 1 - 2m$ contains no poles of the operator–valued function $\tilde{\mathcal{L}}^{-1}(\lambda)$.

Proof. Suppose the line $\operatorname{Im}\lambda = b+1-2m$ contains no poles of $\tilde{\mathcal{L}}^{-1}(\lambda)$; then by Theorem 3.3, the operator $\mathcal{L}(\theta)$ has finite dimensional kernel and closed range.

Let us prove that cokernel of the operator $\mathcal{L}(\theta)$ is of finite dimension. First, we put $l = 0$. By Theorems 2.1 and 6.1, the operators $\tilde{\mathcal{L}}(\lambda)$ and $\tilde{\mathcal{M}}(\lambda)$ are Fredholm and have zero indices. Therefore from Green formula (4.16) and Remark 4.3, it follows that λ_0 is a pole of $\tilde{\mathcal{L}}^{-1}(\lambda)$ iff $\lambda_0' = \bar{\lambda}_0 - 2i(m-1)$ is a pole of $\tilde{\mathcal{M}}^{-1}(\lambda)$. Hence the line $\operatorname{Im}\ \lambda = (-b+2m)+1-2m$ contains no poles of the operator–valued function $\tilde{\mathcal{M}}^{-1}(\lambda)$. Now by Theorem 7.3, kernel of the operator $\mathcal{M}(\theta)$ is of finite dimension. Finally, Lemma 8.1 implies $\dim\ker(\mathcal{L}(\theta)^*) = \dim\ker(\mathcal{M}(\theta)) < \infty$.

Consider the case $l \geq 1$. Suppose $f \in E_a^{l,\,N}(K,\ \gamma)$. By the above, there exists a $u \in E_{a-l}^{2m,\,N}(K)$ such that $\mathcal{L}(\theta)u = f$ iff $(f,\ \Psi_i)_{E_{a-l}^{0,\,N}(K,\ \gamma)} = 0$ for some linearly independent functions $\Psi_i \in E_{a-l}^{0,\,N}(K,\ \gamma)$ $(i = 1,\ \dots,\ J)$. Here $(\cdot,\ \cdot)_{E_{a-l}^{0,\,N}(K,\ \gamma)}$ is the inner product in the Hilbert space $E_{a-l}^{0,\,N}(K,\ \gamma)$. In addition, by Theorem 3.2, we have $u \in E_a^{l+2m,\,N}(K)$.

By virtue of the Schwarz inequality and boundness of the imbeding operator of $E_a^{l,\,N}(K,\ \gamma)$ into $E_{a-l}^{0,\,N}(K,\ \gamma)$, we have

$$(f,\ \Psi_i)_{E_{a-l}^{0,\,N}(K,\ \gamma)} \leq \|f\|_{E_{a-l}^{0,\,N}(K,\ \gamma)}\|\Psi_i\|_{E_{a-l}^{0,\,N}(K,\ \gamma)} \leq$$

$$k_1\|f\|_{E_a^{l,\,N}(K,\ \gamma)}\|\Psi_i\|_{E_{a-l}^{0,\,N}(K,\ \gamma)}$$

for all $f \in E_a^{l,\,N}(K,\ \gamma)$. Therefore, by virtue of the Riesz theorem concerning a general form of a linear functional in a Hilbert space, there exist linearly independent functions $\hat{\Psi}_i \in E_a^{l,\,N}(K,\ \gamma)$ $(i = 1,\ \ldots,\ J)$ such that

$$(f,\ \Psi_i)_{E_{a-l}^{0,\,N}(K,\ \gamma)} = (f,\ \hat{\Psi}_i)_{E_a^{l,\,N}(K,\ \gamma)} \text{ for all } f \in E_a^{l,\,N}(K,\ \gamma).$$

This means that cokernel of the operator $\mathcal{L}(\theta)$ is of the same finite dimension J for all $l \geq 0$.

The second part of the Theorem follows from Theorem 3.3. □

2 Solvability of nonlocal boundary value problems in dihedral angles.

Theorem 9.2. *Put $a = b + l$. Suppose the line* $\operatorname{Im}\lambda = b + 1 - 2m$ *contains no poles of the operator–valued function $\tilde{\mathcal{L}}^{-1}(\lambda)$. Suppose also that for $l = 0$, we have* $\dim\ker(\mathcal{L}(\theta)) = 0$ *for all $\theta \in S^{n-3}$,* $\operatorname{codim}\mathcal{R}(\mathcal{L}(\theta_0)) = 0$ *for some $\theta_0 \in S^{n-3}$; then the operator*

$$\mathcal{L} = \{\mathcal{P}_j(D_y,\ D_z),\ \mathcal{B}_{j\sigma\mu}(D_y,\ D_z)\} : H_a^{l+2m,\,N}(\Omega) \to H_a^{l,\,N}(\Omega,\ \Gamma)$$

is an isomorphism.

Proof. By Theorem 3.3, we have $\dim\ker(\mathcal{L}(\theta)) < \infty$ and range $\mathcal{R}(\mathcal{L}(\theta))$ is closed in $E_a^{l,\,N}(K,\ \gamma)$ for all $\theta \in S^{n-3}$.

Since the operator $\mathcal{L}(\theta)$ is bounded and $\dim\ker(\mathcal{L}(\theta)) = 0$ for $l = 0$, we have

$$k_1\|\mathcal{L}(\theta)u\|_{E_a^{0,\,N}(K,\ \gamma)} \leq \|u\|_{E_a^{2m,\,N}(K)} \leq k_2\|\mathcal{L}(\theta)u\|_{E_a^{0,\,N}(K,\ \gamma)}, \tag{9.1}$$

where $k_1,\ k_2 > 0$ are independent of $\theta \in S^{n-3}$ and u (k_2 does not depend on $\theta \in S^{n-3}$, since the sphere S^{n-3} is compact).

By assumption, there exists a $\theta_0 \in S^{n-3}$ such that the operator $\mathcal{L}(\theta_0)$ has a bounded inverse. Therefore, using estimates (9.1) and the method of continuation with respect to the parameter $\theta \in S^{n-3}$ (see the proof of theorem 7.1 [14, Chapter 2, §7]), we prove that the operator $\mathcal{L}(\theta)$ has a bounded inverse for all $\theta \in S^{n-3}$.

Reduce problem (1.1), (1.2) to problem (1.6), (1.7) doing the Fourier transform with respect to $z : U(y,\ z) \to \hat{U}(y,\ \eta)$ and changing variables: $y' = |\eta| \cdot y$. Now repeating the proof of lemma 7.3 [7, §7] and applying Theorem 3.1 of this work, we complete the proof. □

Theorem 9.3. *Suppose for some $b \in \mathbb{R}$, $l_1 \geq 0$, the operator*

$$\mathcal{L} = \{\mathcal{P}_j(D_y,\ D_z),\ \mathcal{B}_{j\sigma\mu}(D_y,\ D_z)\} : H_{a_1}^{l_1+2m,\,N}(\Omega) \to H_{a_1}^{l_1,\,N}(\Omega,\ \Gamma),\ a_1 = b+l_1,$$

is Fredholm; then the operator

$$\mathcal{L}(\theta) = \{\mathcal{P}_j(D_y,\ \theta),\ \mathcal{B}_{j\sigma\mu}(D_y,\ \theta)\} : E_a^{l+2m,\,N}(K) \to E_a^{l,\,N}(K,\ \gamma),\ a = b+l,$$

is an isomorphism for all $\theta \in S^{n-3}$, $l = 0,\ 1,\ \dots$

Proof. 1) While proving the Theorem, we shall follow the scheme of the paper [7, §8].

Similarly to the proof of lemma 8.1 [7, §8], one can prove that the operator $\mathcal{L}$ is an isomorphism for $l = l_1$, $a = a_1$. Therefore we have

$$\|U\|_{H_{a_1}^{l_1+2m,\,N}(\Omega)} \leq k_1 \|\mathcal{L}U\|_{H_{a_1}^{l_1,\,N}(\Omega,\ \Gamma)}.$$

Substituting $U^p(y,\ z) = p^{1-n/2} e^{i(\theta,\ z)} \varphi(z/p) u(y)$ ($\varphi \in C_0^\infty(\mathbb{R}^{n-2})$, $u \in E_{a_1}^{l_1+2m,\,N}(K)$, $\theta \in S^{n-3}$) into the last inequality and passing to the limit as $p \to \infty$, we get

$$\|u\|_{E_a^{l+2m,\,N}(K)} \leq k_2 \|\mathcal{L}(\theta)u\|_{E_a^{l,\,N}(K,\ \gamma)} \tag{9.2}$$

for $l = l_1$, $a = a_1$. This implies that $\mathcal{L}(\theta)$ has trivial kernel for $l = l_1$, $a = a_1$. But by Theorem 3.2, kernel of $\mathcal{L}(\theta)$ does not depend on l and $a = b + l$; therefore the operator $\mathcal{L}(\theta)$ has trivial kernel for all l and $a = b + l$.

By Theorem 3.3, estimate (9.2) implies that the line $\operatorname{Im} \lambda = b + 1 - 2m$ contains no poles of the operator–valued function $\tilde{\mathcal{L}}^{-1}(\lambda)$. Hence, by Theorem 9.1, the operator $\mathcal{L}(\theta)$ is Fredholm for all l and $a = b + l$. From this and from triviality of $\ker \mathcal{L}(\theta)$, it follows that estimate (9.2) is valid for all l and $a = b + l$.

2) Repeating the proof of lemma 7.3 [7, §7], from estimate (9.2), we get

$$\|U\|_{H_a^{2m,\,N}(\Omega)} \leq k_3 \|\mathcal{L}U\|_{H_a^{0,\,N}(\Omega,\ \Gamma)},$$

where $l = 0$, $a = b$. Therefore, the operator $\mathcal{L} : H_b^{2m,\,N}(\Omega) \to H_b^{0,\,N}(\Omega,\ \Gamma)$ has trivial kernel and closed range. Let us show that its range coincides with $H_b^{0,\,N}(\Omega,\ \Gamma)$. Indeed, since $H_{b+l_1}^{l_1+2m,\,N}(\Omega) \subset H_b^{2m,\,N}(\Omega)$, range $\mathcal{R}(\mathcal{L})_{b+l_1}$ of the operator $\mathcal{L} : H_{b+l_1}^{l_1+2m,\,N}(\Omega) \to H_{b+l_1}^{l_1,\,N}(\Omega,\ \Gamma)$ is contained in range $\mathcal{R}(\mathcal{L})_b$ of the operator $\mathcal{L} : H_b^{2m,\,N}(\Omega) \to H_b^{0,\,N}(\Omega,\ \Gamma)$:

$$\mathcal{R}(\mathcal{L})_{b+l_1} \subset \mathcal{R}(\mathcal{L})_b.$$

By proved in 1), $\mathcal{R}(\mathcal{L})_{b+l_1} = H^{l_1,\,N}_{b+l_1}(\Omega,\ \Gamma)$ which is dense in $H^{0,\,N}_b(\Omega,\ \Gamma)$; hence, $\mathcal{R}(\mathcal{L})_b$ is also dense in $H^{0,\,N}_b(\Omega,\ \Gamma)$. But $\mathcal{R}(\mathcal{L})_b$ is closed; therefore, $\mathcal{R}(\mathcal{L})_b = H^{0,\,N}_b(\Omega,\ \Gamma)$.

So, we have proved that the operator $\mathcal{L} : H^{2m,\,N}_b(\Omega) \to H^{0,\,N}_b(\Omega,\ \Gamma)$ is an isomorphism.

3) Now we shall prove the estimate

$$\|V\|_{\mathcal{H}^{2m,\,N}_{-b+2m}(\Omega)} \le k_4 \|\mathcal{M}V\|_{\mathcal{H}^{0,\,N}_{-b+2m}(\Omega,\ \Gamma)}. \tag{9.3}$$

Denote by $\mathrm{P} : H^{0,\,N}_{b-2m}(\Omega) \to H^{0,\,N}_b(\Omega)$ the unbounded operator corresponding to problem (1.1), (1.2) with homogeneous nonlocal conditions. The operator P is given by

$$\mathrm{Dom}\,(\mathrm{P}) = \{U \in H^{2m,\,N}_b(\Omega) :\ \mathcal{B}_{j\sigma\mu}(D_y,\ D_z)U = 0,$$
$$j = 1,\ \ldots,\ N;\ \sigma = 1,\ R_j + 1;\ \mu = 1,\ \ldots,\ m\},$$

$$\mathrm{P}U = (\mathcal{P}_1(D_y,\ D_z)U_1,\ \ldots,\ \mathcal{P}_N(D_y,\ D_z)U_N), \quad U \in \mathrm{Dom}\,(\mathrm{P}).$$

Denote by $\mathrm{Q} : H^{0,\,N}_{-b}(\Omega) \to H^{0,\,N}_{-b+2m}(\Omega)$ the unbounded operator corresponding to problem (5.1)–(5.3) with homogeneous boundary conditions and homogeneous nonlocal transmission conditions. The operator Q is given by

$$\mathrm{Dom}\,(\mathrm{Q}) = \{V \in \mathcal{H}^{2m,\,N}_{-b+2m}(\Omega) :\ \mathcal{C}_{j\sigma\mu}(D_y,\ D_z)V = 0,\ \mathcal{T}_{jq\nu}(D_y,\ D_z)V = 0,$$
$$j = 1,\ \ldots,\ N;\ \sigma = 1,\ R_j + 1;\ \mu = 1,\ \ldots,\ m;$$
$$q = 2,\ \ldots,\ R_j;\ \nu = 1,\ \ldots,\ 2m\}$$

$$\mathrm{Q}V = (W_1,\ \ldots,\ W_N),\ W_j = \mathcal{Q}_j(D_y,\ D_z)V_{jt} \text{ for } x \in \Omega_{jt},\ V \in \mathrm{Dom}\,(\mathrm{Q}).$$

It is clear that $\mathrm{Dom}\,(\mathrm{P})$ is dense in $H^{0,\,N}_{b-2m}(\Omega)$ and $\mathrm{Dom}\,(\mathrm{Q})$ is dense in $\mathcal{H}^{0,\,N}_{-b}(\Omega)$. From Theorems 3.1 and 7.1, it follows that the operators P and Q are closed. Since the operator $\mathcal{L} : H^{2m,\,N}_b(\Omega) \to H^{0,\,N}_b(\Omega,\ \Gamma)$ is an isomorphism, the operator P is also an isomorphism from $\mathrm{Dom}\,(\mathrm{P})$ onto $H^{0,\,N}_b(\Omega)$.

Denote by $\mathrm{P}^* : H^{0,\,N}_{-b}(\Omega) \to H^{0,\,N}_{-b+2m}(\Omega)$ the operator that is adjoint to P with respect to the inner product $\sum\limits_j (U_j,\ V_j)_{\Omega_j}$ in $\prod\limits_j L_2(\Omega_j)$. Since the operator P is an isomorphism from $\mathrm{Dom}\,(\mathrm{P})$ onto $H^{0,\,N}_b(\Omega)$, the operator P^* is also an isomorphism from $\mathrm{Dom}\,(\mathrm{P}^*)$ onto $H^{0,\,N}_{-b+2m}(\Omega)$ and its domain $\mathrm{Dom}\,(\mathrm{P}^*)$ is dense in $H^{0,\,N}_{-b}(\Omega)$. The operator P^* is given by

$$\sum_j \big(\mathrm{P}_j U_j,\ V_j\big)_{\Omega_j} = \sum_j \big(U_j,\ (\mathrm{P}^*V)_j\big)_{\Omega_j} \text{ for any } U \in \mathrm{Dom}\,(\mathrm{P}), V \in \mathrm{Dom}\,(\mathrm{P}^*).$$

Since the closed operator P^* is an isomorphism from $\mathrm{Dom}\,(\mathrm{P}^*)$ onto $H^{0,\,N}_{-b+2m}(\Omega)$, we have

$$\|V\|_{H^{0,\,N}_{-b}(\Omega)} \le k_5 \|\mathrm{P}^* V\|_{H^{0,\,N}_{-b+2m}(\Omega)} \tag{9.4}$$

for all $V \in \mathrm{Dom}\,(\mathrm{P}^*)$, where $k_5 > 0$ is independent of V.

From Theorem 4.1 and Remark 4.1, it follows that $\mathrm{Q} \subset \mathrm{P}^*$.[7] Therefore using (9.4), we get

$$\|V\|_{\mathcal{H}^{0,\,N}_{-b}(\Omega)} \le k_5 \|\mathrm{Q} V\|_{H^{0,\,N}_{-b+2m}(\Omega)}$$

for all $V \in \mathrm{Dom}\,(\mathrm{Q})$. From the last inequality, Lemma 5.1, and Theorem 7.1, we obtain estimate (9.3).

4) Substituting $V^p(y,\ z) = p^{1-n/2} e^{i(\theta,\ z)} \varphi(z/p) v(y)$ ($\varphi \in C_0^\infty(\mathbb{R}^{n-2})$, $v \in \mathcal{E}^{2m,\,N}_{-b+2m}(K)$, $\theta \in S^{n-3}$) into inequality (9.3) and passing to the limit as $p \to \infty$, we get

$$\|v\|_{\mathcal{E}^{2m,\,N}_{-b+2m}(K)} \le k_6 \|\mathcal{M}(\theta) v\|_{\mathcal{E}^{0,\,N}_{-b+2m}(K,\ \gamma)}.$$

Therefore kernel of the operator $\mathcal{M}(\theta) : \mathcal{E}^{2m,\,N}_{-b+2m}(K) \to \mathcal{E}^{0,\,N}_{-b+2m}(K,\ \gamma)$ is trivial. By virtue of Lemma 8.1, $\dim\ker(\mathcal{L}(\theta)^*) = \dim\ker(\mathcal{M}(\theta)) = 0$. Combining this with 1), we see that the operator $\mathcal{L}(\theta) : E^{2m,\,N}_b(K) \to E^{0,\,N}_b(K,\ \gamma)$ is an isomorphism. Using Theorem 7.2, we prove the Theorem for arbitrary l and $a = b + l$. □

Remark 9.1. *From Theorems 9.1 and 9.3, it follows that the operator* $\mathcal{L} : H^{l+2m,\,N}_a(\Omega) \to H^{l,\,N}_a(\Omega,\ \Gamma)$ *is an isomorphism for all* l *and* $a = b+l$ *whenever* $\mathcal{L} : H^{l_1+2m,\,N}_{a_1}(\Omega) \to H^{l_1,\,N}_{a_1}(\Omega,\ \Gamma)$ *is Fredholm for some* l_1 *and* $a_1 = b + l_1$.

10 One–valued solvability of nonlocal problems for the Poisson equation in dihedral angles

As an application of the results obtained in this work we shall prove the one–valued solvability of nonlocal problems for the Poisson equation in dihedral angles. For this purpose we need to study corresponding auxiliary nonlocal problems in plane angles which is done by reducing them to boundary value problems for differential–difference equations (see [2, 17, 18]).

[7] One can prove that $\mathrm{Q} = \mathrm{P}^*$, but for our purposes, it is sufficient to prove the weaker result.

1 Difference operators in plane angles.

Put

$$K = \{y \in \mathbb{R}^2 : \ r > 0, \ b_1 < \varphi < b_{R+1}\},$$

$$K_t = \{y \in \mathbb{R}^2 : \ r > 0, \ b_t < \varphi < b_{t+1}\} \ (t = 1, \ \dots, \ R),$$

$$\gamma_q = \{y \in \mathbb{R}^2 : \ r > 0, \ \varphi = b_q\} \ (q = 1, \ \dots, \ R+1),$$

where $R \geq 1$ is an integer; $0 < b_1 < b_2 < \dots < b_R < b_{R+1} < 2\pi$; $b_2 - b_1 = \dots = b_{R+1} - b_R = d > 0$.

Consider the difference operator $\mathcal{R} : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ given by

$$(\mathcal{R}w)(y) = \sum_{p=-R+1}^{R-1} e_p \cdot w(r, \ \varphi + pd),$$

where $w(r, \ \varphi)$ is the function $w(y)$ written in the polar coordinates; $e_p \in \mathbb{R}$.

Let $I_K : L_2(K) \to L_2(\mathbb{R}^2)$ be the operator of extension by zero outside K; $P_K : L_2(\mathbb{R}^2) \to L_2(K)$ be the operator of restriction to K. Introduce the operator $\mathcal{R}_K : L_2(K) \to L_2(K)$ given by

$$\mathcal{R}_K = P_K \mathcal{R} I_K.$$

The following statement is obvious.

Lemma 10.1. *The operators $\mathcal{R} : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$, $\mathcal{R}_K : L_2(K) \to L_2(K)$ are bounded.*

$(\mathcal{R}^* w)(x) = \sum\limits_{p=-R+1}^{R-1} e_p \cdot w(r, \ \varphi - pd)$; $\mathcal{R}_K^* = P_K \mathcal{R}^* I_K$.

Introduce an isomorphism of the Hilbert spaces $\mathcal{U} : L_2(K) \to L_2^R(K_1)$ by the formula

$$(\mathcal{U}w)_t(y) = w(r, \ \varphi + b_t - b_1) \quad (y \in K_1; \ t = 1, \ \dots, \ R),$$

where $L_2^R(K_1) = \prod\limits_{t=1}^{R} L_2(K_1)$.

Denote by $\mathcal{R}_1$ the matrix of order $R \times R$ with the elements

$$r_{p_1 p_2} = e_{p_2 - p_1} \ (p_1, \ p_2 = 1, \ \dots, \ R).$$

Lemma 10.2. *The operator $\mathcal{U}\mathcal{R}_K\mathcal{U}^{-1} : L_2^R(K_1) \to L_2^R(K_1)$ is the operator of multiplication by the matrix $\mathcal{R}_1$.*

Lemma 10.3. *Spectrum of the operator $\mathcal{R}_K : L_2(K) \to L_2(K)$ coincides with spectrum of the matrix $\mathcal{R}_1$.*

Lemma 10.4. *The operator $\mathcal{R}_K + \mathcal{R}_K^* : L_2(K) \to L_2(K)$ is positive definite if and only if the matrix $\mathcal{R}_1 + \mathcal{R}_1^*$ is positive definite.*

Lemmas 10.2–10.4 are analogous to lemmas 8.6–8.8 [18, Chapter 2, §8].

Introduce the spaces $W^l(K)$ and $\mathring{W}^l(K)$ as a completion of the sets $C_0^\infty(\bar{K}\backslash\{0\})$ and $C_0^\infty(K)$ correspondingly in the norm $\left(\sum\limits_{|\alpha|\le l}\int\limits_K |D_y^\alpha w(y)|^2 dy\right)^{1/2}$. Similarly, we introduce the space $W^l(K_t)$.

Denote by w_t the restriction of a function w to K_t. Consider the spaces $\mathcal{W}^l(K) = \bigoplus\limits_{t=1}^{R} W^l(K_t)$ and $\mathcal{E}_a^l(K) = \bigoplus\limits_{t=1}^{R} E_a^l(K_t)$ with the norms $\|w\|_{\mathcal{W}^l(K)} = \left(\sum\limits_{t=1}^{R} \|w_t\|^2_{W^l(K_t)}\right)^{1/2}$ and $\|w\|_{\mathcal{E}_a^l(K)} = \left(\sum\limits_{t=1}^{R} \|w_t\|^2_{E_a^l(K_t)}\right)^{1/2}$ correspondingly.

Lemma 10.5. *The operator $\mathcal{R}_K$ maps continuously $\mathring{W}^l(K)$ into $W^l(K)$ and for all $w \in \mathring{W}^l(K)$,*

$$D^\alpha \mathcal{R}_K w = \mathcal{R}_K D^\alpha w \quad (|\alpha| \le l).$$

Lemma 10.5 is analogous to lemma 8.13 [18, Chapter 1, §8].

Lemma 10.6. *The operator $\mathcal{R}_K$ maps continuously $\mathcal{W}^l(K)$ into $\mathcal{W}^l(K)$ and $\mathcal{E}_a^l(K)$ into $\mathcal{E}_a^l(K)$.*

If $\det \mathcal{R}_1 \ne 0$, then the operator $\mathcal{R}_K^{-1}$ also maps continuously $\mathcal{W}^l(K)$ into $\mathcal{W}^l(K)$ and $\mathcal{E}_a^l(K)$ into $\mathcal{E}_a^l(K)$.

The proof follows from Lemmas 10.2, 10.3.

2 Differential–difference operators in plane angles.

Consider the differential–difference equation

$$\mathcal{P}_\mathcal{R} w = -\sum_{i,j=1}^{2} (\mathcal{R}_{ijK} w_{y_j})_{y_i} + \sum_{i=1}^{2} \mathcal{R}_{iK} w_{y_i} + \mathcal{R}_{0K} w = f(y) \quad (y \in K) \quad (10.1)$$

with the boundary conditions

$$w|_{\gamma_1} = w|_{\gamma_{R+1}} = 0, \quad (10.2)$$

where $\mathcal{R}_{ijK}=P_K\mathcal{R}_{ij}I_K$, $\mathcal{R}_{iK}=P_K\mathcal{R}_iI_K$, $\mathcal{R}_{0K}=P_K\mathcal{R}_0I_K$;

$$\mathcal{R}_{ij}w(y)=\sum_{p=-R+1}^{R-1}e_{ijp}\cdot w(r,\ \varphi+pd)\quad (i,\ j=1,\ 2);$$

$$\mathcal{R}_{i}w(y)=\sum_{p=-R+1}^{R-1}e_{ip}\cdot w(r,\ \varphi+pd)\quad (i=0,\ 1,\ 2);$$

$e_{ijp},\ e_{ip}\in\mathbb{R}$; $f\in L_2(K)$.

Denote by $(\cdot,\ \cdot)_K$ the inner product in $L_2(K)$.

Definition 10.1. *We shall say that differential–difference equation (10.1) is strongly elliptic in* $\bar{K}$ *if for all* $w\in C_0^\infty(\bar{K}\backslash\{0\})$,

$$\operatorname{Re}(\mathcal{P}_{\mathcal{R}}w,\ w)_K\geq c_1\|w\|^2_{W^1(K)}-c_2\|w\|^2_{L_2(K)},\tag{10.3}$$

where $c_1>0,\ c_2\geq 0$ *do not depend on* w.

Definition 10.2. *A function* $w\in\mathring{W}^1(K)$ *is called a generalized solution for problem (10.1), (10.2) if for all* $u\in\mathring{W}^1(K)$,

$$\sum_{i,\,j=1}^{2}(\mathcal{R}_{ijK}w_{y_j},\ u_{y_i})_K+\sum_{i=1}^{2}(\mathcal{R}_{iK}w_{y_i},\ u)_K+(\mathcal{R}_{0K}w,\ u)_K=(f,\ u)_K.$$

We define the unbounded operator $\mathbb{P}_{\mathcal{R}}:L_2(K)\to L_2(K)$ with domain $\operatorname{Dom}(\mathbb{P}_{\mathcal{R}})=\{w\in\mathring{W}^1(K):\ \mathcal{P}_{\mathcal{R}}w\in L_2(K)\}$ acting in the space of distributions $D'(K)$ by the formula

$$\mathbb{P}_{\mathcal{R}}w=-\sum_{i,\,j=1}^{2}(\mathcal{R}_{ijK}w_{y_j})_{y_i}+\sum_{i=1}^{2}\mathcal{R}_{iK}w_{y_i}+\mathcal{R}_{0K}w$$

The operator $\mathbb{P}_{\mathcal{R}}$ is called a *differential–difference operator.*

It is easy to show that Definition 10.2 is equivalent to the following one.

Definition 10.3. *A function* $w\in D(\mathbb{P}_{\mathcal{R}})$ *is called a generalized solution for problem (10.1), (10.2) if*

$$\mathbb{P}_{\mathcal{R}}w=f.$$

Denote by $\sigma(\mathbb{P}_{\mathcal{R}})$ spectrum of the operator $\mathbb{P}_{\mathcal{R}} : L_2(K) \to L_2(K)$.

Using the strong ellipticity of the operator $\mathbb{P}_{\mathcal{R}}$ and Lemmas 10.1, 10.2, 10.5, one can prove the following result (cf. theorem 10.1 [18, Chapter 2, §10]).

Theorem 10.1. *Suppose differential–difference equation (10.1) is strongly elliptic; then*

$$\sigma(\mathbb{P}_{\mathcal{R}}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -c_2\},$$

where $c_2 \geq 0$ is a constant in (10.3).

Example 10.1. Consider the equation

$$-\triangle \mathcal{R}_K w(y) + \mathcal{R}_K w(y) = f(y) \quad (y \in K = \{y \in \mathbb{R}^2 : \ r > 0, \ b_1 < \varphi < b_3\}) \tag{10.4}$$

with the boundary conditions

$$w|_{\gamma_1} = w|_{\gamma_3} = 0, \tag{10.5}$$

where $\mathcal{R}w(y) = w(r, \ \varphi) - \alpha w(r, \ \varphi + d) - \beta w(r, \ \varphi - d)$, $d = b_3 - b_2 = b_2 - b_1$; $\alpha, \ \beta \in \mathbb{R}$; $|\alpha + \beta| < 2$.

Clearly, the matrix $\mathcal{R}_1$ has the form

$$\mathcal{R}_1 = \begin{pmatrix} 1 & -\alpha \\ -\beta & 1 \end{pmatrix}.$$

Using Lemma 10.5, for all $w \in C_0^\infty(K \setminus \{0\})$, we get

$$\operatorname{Re}(-\triangle \mathcal{R}_K w + \mathcal{R}_K w, \ w)_K =$$
$$= \frac{1}{2}\sum_{i=1}^{2}((\mathcal{R}_K + \mathcal{R}_K^*)w_{y_i}, \ w_{y_i})_K + \frac{1}{2}((\mathcal{R}_K + \mathcal{R}_K^*)w, \ w)_K.$$

Since $|\alpha + \beta| < 2$, the matrix $\mathcal{R}_1 + \mathcal{R}_1^*$ is positive definite; therefore, by Lemma 10.4, the operator $\mathcal{R}_K + \mathcal{R}_K^*$ is also positive definite. From this and from the last equality, we obtain

$$\operatorname{Re}(-\triangle \mathcal{R}_K w + \mathcal{R}_K w, \ w)_K \geq c_1 \|w\|^2_{W^1(K)}.$$

Hence by Theorem 10.1, boundary value problem (10.4), (10.5) has a unique generalized solution $w \in \mathring{W}^1(K)$ for every $f \in L_2(K)$.

3 Nonlocal problems for the Poisson equation in dihedral angles.

Put

$$\Omega = \{x = (y,\ z):\ r > 0,\ b_1 < \varphi < b_3,\ z \in \mathbb{R}^{n-2}\},$$

$$\Omega_t = \{x = (y,\ z):\ r > 0,\ b_t < \varphi < b_{t+1},\ z \in \mathbb{R}^{n-2}\}\ (t = 1,\ 2),$$

$$\Gamma_q = \{x = (y,\ z):\ r > 0,\ \varphi = b_q,\ z \in \mathbb{R}^{n-2}\}\ (q = 1,\ \dots,\ 3),$$

where $b_2 - b_1 = b_3 - b_2 = d > 0$.

Consider the nonlocal boundary value problem

$$-\triangle U \equiv -\sum_{i=1}^{n} U_{x_i x_i}(x) = f(x) \quad (x \in \Omega), \tag{10.6}$$

$$\begin{array}{c} U|_{\Gamma_1} + \alpha U(r,\ \varphi + d,\ z)|_{\Gamma_1} = g_1(x) \quad (x \in \Gamma_1), \\ U|_{\Gamma_3} + \beta U(r,\ \varphi - d,\ z)|_{\Gamma_3} = g_3(x) \quad (x \in \Gamma_3). \end{array} \tag{10.7}$$

Here $U(r,\ \varphi,\ z)$ is the function $U(x)$ written in the cylindrical coordinates; $\alpha,\ \beta \in \mathbb{R}$; $|\alpha + \beta| < 2$.

For $n = 2$, we put $K = \{y:\ r > 0,\ b_1 < \varphi < b_3\}$, $K_t = \{y:\ r > 0,\ b_t < \varphi < b_{t+1}\}$, $\gamma_q = \{y:\ r > 0,\ \varphi = b_q\}$. Write the corresponding nonlocal problem in the plane angle K :

$$-\triangle u + u \equiv -\sum_{i=1}^{2} u_{y_i y_i}(y) + u(y) = f(y) \quad (y \in K), \tag{10.8}$$

$$\begin{array}{c} u|_{\gamma_1} + \alpha u(r,\ \varphi + d)|_{\gamma_1} = g_1(y) \quad (y \in \gamma_1), \\ u|_{\gamma_3} + \beta u(r,\ \varphi - d)|_{\gamma_3} = g_3(y) \quad (y \in \gamma_3). \end{array} \tag{10.9}$$

Clearly, the corresponding homogeneous problem with parameter λ has the form

$$-\tilde{U}_{\varphi\varphi} + \lambda^2 \tilde{U} = 0 \quad (\varphi \in (b_1,\ b_3)), \tag{10.10}$$

$$\begin{array}{c} \tilde{U}(\varphi)|_{\varphi=b_1} + \alpha \tilde{U}(\varphi + d)|_{\varphi=b_1} = 0, \\ \tilde{U}(\varphi)|_{\varphi=b_3} + \beta \tilde{U}(\varphi - d)|_{\varphi=b_3} = 0. \end{array} \tag{10.11}$$

One can easily find the eigenvalues of problem (10.10), (10.11). If $\alpha + \beta = 0$, then we have

$$\lambda_k = i \frac{\pi}{b_3 - b_1} k \quad (k = \pm 1,\ \pm 2,\ \dots).$$

If $0 < |\alpha+\beta| < 2$, then we have

$$\lambda_k = i\frac{2\pi}{b_3-b_1}k \quad (k=\pm1,\ \pm2,\ \dots),$$

$$\lambda_p = \begin{cases} i\dfrac{\pm2\arctan\dfrac{\sqrt{4-(\alpha+\beta)^2}}{\alpha+\beta}}{b_3-b_1} + i\dfrac{4\pi p}{b_3-b_1} & \text{for } -2<\alpha+\beta<0, \\ i\dfrac{2\pi\pm2\arctan\dfrac{\sqrt{4-(\alpha+\beta)^2}}{\alpha+\beta}}{b_3-b_1} + i\dfrac{4\pi p}{b_3-b_1} & \text{for } 0<\alpha+\beta<2 \end{cases}$$
$$(p=0,\ \pm1,\ \pm2,\ \dots).$$

Obviously, the line $\operatorname{Im}\lambda = 0$ contains no eigenvalues of problem (10.10), (10.11). Therefore by Theorem 9.1, the operator

$$\big(-\triangle u+u,\ u|_{\gamma_1}+\alpha u(r,\ \varphi+d)|_{\gamma_1},\ u|_{\gamma_3}+\beta u(r,\ \varphi-d)|_{\gamma_3}\big): E_1^2(K)\to \\ \to E_1^0(K)\times\prod_{\sigma=1,3}E_1^{3/2}(\gamma_\sigma) \tag{10.12}$$

is Fredholm. Let us show that operator (10.12) has trivial kernel.

Suppose $u\in E_1^2(K)$ is a solution for homogeneous problem (10.8), (10.9). Introduce the difference operator $\mathcal{R}_K = P_K\mathcal{R}I_K$, where

$$\mathcal{R}w(y) = w(r,\ \varphi) - \alpha w(r,\ \varphi+d) - \beta w(r,\ \varphi-d).$$

Put $u=\mathcal{R}_K w$. Since $|\alpha+\beta|<2$, the matrix $\mathcal{R}_1 = \begin{pmatrix} 1 & -\alpha \\ -\beta & 1\end{pmatrix}$ corresponding to the difference operator $\mathcal{R}_K$ is non-singular and

$$\mathcal{R}_1^{-1} = \frac{1}{1-\alpha\beta}\begin{pmatrix} 1 & \alpha \\ \beta & 1\end{pmatrix}.$$

Therefore, by Lemma 10.3, the operator $\mathcal{R}_K$ has the bounded inverse $\mathcal{R}_K^{-1}$ and $w=\mathcal{R}_K^{-1}u$.

Now we shall show that $w\in\mathcal{E}_1^2(K)\cap E_1^1(K)$ and $w|_{\gamma_1}=w|_{\gamma_3}=0$. Indeed, by Lemma 10.6, $w\in\mathcal{E}_1^2(K)$. Further, using the isomorphism $\mathcal{U}$, the matrix

$\mathcal{R}_1^{-1}$, and Lemma 10.2, we get

$$\begin{aligned} w_1|_{\gamma_2} = [\mathcal{U}w]_1(r,\ b_2) = \frac{1}{1-\alpha\beta}([\mathcal{U}u]_1(r,\ b_2) + \alpha[\mathcal{U}u]_2(r,\ b_2)) = \\ = \frac{1}{1-\alpha\beta}(u(r,\ b_2) + \alpha u(r,\ b_3)), \\ w_2|_{\gamma_2} = [\mathcal{U}w]_2(r,\ b_1) = \frac{1}{1-\alpha\beta}(\beta[\mathcal{U}u]_1(r,\ b_1) + [\mathcal{U}u]_2(r,\ b_1)) = \\ = \frac{1}{1-\alpha\beta}(\beta u(r,\ b_1) + u(r,\ b_2)). \end{aligned} \tag{10.13}$$

But the function u satisfies homogeneous conditions (10.9) and therefore $\alpha u(r,\ b_3) = \beta u(r,\ b_1)$. Combining this with (10.13), we see that $w_1|_{\gamma_2} = w_2|_{\gamma_2}$, i.e., $w \in E_1^1(K)$.

Similarly,

$$\begin{aligned} w_1|_{\gamma_1} = [\mathcal{U}w]_1(r,\ b_1) = \frac{1}{1-\alpha\beta}([\mathcal{U}u]_1(r,\ b_1) + \alpha[\mathcal{U}u]_2(r,\ b_1)) = \\ = \frac{1}{1-\alpha\beta}(u(r,\ b_1) + \alpha u(r,\ b_2)) = 0, \end{aligned}$$

$$\begin{aligned} w_2|_{\gamma_3} = [\mathcal{U}w]_2(r,\ b_2) = \frac{1}{1-\alpha\beta}(\beta[\mathcal{U}u]_1(r,\ b_2) + [\mathcal{U}u]_2(r,\ b_2)) = \\ = \frac{1}{1-\alpha\beta}(\beta u(r,\ b_2) + u(r,\ b_3)) = 0, \end{aligned}$$

since the function u satisfies homogeneous conditions (10.9).

Therefore from the imbedding $\mathcal{E}_1^2(K) \cap E_1^1(K) \subset W^1(K)$, it follows that $w \in \mathring{W}^1(K)$ and w is a generalized solution to boundary value problem (10.4), (10.5) for $f = 0$. In Example 10.1, it is shown that $w = 0$ which implies $u = \mathcal{R}_K w = 0$.

In order to prove that range of operator (10.12) coincides with $E_1^0(K) \times \prod\limits_{\sigma=1,\,3} E_1^{3/2}(\gamma_\sigma)$, we study the problems that are formally adjoint to problems (10.6), (10.7) and (10.8), (10.9) with respect to the Green formulas. Similarly to Example 4.1, we obtain the following nonlocal transmission problems:

$$-\triangle V_t + V_t = f(x) \quad (x \in \Omega_t;\ t = 1,\ 2) \tag{10.14}$$

$$\begin{aligned} V_1|_{\Gamma_1} = g_1(x) \quad (x \in \Gamma_1), \\ V_2|_{\Gamma_3} = g_3(x) \quad (x \in \Gamma_3), \end{aligned} \tag{10.15}$$

$$
\begin{gathered}
V_1|_{\Gamma_2} - V_2|_{\Gamma_2} = h_{21}(x) \quad (x \in \Gamma_2), \\
\frac{\partial V_1}{\partial n_2}\Big|_{\Gamma_2} - \frac{\partial V_2}{\partial n_2}\Big|_{\Gamma_2} + \alpha \frac{\partial V_1}{\partial n_1}(r,\ \varphi - d,\ z)\Big|_{\Gamma_2} + \beta \frac{\partial V_2}{\partial n_3}(r,\ \varphi + d,\ z)\Big|_{\Gamma_2} = \\
= h_{22}(x) \quad (x \in \Gamma_2)
\end{gathered}
\tag{10.16}
$$

and

$$
-\triangle v_t + v_t = f(y) \quad (y \in K_t;\ t = 1,\ 2) \tag{10.17}
$$

$$
\begin{gathered}
v_1|_{\gamma_1} = g_1(y) \quad (y \in \gamma_1), \\
v_2|_{\gamma_3} = g_3(y) \quad (y \in \gamma_3),
\end{gathered}
\tag{10.18}
$$

$$
\begin{gathered}
v_1|_{\gamma_2} - v_2|_{\gamma_2} = h_{21}(y) \quad (y \in \gamma_2), \\
\frac{\partial v_1}{\partial n_2}\Big|_{\gamma_2} - \frac{\partial v_2}{\partial n_2}\Big|_{\gamma_2} + \alpha \frac{\partial v_1}{\partial n_1}(r,\ \varphi - d)\Big|_{\gamma_2} + \beta \frac{\partial v_2}{\partial n_3}(r,\ \varphi + d)\Big|_{\gamma_2} = \\
= h_{22}(y) \quad (y \in \gamma_2).
\end{gathered}
\tag{10.19}
$$

Here n_1 is the unit normal vector to Γ_1 (γ_1) direct inside Ω_1 (K_1); n_2 and n_3 are the unit normal vectors to Γ_2 (γ_2) Γ_3 (γ_3) correspondingly directed inside Ω_2 (K_2). As we have notices in the proof of Theorem 9.1, λ_0 is an eigenvalue of problem (10.10), (10.11) iff $\lambda_0' = \bar{\lambda}_0$ is an eigenvalue of nonlocal transmission problem with parameter λ corresponding to problem (10.14)–(10.16) (which can be written in the obvious way). Hence this problem also has no eigenvalues on the line $\operatorname{Im} \lambda = 0$. Then by Theorem 7.3, the operator

$$
\begin{gathered}
\big(-v_\triangle + v,\ v_1|_{\gamma_1},\ v_2|_{\gamma_3},\ v_1|_{\gamma_2} - v_2|_{\gamma_2},\ \frac{\partial v_1}{\partial n_2}\Big|_{\gamma_2} - \frac{\partial v_2}{\partial n_2}\Big|_{\gamma_2} + \\
+ \alpha \frac{\partial v_1}{\partial n_1}(r,\ \varphi - d)\Big|_{\gamma_2} + \beta \frac{\partial v_2}{\partial n_3}(r,\ \varphi + d)\Big|_{\gamma_2}\big) : \\
\mathcal{E}_1^2(K) \to \mathcal{E}_1^0(K) \times \prod_{\sigma=1,\,3} E_1^{3/2}(\gamma_\sigma) \times \prod_{\nu=1}^{2} E_1^{2-\nu+1/2}(\gamma_2)
\end{gathered}
\tag{10.20}
$$

has finite dimensional kernel. Here $v_\triangle(y) = \triangle v_t(y)$ for $y \in K_t$, $t = 1,\ 2$. Let us show that kernel of operator (10.20) is trivial.

Suppose $v \in \mathcal{E}_1^2(K)$ is a solution for homogeneous problem (10.17)–(10.19). Consider the adjoint difference operator $\mathcal{R}_K^*$. The matrix $\mathcal{R}_1^* = \begin{pmatrix} 1 & -\beta \\ -\alpha & 1 \end{pmatrix}$ corresponds to the difference operator $\mathcal{R}_K^*$. Since $|\alpha + \beta| < 2$, the matrix $\mathcal{R}_1^*$ is non-singular and by Lemma 10.3, there exists the inverse operator $(\mathcal{R}_K^*)^{-1}$. Put $v = (\mathcal{R}_K^*)^{-1} w$; hence $w = \mathcal{R}_K^* v$.

Let us show that $w \in E_1^2(K)$ and $w|_{\gamma_1} + \beta w(r,\ \varphi + d)|_{\gamma_1} = 0$, $w|_{\gamma_3} + \alpha w(r,\ \varphi - d)|_{\gamma_3} = 0$. Indeed, by Lemma 10.6, $w \in \mathcal{E}_1^2(K)$. Further using the isomorphism $\mathcal{U}$, the matrix $\mathcal{R}_1^*$, and Lemma 10.2, we get

$$\begin{gathered}
w_1|_{\gamma_2} = [\mathcal{U}w]_1(r,\ b_2) = [\mathcal{U}v]_1(r,\ b_2) - \beta[\mathcal{U}v]_2(r,\ b_2) = \\
= v_1(r,\ b_2) - \beta v_2(r,\ b_3) = v_1(r,\ b_2), \\
w_2|_{\gamma_2} = [\mathcal{U}w]_2(r,\ b_1) = -\alpha[\mathcal{U}v]_1(r,\ b_1) + [\mathcal{U}v]_2(r,\ b_1) = \\
= -\alpha v_1(r,\ b_1) + v_2(r,\ b_2) = v_2(r,\ b_2),
\end{gathered} \tag{10.21}$$

since v satisfies homogeneous conditions (10.18). From (10.21) and homogeneous conditions (10.19), we get $w_1|_{\gamma_2} = w_2|_{\gamma_2}$.

Similarly,

$$\begin{gathered}
\frac{\partial w_1}{\partial \varphi}\Big|_{\gamma_2} = \frac{\partial v_1}{\partial \varphi}(r,\ b_2) - \beta\frac{\partial v_2}{\partial \varphi}(r,\ b_3), \\
\frac{\partial w_2}{\partial \varphi}\Big|_{\gamma_2} = -\alpha\frac{\partial v_1}{\partial \varphi}(r,\ b_1) + \frac{\partial v_2}{\partial \varphi}(r,\ b_2).
\end{gathered} \tag{10.22}$$

Taking into account that $\frac{\partial}{\partial n_i} = \frac{1}{r}\frac{\partial}{\partial \varphi}$ $(i = 1,\ 2)$ and $\frac{\partial}{\partial n_3} = -\frac{1}{r}\frac{\partial}{\partial \varphi}$, from (10.22) and homogeneous conditions (10.19), we obtain $\frac{\partial w_1}{\partial n_2}\Big|_{\gamma_2} = \frac{\partial w_2}{\partial n_2}\Big|_{\gamma_2}$. Therefore, $w \in E_1^2(K)$. Analogously one can show that $w|_{\gamma_1} + \beta w(r,\ \varphi + d)|_{\gamma_1} = 0$, $w|_{\gamma_3} + \alpha w(r,\ \varphi - d)|_{\gamma_3} = 0$.

This means that $w \in E_1^2(K)$ is a solution for the problem

$$-\triangle w + w = 0 \quad (y \in K), \tag{10.23}$$

$$\begin{gathered}
w|_{\gamma_1} + \beta w(r,\ \varphi + d)|_{\gamma_1} = 0 \quad (y \in \gamma_1), \\
w|_{\gamma_3} + \alpha w(r,\ \varphi - d)|_{\gamma_3} = 0 \quad (y \in \gamma_3).
\end{gathered} \tag{10.24}$$

But problem (10.23), (10.24) is a nonlocal boundary value problem of type (10.8), (10.9) (one must replace α by β and β by α). Hence, by the above, $w = 0$ if $|\alpha + \beta| < 2$. This implies $v = \bar{\mathcal{R}}_K w = 0$.

From Lemma 8.1, it follows that dimension of cokernal of operator (10.12) is equal to dimension of kernel of operator (10.20). Therefore cokernal of operator (10.12) is trivial. Finally applying Theorem 9.2, we obtain that

nonlocal boundary value problem (10.6), (10.7) has a unique solution $U \in H_{1+l}^{l+2}(\Omega)$ *for every right-hand side* $\{f,\ g_1,\ g_3\} \in H_{1+l}^{l}(K) \times \prod_{\sigma=1,3} H_{1+l}^{l+3/2}(\Gamma_\sigma)$.

A A priori estimates for the operator L^* in $\mathbb{R}^n$

1 Some approaches for ordinary differential equations.

Let $\mathcal{P}(\xi',\ -i\dfrac{d}{dx_n})$ and $B_\nu(\xi',\ -i\dfrac{d}{dx_n})$ ($\nu = 1,\ \ldots,\ J;\ J \geq 1$) be differential operators with constant coefficients and parameter $\xi' = (\xi_1,\ \ldots,\ \xi_{n-1}) \in \mathbb{R}^{n-1}$ such that after replacing $-i\dfrac{d}{dx_n}$ by ξ_n, we get polynomials of orders $2m$ and $m_\nu \leq 2m-1$ that are homogeneous with respect to $(\xi',\ \xi_n)$ correspondingly.

Let the following condition hold.

Condition A.1. $\mathcal{P}(\xi',\ \xi_n) \neq 0$ *for all* $(\xi',\ \xi_n) \neq 0$.

Consider the bounded operator $L_{\xi'} : W^{2m}(\mathbb{R}) \to L_2(\mathbb{R}) \times \mathbb{C}^J$ given by

$$L_{\xi'}u = (\mathcal{P}(\xi',\ -i\frac{d}{dx_n})u,\ B_1(\xi',\ -i\frac{d}{dx_n})u|_{x_n=0},\ \ldots,\ B_J(\xi',\ -i\frac{d}{dx_n})u|_{x_n=0}).$$

Introduce the adjoint operator $L^*_{\xi'} : L_2(\mathbb{R}) \times \mathbb{C}^J \to W^{-2m}(\mathbb{R})$ that takes $\Psi = (\psi,\ d_1,\ \ldots,\ d_J) \in L_2(\mathbb{R}) \times \mathbb{C}^J$ to $L^*_{\xi'}\Psi$ by the rule

$$< u,\ L^*_{\xi'}\Psi > = < \mathcal{P}(\xi',\ -i\frac{d}{dx_n})u,\ \psi > + \sum_{\nu=1}^{J} < B_\nu(\xi',\ -i\frac{d}{dx_n})u|_{x_n=0},\ d_\nu >$$

for all $u \in W^{2m}(\mathbb{R})$.

Lemma A.1. *Suppose $n \geq 2$; then for all $\xi' \in \mathbb{R}^{n-1}$, $\xi' \neq 0$, the operator $L_{\xi'}$ is Fredholm, its kernel is trivial.*

Proof. Since $n \geq 2$, condition A.1 implies that

$$k_1(1 + |\xi_n|^2)^{2m} \leq |\mathcal{P}(\xi',\ \xi_n)|^2 \leq k_2(1 + |\xi_n|^2)^{2m} \text{ for } \xi' \neq 0. \qquad \text{(A.1)}$$

Here $k_1,\ k_2$ depend on ξ' and do not depend on ξ_n. Multiplying the first inequality in (A.1) by $|\tilde{u}(\xi_n)|^2$ ($\tilde{u}$ is the Fourier transform of the function u with respect to x_n) and integrating over $\mathbb{R}$, we obtain

$$\|u\|_{W^{2m}(\mathbb{R})} \leq k_3\|\mathcal{P}(\xi',\ \xi_n)u\|_{L_2(\mathbb{R})},$$

where $k_3 > 0$ depend only on ξ' and do not depend on u. The last inequality implies that the operator $L_{\xi'}$ has trivial kernel and closed range.

Let us show that cokernel of the operator $L_{\xi'}$ is of finite dimension. Using the Fourier transform and inequality (A.1), one can easily check that for $n \geq 2$, $\xi' \neq 0$, the operator $\mathcal{P}(\xi',\ -i\dfrac{d}{dx_n})$ maps $W^{2m}(\mathbb{R})$ onto $L_2(\mathbb{R})$. Therefore the operator $L_{\xi'}$ maps $W^{2m}(\mathbb{R})$ onto $L_2(\mathbb{R})\times\mathbb{M}^J$, where $\mathbb{M}^J$ is a closed (since range of $L_{\xi'}$ is closed) subspace of $\mathbb{C}^J$. But $\mathbb{C}^J$ is a finite dimensional space; hence cokernel of the operator $L_{\xi'}$ is also finite dimensional. □

Lemma A.2. *Suppose $n \geq 2$; then for all $\xi' \in \mathbb{R}^{n-1}$, $\xi' \neq 0$, we have*

I) the operator $L^*_{\xi'}$ is Fredholm, its range coincides with $W^{-2m}(\mathbb{R})$;

II) for all $\Psi = (\psi,\ d_1,\ \ldots,\ d_J) \in L_2(\mathbb{R}) \times \mathbb{C}^J$, the following estimate holds:

$$\|\psi\|_{L_2(\mathbb{R})} \leq c_{\xi'}\big(\|L^*_{\xi'}\Psi\|_{W^{-2m}(\mathbb{R})} + \sum_{\nu=1}^{J} |d_\nu|\big), \tag{A.2}$$

where $c_{\xi'} > 0$ depends on ξ' and does not depend on Ψ;

III) if $\xi' \in \mathbb{K} \subset \mathbb{R}^{n-1}$, where $\mathbb{K}$ is a compactum such that $\mathbb{K} \cap \{0\} = \varnothing$, then inequality (A.2) holds with a constant c that does not depend on ξ'.

Proof. I) follows from Lemma A.1. Let us prove II). Denote by $\ker(L^*_{\xi'})$ kernel of the operator $L^*_{\xi'}$. Since $L^*_{\xi'}$ is Fredholm, $\ker(L^*_{\xi'})$ is of finite dimension.

Let us show that in the space $\ker(L^*_{\xi'})$, we can introduce the norm

$$\|\hat\Psi\|_{\ker(L^*_{\xi'})} = \left(\sum_{\nu=1}^{J} |\hat d_\nu|^2\right)^{1/2},\ \hat\Psi = (\hat\psi,\ \hat d_1,\ \ldots,\ \hat d_J) \in \ker(L^*_{\xi'}) \subset L_2(\mathbb{R})\times\mathbb{C}^J,$$

which is equivalent to the standart norm in $L_2(\mathbb{R}) \times \mathbb{C}^J$. Among all of the properties of a norm, the following one is not obvious: $\hat\Psi = 0$ whenever $\|\hat\Psi\|_{\ker(L^*_{\xi'})} = 0$. Check it. Suppose $\|\hat\Psi\|_{\ker(L^*_{\xi'})} = 0$; then $\hat\Psi = (\hat\psi,\ 0,\ \ldots,\ 0)$. Since $\hat\Psi \in \ker(L^*_{\xi'})$, it follows from the definition of the operator $L^*_{\xi'}$ that

$$< \mathcal{P}(\xi',\ -i\frac{d}{dx_n})u,\ \hat\psi >= 0 \tag{A.3}$$

for all $u \in W^{2m}(\mathbb{R})$.

As we have already mentioned in the proof of Lemma A.1, the operator $\mathcal{P}(\xi', -i\frac{d}{dx_n})$ maps $W^{2m}(\mathbb{R})$ onto $L_2(\mathbb{R})$ if $n \geq 2$, $\xi' \neq 0$. From this and (A.3), it follows that $\hat{\psi} = 0$; hence, $\hat{\Psi} = 0$.

Now we get that the norm $\|\cdot\|_{\ker (L^*_{\xi'})}$ is equivalent to the norm $\|\cdot\|_{L_2(\mathbb{R})\times\mathbb{C}^J}$, since the space $\ker (L^*_{\xi'})$ is of finite dimension.

The operator $L^*_{\xi'}$ is closed and range of $L^*_{\xi'}$ is closed; hence from theorem 2.3 [15, §2], it follows that for any $\Psi = (\psi, d_1, \ldots, d_J) \in L_2(\mathbb{R}) \times \mathbb{C}^J$, there exists a $\Phi \in L_2(\mathbb{R}) \times \mathbb{C}^J$ such that $L^*_{\xi'}\Psi = L^*_{\xi'}\Phi$ and

$$\|\Phi\|_{L_2(\mathbb{R})\times\mathbb{C}^J} \leq k_1 \|L^*_{\xi'}\Psi\|_{W^{-2m}(\mathbb{R})},$$

where $k_1 > 0$ depends on ξ' and does not depend on Φ and Ψ. But $\Psi = \Phi + \hat{\Psi}$, where $\hat{\Psi} = (\hat{\psi}, \hat{d}_1, \ldots, \hat{d}_J) \in \ker (L^*_{\xi'})$; therefore,

$$\|\Psi\|_{L_2(\mathbb{R})\times\mathbb{C}^J} \leq k_1 \|L^*_{\xi'}\Psi\|_{W^{-2m}(\mathbb{R})} + \|\hat{\Psi}\|_{L_2(\mathbb{R})\times\mathbb{C}^J}.$$

By proved, the norms $\|\cdot\|_{\ker (L^*_{\xi'})}$ and $\|\cdot\|_{L_2(\mathbb{R})\times\mathbb{C}^J}$ are equivalent; this implies

$$\begin{aligned} \|\psi\|_{L_2(\mathbb{R})} \leq \|\Psi\|_{L_2(\mathbb{R})\times\mathbb{C}^J} \leq k_1 \|L^*_{\xi'}\Psi\|_{W^{-2m}(\mathbb{R})} + k_2 \sum_{\nu=1}^{J} |\hat{d}_\nu| \leq \\ \leq k_1 \|L^*_{\xi'}\Psi\|_{W^{-2m}(\mathbb{R})} + k_2 \sum_{\nu=1}^{J} |d_\nu| + k_2 \|\Phi\|_{L_2(\mathbb{R})\times\mathbb{C}^J} \leq \\ \leq k_1 \|L^*_{\xi'}\Psi\|_{W^{-2m}(\mathbb{R})} + k_2 \sum_{\nu=1}^{J} |d_\nu| + k_1 k_2 \|L^*_{\xi'}\Psi\|_{W^{-2m}(\mathbb{R})} \leq \\ \leq c_{\xi'} (\|L^*_{\xi'}\Psi\|_{W^{-2m}(\mathbb{R})} + \sum_{\nu=1}^{J} |d_\nu|), \end{aligned}$$

where $c_{\xi'} = \max(k_1 + k_1 k_2, k_2)$.

Let us prove III). Suppose III) does not hold; then there exist sequences $\{(\xi')^k\} \subset \mathbb{K}$, $\{\Psi_k\} = \{(\psi_k, d_1^k, \ldots, d_J^k)\}$, $k = 1, 2, \ldots$, such that $\|\psi_k\|_{L_2(\mathbb{R})} = 1$,

$$\|L^*_{(\xi')^k}\Psi_k\|_{W^{-2m}(\mathbb{R})} + \sum_{\nu=1}^{J} |d_\nu^k| \to 0 \text{ as } k \to \infty. \tag{A.4}$$

Choose from $\{(\xi')^k\}$ a subsequence (we shall denote it $\{(\xi')^k\}$ too) that converges to a $(\xi')^0 \in \mathbb{K}$. By assumption, $(\xi')^0 \neq 0$; therefore by proved, estimate (A.2) holds for $\xi' = (\xi')^0$.

Notice that

$$\|L^*_{(\xi')^0}\Psi_k\|_{W^{-2m}(\mathbb{R})} \le \|L^*_{(\xi')^k}\Psi_k\|_{W^{-2m}(\mathbb{R})}+$$
$$+\|L^*_{(\xi')^k} - L^*_{(\xi')^0}\|_{L_2(\mathbb{R})\times\mathbb{C}^J\to W^{-2m}(\mathbb{R})} \cdot \|\Psi_k\|_{L_2(\mathbb{R})\times\mathbb{C}^J}.$$

From (A.4), it follows that $\|L^*_{(\xi')^k}\Psi_k\|_{W^{-2m}(\mathbb{R})} \to 0$. Further, $\|L^*_{(\xi')^k} - L^*_{(\xi')^0}\|_{L_2(\mathbb{R})\times\mathbb{C}^J\to W^{-2m}(\mathbb{R})} \to 0$, since $L_{\xi'}$ depends on ξ' polynomially. Finally, $\|\Psi_k\|_{L_2(\mathbb{R})\times\mathbb{C}^J}$ is uniformly bounded by a constant not depending on k which follows from (A.4) and relation $\|\psi_k\|_{L_2(\mathbb{R})} = 1$. Hence, $\|L^*_{(\xi')^0}\Psi_k\|_{W^{-2m}(\mathbb{R})} \to 0$ as $k \to \infty$. Combining this with (A.4), we obtain

$$\|L^*_{(\xi')^0}\Psi_k\|_{W^{-2m}(\mathbb{R})} + \sum_{\nu=1}^{J} |d^k_\nu| \to 0 \text{ as } k \to \infty. \tag{A.5}$$

Now applying estimate (A.2) to the sequence $\{\Psi_k\}$ and $\xi' = (\xi')^0$, from (A.5), we eventually get

$$\|\psi_k\|_{L_2(\mathbb{R})} \to 0 \text{ as } k \to \infty.$$

This contradicts the assumption $\|\psi_k\|_{L_2(\mathbb{R})} = 1$. □

2 A priori estimates in $\mathbb{R}^n$.

Write a point $x \in \mathbb{R}^n$ $(n \ge 2)$ in the form $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. Similarly, write a point $\xi \in \mathbb{R}^n$ $(n \ge 2)$ in the form $\xi = (\xi', \xi_n)$, where $\xi' = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, $\xi_n \in \mathbb{R}$.

Let $\mathcal{P}(D) = \mathcal{P}(D_{x'}, D_{x_n})$, $B_\nu(D) = B_\nu(D_{x'}, D_{x_n})$ $(\nu = 1, \dots, J;\ J \ge 1)$ be differential operators with constant coefficients such that after replacing $D = (D_{x'}, D_{x_n})$ by $\xi = (\xi', \xi_n)$, we get polynomials $\mathcal{P}(\xi) = \mathcal{P}(\xi', \xi_n)$, $B_\nu(\xi) = B_\nu(\xi', \xi_n)$ of orders $2m$ and $m_\nu \le 2m-1$ correspondingly that are homogeneous with respect to $\xi = (\xi', \xi_n)$. We shall suppose that the polynomial $\mathcal{P}(\xi)$ satisfies condition A.1.

Consider the bounded operator

$$L : W^{2m}(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \times \prod_{\nu=1}^{J} W^{2m-m_\nu-1/2}(\mathbb{R}^{n-1})$$

given by

$$LU = (\mathcal{P}(D)U,\ B_1(D)U|_{x_n=0},\ \dots,\ B_J(D)U|_{x_n=0}).$$

Notice that the problem corresponding to the operator L is quite artificial. This is not a boundary value problem, since a solution U is considered in $\mathbb{R}^n$. And this is not a transmission problem, since we impose the trace conditions on the hyperplane $\{x_n = 0\}$, but not transmission conditions. Moreover the operators $B_J(D)$ do not cover the operator $\mathcal{P}(D)$ on the hyperplane $\{x_n = 0\}$. Nevertheless we need this problem for getting a priori estimates of solutions to adjoint nonlocal problems (§8). This is explained by the specific character of our method, which may be called “separation of nonlocality”.

Introduce the adjoint operator $L^* : L_2(\mathbb{R}^n) \times \prod_{\nu=1}^{J} W^{-2m+m_\nu+1/2}(\mathbb{R}^{n-1}) \to W^{-2m}(\mathbb{R}^n)$. The operator L^* takes $F = (f_0,\ g_1,\ \dots,\ g_J) \in L_2(\mathbb{R}^n) \times \prod_{\nu=1}^{J} W^{-2m+m_\nu+1/2}(\mathbb{R}^{n-1})$ to L^*F by the rule

$$< U,\ L^*F > = < \mathcal{P}(D)U,\ f_0 > + \sum_{\nu=1}^{J} < B_\nu(D)U|_{x_n=0},\ g_\nu >$$
$$\text{for all } U \in W^{2m}(\mathbb{R}^n).$$

Denote $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n :\ \ x_n > 0\}$, $\mathbb{R}^n_- = \{x \in \mathbb{R}^n :\ \ x_n < 0\}$. Consider the space $\mathcal{W}^l(\mathbb{R}^n) = W^l(\mathbb{R}^n_+) \oplus W^l(\mathbb{R}^n_-)$ with the norm $\|U\|_{\mathcal{W}^l(\mathbb{R}^n)} = \Big(\|U_+\|^2_{W^l(\mathbb{R}^n_+)} + \|U_-\|^2_{W^l(\mathbb{R}^n_-)}\Big)^{1/2}$.

Theorem A.1. *Suppose*

$$F = (f_0,\ g_1,\ \dots,\ g_J) \in L_2(\mathbb{R}^n) \times \prod_{\nu=1}^{J} W^{-2m+l+m_\nu+1/2}(\mathbb{R}^{n-1}),$$

$$L^*F \in \begin{cases} W^{-2m+l}(\mathbb{R}^n) \text{ for } l < 2m, \\ \mathcal{W}^{-2m+l}(\mathbb{R}^n) \text{ for } l \geq 2m; \end{cases}$$

then $f_0 \in \mathcal{W}^l(\mathbb{R}^n)$ *and*

$$\|f_0\|_{\mathcal{W}^l(\mathbb{R}^n)} \leq c_l\big(\|L^*F\|_{-2m+l} + \|f_0\|_{W^{-1}(\mathbb{R}^n)} + \sum_{\nu=1}^{J} \|g_\nu\|_{W^{-2m+l+m_\nu+1/2}(\mathbb{R}^{n-1})}\big), \tag{A.6}$$

where $\|\cdot\|_{-2m+l} = \begin{cases} \|\cdot\|_{W^{-2m+l}(\mathbb{R}^n)} \text{ for } l < 2m, \\ \|\cdot\|_{\mathcal{W}^{-2m+l}(\mathbb{R}^n)} \text{ for } l \geq 2m, \end{cases}$ $c_l > 0$ *depends on* $l \geq 0$ *and does not depend on* F.

Proof. Suppose $l = 0$. Then using Fourier transform of the functions f_0, g_ν and L^*F with respect to x' we derive estimate (A.6) from Lemma A.2 (in the same way as estimate (4.27) [12, Chapter 2, §4.4] follows from (4.18) [12, Chapter 2, §4.2], see the proof of theorem 4.1 [12, Chapter 2, §4.4]).

If $l \geq 1$, then we prove that $f_0 \in \mathcal{W}^l(\mathbb{R}^n)$ and obtain estimate (A.6) using (A.6) for $l = 0$, the finite difference method, and condition A.1 (in the same way as estimate (4.40) [12, Chapter 2, §4.5] is derived from (4.40′) [12, Chapter 2, §4.5], see the proof of theorem 4.3 [12, Chapter 2, §4.5]). □

Remark A.1. *Unlike model problems in* $\mathbb{R}^n$ *(see [12, Chapter 2, §3]), our operator* L^* *contains distributions with support on the hyperplane* $\{x_n = 0\}$. *That is why smoothness of the function* f_0 *can be violated on the hyperplane* $\{x_n = 0\}$ *even if* L^*F *is infinitely smooth in* $\mathbb{R}^n$. *Moreover, Theorem A.1 shows that if we want the function* f_0 *to be more smooth in* $\mathbb{R}^n_+$ *and* $\mathbb{R}^n_-$, *then we must consider more smooth function* L^*F *and* **more smooth distributions g_ν as well**.

B Some properties of weighted spaces

Introduce the space $H_a^l(\Omega)$ as a completion of the set $C_0^\infty(\bar{\Omega}\backslash M)$ in the norm

$$\|U\|_{H_a^l(\Omega)} = \left(\sum_{|\alpha|\leq l} \int\limits_{\Omega} r^{2(a+|\alpha|-l)} |D_x^\alpha U(x)|^2 dx\right)^{1/2},$$

where $\Omega = \{x = (y,\ z) :\ \ r > 0,\ 0 < b_1 < \varphi < b_2 < 2\pi,\ z \in \mathbb{R}^{n-2}\}$, $M = \{x = (y,\ z) :\ \ y = 0,\ z \in \mathbb{R}^{n-2}\}$. Denote by $H_a^{l-1/2}(\Gamma)$ $(l \geq 1)$ the space of traces on the $(n-1)$-dimensional half-plane $\Gamma = \{x = (y,\ z) :\ r > 0,\ \varphi = b,\ z \in \mathbb{R}^{n-2}\}$ $(b_1 \leq b \leq b_2)$ with the norm

$$\|\Psi\|_{H_a^{l-1/2}(\Gamma)} = \inf \|U\|_{H_a^l(\Omega)} \quad (U \in H_a^l(\Omega) :\ U|_\Gamma = \Psi).$$

Introduce the space $E_a^l(K)$ as a completion of the set $C_0^\infty(\bar{K}\backslash\{0\})$ in the norm

$$\|u\|_{E_a^l(K)} = \left(\sum_{|\alpha|\leq l} \int\limits_{K} r^{2a}(r^{2(|\alpha|-l)} + 1)|D_y^\alpha u(y)|^2 dy\right)^{1/2},$$

where $K = \{y \in \mathbb{R}^2 :\ r > 0,\ 0 < b_1 < \varphi < b_2 < 2\pi\}$. By $E_a^{l-1/2}(\gamma)$ $(l \geq 1)$ we denote the space of traces on the ray $\gamma = \{y :\ r > 0,\ \varphi = b\}$ $(b_1 \leq b \leq b_2)$ with the norm

$$\|\psi\|_{E_a^{l-1/2}(\gamma)} = \inf \|u\|_{E_a^l(K)} \quad (u \in E_a^l(K) :\ u|_\gamma = \psi).$$

Our aim is to prove the following two theorems.

Theorem B.1. *For all $\Psi \in H_a^{l-1/2}(\Gamma)$, we have*

$$\left(\int\limits_\Gamma r^{2(a-(l-1/2))} |\Psi|^2 \, d\Gamma \right)^{1/2} \leq c \|\Psi\|_{H_a^{l-1/2}(\Gamma)},$$

where $c > 0$ is independent of Ψ.

Theorem B.2. *For all $\psi \in E_a^{l-1/2}(\gamma)$, we have*

$$\left(\int\limits_\gamma r^{2(a-(l-1/2))} |\psi|^2 \, d\gamma \right)^{1/2} \leq c \|\psi\|_{E_a^{l-1/2}(\gamma)},$$

where $c > 0$ is independent of ψ.

At first, let us formulate two lemmas (see [8, Chapter 6, §1.3].

Lemma B.1. *The norm $\|U\|_{H_a^l(\Omega)}$ is equivalent to the norm*

$$\left(\int\limits_{\mathbb{R}^{n-2}} |\eta|^{2(l-a)-2} \|W(\cdot,\ \eta)\|^2_{E_a^l(K)} \, d\eta \right)^{1/2},$$

where $W(y,\ \eta) = \hat{U}(|\eta|^{-1}y,\ \eta)$, $\hat{U}(y,\ \eta)$ is the Fourier transform of $U(y,\ z)$ with respect to z.

Lemma B.2. *The norm $\|u\|_{E_a^l(\Omega)}$ is equivalent to the norm*

$$\left(\sum_{k=0}^{l} \int\limits_0^\infty r^{2(a-(l-1/2))} \sum_{j=0}^{l-k} (1+r)^{2(l-k-j)} \|(rD_r)^k u(r,\ \cdot)\|^2_{W^j(b_1,\ b_2)} \, dr \right)^{1/2},$$

$u(r,\ \varphi)$ is the function $u(y)$ written in the polar coordinates.

Let us prove Theorem B.2. Take a function $u \in E_a^l(K)$ such that $u|_\gamma = \psi$, $\|u\|_{E_a^l(K)} \le 2\|\psi\|_{E_a^{l-1/2}(\gamma)}$. Since $u(r,\ \varphi)|_{\varphi=b} = \psi(r)$ and the trace operator in Sobolev spaces is bounded, we have $|\psi(r)|^2 \le k_1\|u(r,\ \cdot)\|^2_{W^l(b_1,\ b_2)}$. Therefore by Lemma B.2, we get

$$\int\limits_\gamma r^{2(a-(l-1/2))}|\psi|^2\,d\gamma \le k_1\int\limits_0^\infty r^{2(a-(l-1/2))}\|u(r,\ \cdot)\|^2_{W^l(b_1,\ b_2)}\,dr \le k_2\|u\|^2_{E_a^l(K)}. \tag{B.1}$$

Now Theorem B.2 follows from (B.1) and the inequality $\|u\|_{E_a^l(K)} \le 2\|\psi\|_{E_a^{l-1/2}(\gamma)}$.

Let us prove Theorem B.1. Take a function $U \in H_a^l(\Omega)$ such that $U|_\Gamma = \Psi$, $\|U\|_{H_a^l(\Omega)} \le 2\|\Psi\|_{H_a^{l-1/2}(\Gamma)}$. Using the Fourier transform with respect to z and the Parseval equality, we have

$$\int\limits_\Gamma r^{2(a-(l-1/2))}|\Psi|^2\,d\Gamma = \int\limits_{\mathbb{R}^{n-2}}\int\limits_{\mathbb{R}^1} r^{2(a-(l-1/2))}|\hat\Psi(r,\ \eta)|^2\,drd\eta,$$

where $\hat\Psi(r,\ \eta)$ is the Fourier transformation of the function $\Psi(r,\ z)$ with respect to z. Doing change of variables $r = |\eta|^{-1}r'$ in the last integral and using (B.1), we obtain

$$\begin{gathered}\int\limits_\Gamma r^{2(a-(l-1/2))}|\Psi|^2\,d\Gamma = \\ \le \int\limits_{\mathbb{R}^{n-2}}\int\limits_{\mathbb{R}^1} |\eta|^{-2(a-(l-1/2))-1}(r')^{2(a-(l-1/2))}|\hat\Psi(\eta^{-1}r',\ \eta)|^2\,dr'd\eta \le \\ \le k_2\int\limits_{\mathbb{R}^{n-2}} |\eta|^{2(l-a)-2}\|W(\cdot,\ \eta)\|^2_{E_a^l(K)}\,d\eta,\end{gathered} \tag{B.2}$$

where $W(y,\ \eta) = \hat U(|\eta|^{-1}y,\ \eta)$. Now Theorem B.1 follows from (B.2), Lemma B.1, and the inequality $\|U\|_{H_a^l(\Omega)} \le 2\|\Psi\|_{H_a^{l-1/2}(\Gamma)}$.

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