DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **29**, Number **3**, March **2011**

doi:10.3934/dcds.2011.29.1041

pp. **1041–1083**

PERIODIC SOLUTIONS OF PARABOLIC PROBLEMS WITH HYSTERESIS ON THE BOUNDARY

Pavel Gurevich

Free University Berlin - Institute for Mathematics 1
Arnimallee 2-6
14195 Berlin, Germany

(Communicated by Hans-Otto Walther)

Abstract. We consider a parabolic problem with discontinuous hysteresis on the boundary, arising in modelling various thermal control processes. By reducing the problem to an infinite dynamical system, sufficient conditions for the existence and uniqueness of a periodic solution are found. Global stability of the periodic solution is proved.

1. **Introduction.** Hysteresis operators naturally arise in mathematical description of many physical processes [17, 29, 5]. Models involving ordinary differential equations with hysteresis were considered by many authors and are nowadays relatively well investigated (see e.g., [2, 17, 27, 7, 28, 3, 22]). Partial differential equations with hysteresis have also been actively studied during the last decades (see [29, 5] and references therein), but many questions remain open, especially those related to the periodicity and large-time behavior of solutions.

In this paper, we deal with parabolic problems containing a discontinuous hysteresis operator in the boundary condition. Such problems describe various processes of thermal control, where the temperature regulation in a domain is performed via heating (or cooling) elements on the boundary of the domain. The regime of the heating elements on the boundary is based on the registration of thermal sensors inside the domain and obeys the hysteresis law.

Let $v(x,t)$ denotes the temperature at the point x of a bounded domain $Q\subset\mathbb{R}^n$ at the moment t. We define the *mean temperature* $\hat v(t)$ by the formula

$$\hat v(t)=\int_Q m(x)v(x,t)\,dx,$$

where $m\in L_2(Q)$ (see Condition 2.1 for another technical assumption on $m(x)$).

We assume that the function $v(x,t)$ satisfies the heat equation

$$v_t(x,t)=\Delta v(x,t)\quad (x\in Q,\ t>0) \tag{1.1}$$

and a boundary condition which involves a hysteresis operator $\mathcal{H}$ depending on the mean temperature $\hat v$.

2000 *Mathematics Subject Classification.* Primary: 35K10, 47J40, 35B10; Secondary: 35B41.
Key words and phrases. Parabolic problems, Hysteresis, Periodic solutions, Global attractor.
The author is supported by DFG in the framework of SFB 555, by DAAD via G-RISC project, and by RFBR grant 10-01-00395-a.

The hysteresis $\mathcal{H}(\hat{v})(t)$ is defined as follows (cf. [17, 29] and the accurate definition and Fig. 2.1 in Sec. 2). One fixes two temperature thresholds α and β ($\alpha < \beta$). If $\hat{v}(t) \le \alpha$, then $\mathcal{H}(\hat{v})(t) = 1$ (the heating is switched on); if $\hat{v}(t) \ge \beta$, then $\mathcal{H}(\hat{v})(t) = -1$ (the cooling is switched on); if the mean temperature $\hat{v}(t)$ is between α and β, then $\mathcal{H}(\hat{v})(t)$ takes the same value as "just before." We say that the hysteresis operator *switches* when it jumps from 1 to -1 or from -1 to 1. The corresponding time moment is called the *switching moment.* Note that the hysteresis phenomenon takes place along with the nonlocal effect caused by the fact that the mean temperature $\hat{v}(t)$ is the integral of the temperature $v(x,t)$ over Q.

To be definite, let us assume that one regulates the heat flux through the boundary ∂Q. Then the boundary condition is of the form

$$\frac{\partial v}{\partial \nu} = K(x)u(t) \quad (x \in \partial Q,\ t > 0), \tag{1.2}$$

where ν is the outward normal to ∂Q at the point x, $K \in C^\infty(\partial Q)$ is a given real-valued function (distribution of the heating elements on the boundary), and $u(t)$ satisfies the ordinary differential equation

$$au'(t) + u(t) = \mathcal{H}(\hat{v})(t) \tag{1.3}$$

with $a \ge 0$. Thus, if $a = 0$ the heat flux through the boundary changes by jump, whereas if $a > 0$, it changes continuously.

A similar mathematical model was originally proposed in [10, 11]. Generalizations to various phase-transition problems with hysteresis were studied in [6, 8, 15, 18, 5]. Some related issues of optimal control were considered in [4]. The most important questions here concern the existence and uniqueness of solutions, the existence of periodic solutions, and large-time behavior of solutions. The latter two questions are especially difficult.

In the case of a *one-dimensional* domain Q (a finite interval, $n = 1$), the periodicity was studied in [9, 26, 12]. Thermocontrol problems in *multidimensional* domains ($n \ge 2$) turn out to be much more complicated. Although one can relatively easily prove the existence (and sometimes uniqueness) of solutions, the issue of finding periodic solutions is still an open question. The main obstacle is the possible failure of the transversality at a switching moment (cf. [3], where the same phenomenon occurs for ordinary differential equations). This means that if the mean temperature $\hat{v}(t)$ has the zero derivative at a switching moment, then the continuous dependence of the solutions on the initial data may fail (see Fig. 3.2). As a result, most methods based on fixed-point theorems do not apply to the corresponding Poincaré maps.

One possible way to overcome the nontransversality is to consider a continuous model of the hysteresis operator. This was done in [13], where a thermocontrol problem with the Preisach hysteresis operator in the boundary condition was considered and the existence of periodic solutions and global attractors were established. Note that the periodicity and the large-time behavior of solutions were also studied in [31, 16] in the situation where a continuous hysteresis operator enters a parabolic equation itself (see also [30] and references therein).

The first results about periodic solutions of thermocontrol problems in *multi-dimensional* domains with *discontinuous* hysteresis were obtained in [14]. It was proved that if a solution with periodic mean temperature exists, then there exists a (possibly another) solution *periodic at each point*, with the same mean temperature. In the case of the Neumann boundary condition and the uniform distribution of the thermal sensors ($m(x) \equiv \text{const}$), a solution with periodic mean temperature was found; thus, the existence of a periodic solution was proved. But it was unclear how to find solutions with periodic mean temperature in the general situation of the Dirichlet or Robin boundary conditions or when $m(x) \not\equiv \text{const}$.

In the present work, we develop a new approach to the study of periodicity and large-time behavior of solutions of thermocontrol problems with *discontinuous* hysteresis on the boundary. Our approach is based on regarding the problem as an infinite-dimensional dynamical system. By using the Fourier method, we reduce the boundary-value problem for the heat equation to infinitely many ordinary differential equations, whose solutions are coupled with each other via the hysteresis operator. To be definite, we consider only the Neumann boundary condition. We also restrict ourself by studying the case $a = 0$, which means that the ordinary differential equation (1.3) is absent and the boundary condition (1.2) reduces to

$$\frac{\partial v}{\partial \nu} = K(x)\mathcal{H}(\hat{v})(t) \quad (x \in \partial Q,\ t > 0).$$

However, the method developed can also be applied to the study of other types of boundary conditions and to the case $a > 0$ where the heat equation (1.1) is coupled with the ordinary differential equation (1.3). We note that the case $m(x) \equiv \text{const}$ (see [14]) appears to be a particular case in which our infinite-dimensional dynamical system reduces to finite-dimensional, namely to one ordinary differential equation if $a = 0$ and to two if $a > 0$.

Analysis of the dynamical system allows us to find sufficient conditions of existence and uniqueness of a periodic solution of the thermocontrol problem. Moreover, we prove that it is a global attractor which attracts any solution exponentially fast. One of the sufficient conditions requires that the difference between the temperature thresholds $\beta - \alpha$ is not too small. Another sufficient condition requires instead that the weight function $m(x)$ is close to a constant in a certain sense.

The paper is organized as follows. In Sec. 2, we formulate the thermocontrol problem, define the hysteresis operator, introduce mild and strong solutions, and prove their existence and uniqueness. In this section, we also reduce the problem to the infinite-dimensional dynamical system and establish its basic properties.

In Sec. 3, we find a sufficient condition (in terms of the difference between the temperature thresholds $\beta - \alpha$) for the existence of a periodic solution. To do so, we introduce the Poincaré map $\mathbf{P}$ as follows. For any function $\varphi = v(\cdot, 0)$ from the hyperspace

$$\int_Q m(x)\varphi(x)\, dx = \hat{v}(0) = \alpha,$$

we show that there is the first switching moment $t_1 > 0$ such that $\hat{v}(t_1) = \beta$ and the second switching moment $t_2 > t_1$ such that $\hat{v}(t_2) = \alpha$ again. We set $\mathbf{P}(\varphi) = v(\cdot, t_2)$ (Fig. 3.1). We find a bounded convex region $B_{\alpha,0}$ which is mapped by $\mathbf{P}$ into itself. Then we show that the above-mentioned condition on the difference $\beta - \alpha$ (or on $m(x)$) guarantees the transversality of the mean temperature $\hat{v}(t)$ at the switching moment t_1. This allows us to prove that the Poincaré map $\mathbf{P}$ is a compact continuous (and even continuously Fréchet-differentiable) operator on $B_{\alpha,0}$, and the periodic solution exists by the Schauder fixed point theorem.

In Sec. 4, we consider a suitable projection Π of the Poincaré map $\mathbf{P}$. Under assumptions that are slightly stronger than those in Sec. 3, we show that Π is a contraction mapping. This fact, combined with the contraction mapping principle, allows us to prove that the periodic solution is unique, stable, and a global attractor.

In Sec. 5, we show that the above results are also true for any α and β, but provided that the weight function $m(x)$ is close to a constant in a certain sense.

2. Setting of the problem. Existence and uniqueness of solution.

2.1. Setting of a thermocontrol problem. Let $Q \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded domain with boundary ∂Q of class C^∞, $Q_T = Q \times (0,T)$, $T > 0$. Let $v(x,t)$ denote the temperature at the point $x \in Q$ at the moment $t \geq 0$ satisfying the heat equation

$$v_t(x,t) = \Delta v(x,t) \quad ((x,t) \in Q_T) \tag{2.1}$$

with the initial condition

$$v(x,0) = \varphi(x) \quad (x \in Q) \tag{2.2}$$

and the boundary condition

$$\frac{\partial v}{\partial \nu} = K(x)\mathcal{H}(\hat{v})(t) \quad (x \in \partial Q,\ t \in (0,T)). \tag{2.3}$$

Here ν is the outward normal to $\partial Q \times (0,T)$ at the point (x,t) and $K \in C^\infty(\partial Q)$ is a real-valued function[1]. For any function $v(x,t)$, we denote by $\hat{v} = \hat{v}(t)$ the averaged function (the "mean" temperature) given by

$$\hat{v}(t) = \int_Q m(x)v(x,t)\,dx,$$

where $m \in L_2(Q)$ is a real-valued weight function determined by characteristics of the thermal sensors; $\mathcal{H}$ is a hysteresis operator, which we now define.

We denote by $BV(0,T)$ the Banach space of real-valued functions having finite total variation on the segment $[0,T]$ and by $C_r[0,T)$ the linear space of functions which are continuous on the right in $[0,T)$. We fix the numbers $\alpha < \beta$ and introduce the *hysteresis operator* (cf. [17, 29])

$$\mathcal{H} : C[0,T] \to BV(0,T) \cap C_r[0,T)$$

by the following rule. For any $g \in C[0,T]$, the function $z = \mathcal{H}(g) : [0,T] \to \{-1, 1\}$ is defined as follows. Let $X_t = \{t' \in (0,t] : g(t') = \alpha \text{ or } \beta\}$; then

$$z(0) = \begin{cases} 1 & \text{if } g(0) < \beta, \\ -1 & \text{if } g(0) \geq \beta \end{cases}$$

and for $t \in (0,T]$

$$z(t) = \begin{cases} z(0) & \text{if } X_t = \varnothing, \\ 1 & \text{if } X_t \neq \varnothing \text{ and } g(\max X_t) = \alpha, \\ -1 & \text{if } X_t \neq \varnothing \text{ and } g(\max X_t) = \beta \end{cases}$$

(see Fig. 2.1). We stress that, by this definition, $\mathcal{H}(g)(0) = 1$ if $g(0) \in (\alpha, \beta)$. A point τ such that $\mathcal{H}(g)(\tau) \neq \mathcal{H}(g)(\tau - 0)$ is called a *switching moment of* $\mathcal{H}(g)$.

We assume throughout that the following condition holds.

[1] All the results of this paper are also true for $K(x)$ from the Sobolev space $H^{1/2}(\partial Q)$ defined in Sec 2.2.

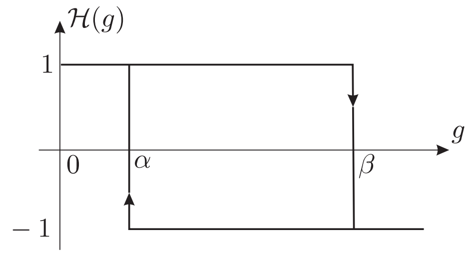

FIGURE 2.1. The hysteresis operator $\mathcal{H}$

Condition 2.1. *The coefficient $K(x)$ in the boundary condition (2.3) and the weight function $m(x)$ satisfy*

$$\int_{\partial Q} K(x)\,d\Gamma > 0, \qquad \int_Q m(x)\,dx > 0. \tag{2.4}$$

The goal of this section is to establish the existence and uniqueness of solutions of problem (2.1)–(2.3) (Sec. 2.3) and to discuss a proper framework for the study of the large-time behavior of solutions (Sec. 2.4).

2.2. **Reduction to infinite dynamical system and auxiliary results.** First, we formulate some auxiliary (well-known) results for the parabolic initial boundary-value problem

$$v_t(x,t) = \Delta v(x,t) \quad ((x,t) \in Q_T), \tag{2.5}$$

$$v(x,0) = \varphi(x) \quad (x \in Q), \tag{2.6}$$

$$\frac{\partial v}{\partial \nu} = K(x) \quad (x \in \partial Q,\ t \in (0,T)). \tag{2.7}$$

Let $L_2 = L_2(Q)$. Denote by $H^k = H^k(Q)$ $(k \in \mathbb{N})$ the Sobolev space with the norm

$$\|\psi\|_{H^k} = \left(\sum_{|\alpha| \le k} \int_Q |D^\alpha \psi(x)|^2\,dx \right)^{1/2}.$$

Let $H^{k-1/2} = H^{k-1/2}(\partial Q)$ $(k \in \mathbb{N})$ be the space of traces on ∂Q of the functions from H^k.

For any Banach space B, denote by $C([a,b];B)$ $(a < b)$ the space of B-valued functions continuous on the segment $[a,b]$ with the norm

$$\|u\|_{C([a,b];B)} = \max_{t \in [a,b]} \|u(t)\|_B$$

and by $L_2((a,b);B)$ the space of L_2-integrable B-valued functions with the norm

$$\|u\|_{L_2((a,b);B)} = \left(\int_a^b \|u(t)\|_B^2\,dt \right)^{1/2}.$$

We introduce the anisotropic Sobolev space

$$H^{2,1}(Q \times (a,b)) = \{ v \in L_2((a,b);H^2) : v_t \in L_2((a,b);L_2) \}$$

with the norm

$$\|v\|_{H^{2,1}(Q \times (a,b))} = \left(\int_a^b \|v(\cdot,t)\|_{H^2}^2\,dt + \int_a^b \|v_t(\cdot,t)\|_{L_2}^2\,dt \right)^{1/2}.$$

Taking into account the results of the interpolation theory (see, e.g., [20, Chap. 1, Secs. 1–3, 9], we make the following remarks.

Remark 2.1. The continuous embedding $H^{2,1}(Q \times (a,b)) \subset C([a,b],H^1)$ takes place. In particular, for any $v \in H^{2,1}(Q\times(a,b))$ and $\tau \in [a,b]$, the trace $v|_{t=\tau} \in H^1$ is well defined and is a bounded operator from $H^{2,1}(Q \times (a,b))$ to H^1.

Remark 2.2. Consider two functions $v_1 \in H^{2,1}(Q\times(a,b))$ and $v_2 \in H^{2,1}(Q\times(b,c))$, where $a < b < c$. Let $v(\cdot,t) = v_1(\cdot,t)$ for $t \in (a,b)$ and $v(\cdot,t) = v_2(\cdot,t)$ for $t \in (b,c)$. Then $v \in H^{2,1}(Q \times (a,c))$ if and only if $v_1|_{t=b} = v_2|_{t=b}$.

Let us introduce the notion of a mild solution. To do so, we reduce problem (2.5)–(2.7) to a problem with the homogeneous boundary condition. Let $v_K \in H^2$ be the solution of the boundary-value problem

$$\Delta v_K(x) = f \quad (x \in Q), \qquad \frac{\partial v_K}{\partial \nu}\Big|_{\partial Q} = K(x) \quad (x \in \partial Q), \tag{2.8}$$

where $f = \dfrac{1}{\operatorname{mes} Q} \int_{\partial Q} K(x)\,d\Gamma$, such that

$$\int_Q v_K(x)\,dx = 0.$$

(Note that f is a positive constant.) It is well known that such a solution exists, is unique, and satisfies the estimate

$$\|v_K\|_{H^2} \le c\|K\|_{H^{1/2}}, \tag{2.9}$$

where $c > 0$ does not depend on $K(x)$.

Then the function

$$w(x,t) = v(x,t) - v_K(x)$$

must satisfy the relations

$$w_t(x,t) = \Delta w(x,t) + f \quad ((x,t) \in Q_T), \tag{2.10}$$

$$w(x,0) = \varphi(x) - v_K(x) \quad (x \in Q), \tag{2.11}$$

$$\frac{\partial w}{\partial \nu} = 0 \quad (x \in \partial Q,\ t \in (0,T)). \tag{2.12}$$

We introduce the unbounded linear operator $P : \mathrm{D}(P) \subset L_2 \to L_2$ given by

$$P\psi = \Delta\psi, \qquad \mathrm{D}(P) = \left\{\psi \in H^2 : \frac{\partial \psi(x)}{\partial \nu}\Big|_{\partial Q} = 0\right\}. \tag{2.13}$$

It is well known that the operator P is a generator of an analytic semigroup $\mathbf{S}_t : L_2 \to L_2$, $t \ge 0$.

Definition 2.1. A function $v \in C([0,T];L_2)$ is called a *mild solution* of problem (2.5)–(2.7) in Q_T with the initial data $\varphi \in L_2$ if $v(x,t) = w(x,t) + v_K(x)$, where $w \in C([0,T];L_2)$ is a mild solution of problem (2.10)–(2.12), i.e.,

$$w(\cdot,t) = \mathbf{S}_t(\varphi - v_K) + ft.$$

It follows from this definition that the mild solution of problem (2.5)–(2.7) is given by

$$v(\cdot,t) = \mathbf{S}_t(\varphi - v_K) + ft + v_K. \tag{2.14}$$

Definition 2.2. A function $v(x,t)$ is called a (*strong*) *solution* of problem (2.5)–(2.7) in Q_T if $v \in H^{2,1}(Q_T)$ and v satisfies Eq. (2.5) a.e. in Q_T and conditions (2.6), (2.7) in the sense of traces.

In what follows, we omit the term "strong" whenever it leads to no confusion.

In the case of the heat equation, one can give a convenient representation of mild and strong solutions in terms of the Fourier series.

Let $\{\lambda_j\}_{j=0}^{\infty}$ and $\{e_j(x)\}_{j=0}^{\infty}$ denote the sequence of eigenvalues and the corresponding system of real-valued eigenfunctions (infinitely differentiable in $\overline{Q}$ and orthonormal in L_2) of the spectral problem

$$-\Delta e_j(x) = \lambda_j e_j(x) \quad (x \in Q), \qquad \frac{\partial e_j}{\partial \nu}\Big|_{\partial Q} = 0. \tag{2.15}$$

It is well known that $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \ldots$, $e_0(x) \equiv (\operatorname{mes} Q)^{-1/2} > 0$, and the system of eigenfunctions $\{e_j\}_{j=0}^{\infty}$ forms an orthonormal basis for L_2. Furthermore, the functions $e_j/\sqrt{\lambda_j+1}$ form an orthonormal basis for H^1.

Remark 2.3. In what follows, we will use the well-known asymptotics for the eigenvalues $\lambda_j = Lj^{2/n} + o(j^{2/n})$ as $j \to \infty$ ($L > 0$ and n is the dimension of Q).

Any function $\psi \in L_2$ can be expanded into the Fourier series with respect to $e_j(x)$, which converges in L_2:

$$\psi(x) = \sum_{j=0}^{\infty} \psi_j e_j(x), \qquad \|\psi\|_{L_2}^2 = \sum_{j=0}^{\infty} |\psi_j|^2, \tag{2.16}$$

where $\psi_j = \int_Q \psi(x) e_j(x)\,dx$. If $\psi \in H^1$, then the first series in (2.16) converges to ψ in H^1 and

$$\|\psi\|_{H^1}^2 = \sum_{j=0}^{\infty} (1+\lambda_j)|\psi_j|^2.$$

If $\psi \in H^2$ and $\frac{\partial \psi}{\partial \nu}\Big|_{\partial Q} = 0$, then the first series in (2.16) converges to ψ in H^2 and

$$\|\psi\|_{H^2}^2 \approx \sum_{j=0}^{\infty} (1+\lambda_j^2)|\psi_j|^2,$$

where "$\approx$" means the equivalence of the norms. The above number series is convergent if and only if $\psi \in H^2$ and $\frac{\partial \psi}{\partial \nu}\Big|_{\partial Q} = 0$.

Furthermore, the semigroup $\mathbf{S}_t$ ($t \ge 0$) and its derivative $\mathbf{S}_t'$ ($t > 0$) are given by

$$\mathbf{S}_t \psi = \sum_{j=0}^{\infty} e^{-\lambda_j t} \psi_j e_j(x) \quad (t \ge 0), \qquad \mathbf{S}_t' \psi = -\sum_{j=0}^{\infty} \lambda_j e^{-\lambda_j t} \psi_j e_j(x) \quad (t > 0), \tag{2.17}$$

where $\psi_j = \int_Q \psi(x) e_j(x)\,dx$.

Denote for j=0,1,2,...

$$m_j = \int_Q m(x) e_j(x)\,dx, \qquad K_j = \int_{\partial Q} K(x) e_j(x)\,dx,$$

$$v_{Kj} = \int_Q v_K(x) e_j(x)\,dx, \qquad f_j = \int_Q f e_j(x)\,dx, \qquad \varphi_j = \int_Q \varphi(x) e_j(x)\,dx. \tag{2.18}$$

Note that $v_{K0} = 0$ and $f_j = 0$ for $j = 1, 2, \ldots$ (by the definition of v_K and f).

The following lemma establishes the connection between K_j, v_{Kj}, f_j, and λ_j.

Lemma 2.1. *Let λ_j be the eigenvalues of problem (2.15). Then*

1. *the constants in (2.18) satisfy the relations*
$$K_0 = f_0, \qquad K_j = \lambda_j v_{Kj} \quad (j = 1, 2, \dots), \tag{2.19}$$
2. *the following inequality holds*:
$$\sum_{j=1}^{\infty} \left(\frac{|K_j|^2}{\lambda_j^2} + \frac{|K_j|^2}{\lambda_j} \right) \le c^2 \|K\|_{H^{1/2}}^2,$$
where $c > 0$ is the constant from (2.9).

Proof. 1. By using the definition of $v_K(x), e_j(x), f_j, v_{Kj}$ and the formula of integration by parts, we obtain

$$f_j = \int_Q \Delta v_K(x) e_j(x)\, dx = \int_{\partial Q} \frac{\partial v_K}{\partial \nu} e_j(x)\, d\Gamma - \int_Q \nabla v_K(x) \nabla e_j(x)\, dx$$
$$= \int_{\partial Q} K(x) e_j(x)\, d\Gamma + \int_Q v_K(x) \Delta e_j(x) = K_j - \lambda_j v_{Kj},$$

which yields (2.19) because $\lambda_0 = v_{K0} = 0$ and $f_j = 0$ for $j = 1, 2, \dots$.

2. Using (2.19), we have
$$\sum_{j=1}^{\infty} \left(\frac{|K_j|^2}{\lambda_j^2} + \frac{|K_j|^2}{\lambda_j} \right) = \sum_{j=1}^{\infty} (1 + \lambda_j)|v_{Kj}|^2 = \|v_K\|_{H^1}^2 \le \|v_K\|_{H^2}^2,$$
and part 2 of the lemma follows from (2.9). ◻

The following two lemmas summarize the results about problem (2.5)–(2.7) which we need further.

Lemma 2.2. *Let $\varphi \in L_2$. Then the following assertions hold.*

1. *The mild solution $v(x,t)$ of problem (2.5)–(2.7) belongs to $C^\infty((0,T]; H^2)$ and satisfies the inequality*
$$\|v(\cdot, T)\|_{H^2} \le c_1 \left(\|\varphi\|_{L_2} + \|K\|_{H^{1/2}} \right), \tag{2.20}$$
where $c_1 = c_1(T) > 0$ does not depend on φ and is bounded on any segment $[T_1, T_2]$ $(0 < T_1 < T_2)$.
2. *The mild solution $v(x,t)$ can be represented as the series*
$$v(x,t) = \sum_{j=0}^{\infty} v_j(t) e_j(x), \quad t \ge 0, \tag{2.21}$$
where $v_j(t) = \int_Q v(x,t) e_j(x)\, dx$ and $v_j(t)$ satisfy the Cauchy problem
$$\dot v_j(t) = -\lambda_j v_j(t) + K_j, \qquad v_j(0) = \varphi_j \qquad (\dot{} = d/dt,\ j = 0, 1, 2 \dots), \tag{2.22}$$
and the estimates
$$|v_j(t)| \le \max\left(|\varphi_j|, \frac{|K_j|}{\lambda_j} \right) \quad (t > 0,\ j = 1, 2, \dots) \tag{2.23}$$
hold. The series in (2.21) converges in L_2 for $t = 0$ and in H^1 for $t > 0$.
3. *The mean temperature $\hat v(t)$ is represented by the absolutely convergent series*
$$\hat v(t) = \sum_{j=0}^{\infty} m_j v_j(t), \quad t \ge 0. \tag{2.24}$$

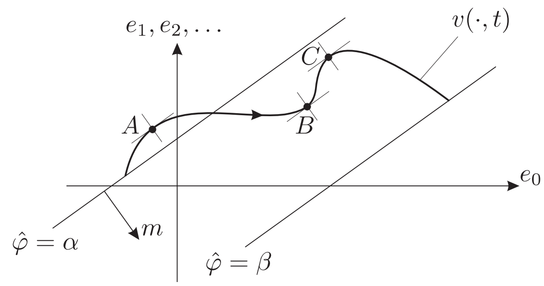

FIGURE 2.2. The plane spanned by $e_0 = (1,0,0,\dots)$ and $m = (m_0, m_1, m_2, \dots)$; $\dfrac{d\hat v(t)}{dt} = 0$ at the points A, B, C

4. The function $\hat v(t)$ is continuously differentiable for $t > 0$. For any $R > 0$, there is a number $t^* = t^*(R) > 0$ such that if $\left(\sum_{j=1}^{\infty} |\varphi_j|^2\right)^{1/2} \le R$, then

$$\frac{m_0 K_0}{2} \le \frac{d\hat v(t)}{dt} \le \frac{3m_0K_0}{2} \qquad \forall t \ge t^*. \tag{2.25}$$

Before we prove the lemma, let us give a geometrical interpretation of the dynamics of $v(\cdot,t)$ in L_2 (or in H^1, see Lemma 2.3 below). We choose the orthonormal basis in L_2 (and orthogonal in H^1) consisting of the eigenfunctions $e_0, e_1, e_2, \dots$. Then, in the coordinate form, we have

$$e_0 = (1,0,0,0,\dots), \quad e_1 = (0,1,0,0,\dots), \quad e_2 = (0,0,1,0,\dots), \quad \dots$$

and (cf. (2.21))

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots), \qquad v(\cdot,t) = (v_0(t), v_1(t), v_2(t), \dots).$$

Consider the plane going through the origin and spanned by the vector $e_0 = (1,0,0,\dots)$ and the vector $m = (m_0, m_1, m_2, \dots)$ (if they are parallel, i.e., $m_1 = m_2 = \dots = 0$, then we consider an arbitrary plane containing e_0). We note that the angle between the vectors m and e_0 is acute (their scalar product is equal to $m_0 > 0$). Clearly, the orthogonal projection of the hyperspace $\hat\varphi = \sum_{j=0}^{\infty} m_j\varphi_j = \alpha$ (or β) on this plane is a line (see Fig. 2.2).

It follows from (2.24) that the trajectory of $v(\cdot,t)$ is orthogonal to the vector m at the points where $\dfrac{d\hat v(t)}{dt} = 0$. These are the points A, B, and C in Fig. 2.2.

Due to (2.22), $v_0(t)$ "goes" from the left to the right with the constant speed $K_0 > 0$, while $v_j(t)$ exponentially converge to K_j/λ_j (see Fig. 2.3).

Proof of Lemma 2.2. 1. The inclusion $v \in C^\infty((0,T]; H^2)$ and estimate (2.20) follow from the general theory of analytic semigroups (see, e.g., [24, Chap. 4])

2. Formally, relations (2.22) can be obtained by multiplying (2.5) by $e_j(x)$, integrating by parts over Q, and substituting $v(x,t) = \sum_{j=0}^{\infty} v_j(t)e_j(x)$. To give a

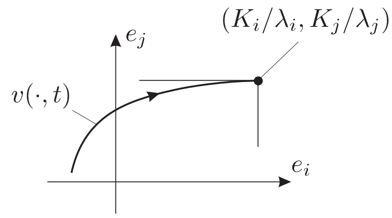

FIGURE 2.3. The plane spanned by e_i and e_j, $i \neq j$, $i, j \geq 1$

rigorous proof, we note that the representations (2.14) and (2.17) imply

$$\begin{aligned} v(x,t) &= \sum_{j=0}^{\infty} e^{-\lambda_j t}(\varphi_j - v_{Kj})e_j(x) + \int_0^t \sum_{j=0}^{\infty} e^{-\lambda_j \sigma} f_j e_j(x)\, d\sigma + \sum_{j=0}^{\infty} v_{Kj} e_j(x) \\ &= \sum_{j=0}^{\infty} v_j(t) e_j(x), \end{aligned}$$

where

$$v_j(t) = e^{-\lambda_j t}(\varphi_j - v_{Kj}) + \int_0^t e^{-\lambda_j \sigma} f_j \, d\sigma + v_{Kj}.$$

One can easily verify that these functions satisfy the relations

$$\dot{v}_j(t) = -\lambda_j v_j + (f_j + \lambda_j v_{Kj}), \qquad v_j(0) = \varphi_j.$$

To complete the proof of part 2, it remains to apply part 1 of Lemma 2.1.

3. Representation (2.24) follows from (2.21) and the definition of m_j.

4. Using (2.24) and (2.22), we have for $t \geq 0$

$$\hat{v}(t) = \sum_{j=0}^{\infty} m_j v_j(t) = m_0(\varphi_0 + K_0 t) + \sum_{j=1}^{\infty} m_j \left(\left(\varphi_j - \frac{K_j}{\lambda_j} \right) e^{-\lambda_j t} + \frac{K_j}{\lambda_j} \right). \quad (2.26)$$

Formally differentiating, we obtain

$$\frac{d\hat{v}(t)}{dt} = m_0 K_0 + \sum_{j=1}^{\infty} m_j (K_j - \lambda_j \varphi_j) e^{-\lambda_j t}. \quad (2.27)$$

Note that $m_0 K_0 > 0$ due to Condition 2.1. Thus, it suffices to show that the series in (2.27) is uniformly absolutely convergent for $t \geq \delta$ for any $\delta > 0$ and

$$\sum_{j=1}^{\infty} \left| m_j (K_j - \lambda_j \varphi_j) e^{-\lambda_j t} \right| \leq \frac{m_0 K_0}{2} \qquad \text{for } t \geq t^*. \quad (2.28)$$

Taking into account Lemma 2.1, we have for $t > 0$

$$\begin{aligned} &\sum_{j=1}^{\infty} \left| m_j(K_j - \lambda_j \varphi_j) e^{-\lambda_j t} \right| \\ &\quad \le \left(\left(\sum_{j=1}^{\infty} \frac{|K_j|^2}{\lambda_j^2} \right)^{1/2} + \left(\sum_{j=1}^{\infty} |\varphi_j|^2 \right)^{1/2} \right) \left(\sum_{j=1}^{\infty} |m_j|^2 \lambda_j^2 e^{-2\lambda_j t} \right)^{1/2} \\ &\quad \le (c\|K\|_{H^{1/2}} + R) \left(\sum_{j=1}^{\infty} |m_j|^2 \lambda_j^2 e^{-2\lambda_j t} \right)^{1/2}. \end{aligned} \tag{2.29}$$

On the other hand,

$$\lambda_j^2 e^{-2\lambda_j t} \le t^{-2} e^{-\lambda_1 t} \cdot (\lambda_j t)^2 e^{-\lambda_j t} \le 4(et)^{-2} e^{-\lambda_1 t}. \tag{2.30}$$

It follows from (2.29) and (2.30) that

$$\sum_{j=1}^{\infty} \left| m_j(K_j - \lambda_j \varphi_j) e^{-\lambda_j t} \right| \le k_1 t^{-1} e^{-\lambda_1 t/2},$$

where $k_1 = k_1(R) > 0$ does not depend on t. Therefore, $\hat{v}(t)$ is continuously differentiable for $t > 0$ and one can find the desired $t^* = t^*(R)$. □

Lemma 2.3. *Let $\varphi \in H^1$. Then the following assertions hold.*

1. *There exists a unique solution $v \in H^{2,1}(Q_T)$ of problem (2.5)–(2.7). It satisfies the inequality*
$$\|v\|_{H^{2,1}(Q_T)} \le c_1(\|\varphi\|_{H^1} + \|K\|_{H^{1/2}}), \tag{2.31}$$
where $c_1 = c_1(T) > 0$ does not depend on φ and is bounded on any segment $[T_1, T_2]$ $(0 < T_1 < T_2)$.
2. *For any $0 \le s < t \le T$*
$$|\hat{v}(s) - \hat{v}(t)| \le c_1 \|m\|_{L_2} (\|\varphi\|_{H^1} + \|K\|_{H^{1/2}})(t - s)^{1/2}. \tag{2.32}$$

Proof. 1. Theorem 5.3 in [21] implies that there is a unique solution $w \in H^{2,1}(Q_T)$ of problem (2.10)–(2.12) and, hence, a unique solution $v \in H^{2,1}(Q_T)$ of problem (2.5)–(2.7) and inequality (2.31) holds.

2. Applying the Schwartz inequality and using (2.31), we obtain for any $0 \le s < t \le T$

$$\begin{aligned} |\hat{v}(t) - \hat{v}(s)| &= \left| \int_Q m(x)\, dx \int_s^t v_t(x,t)\, dt \right| \le \|m\|_{L_2} \|v_t\|_{L_2(Q \times (s,t))} (t-s)^{1/2} \\ &\le c_1 \|m\|_{L_2} (\|\varphi\|_{H^1} + \|K\|_{H^{1/2}})(t-s)^{1/2}. \end{aligned}$$

□

2.3. **Solvability of the thermocontrol problem.** We define mild and strong solutions of problem (2.1)–(2.3) as follows.

First, we define a mild solution, assuming that $\hat{\varphi} < \beta$ (if $\hat{\varphi} \ge \beta$, the modifications are obvious).

Definition 2.3. A function $v \in C([0,T]; L_2)$ is called a *mild solution* of problem (2.1)–(2.3) in Q_T with the initial data $\varphi \in L_2$ $(\hat{\varphi} < \beta)$ if

1. $\mathcal{H}(\hat{v})$ has finitely many switching moments $t_1 < t_2 < \ldots < t_J$ on $(0,T)$ (or no switchings),
2. $v(x,t) = v^{(1)}(x,t)$ for $t \in (0,t_1)$, where $v^{(1)}(x,t)$ is a mild solution of problem (2.5)–(2.7) in Q_{t_1} with the initial data $\varphi \in L_2$,
3. $v(x,t) = v^{(i)}(x, t - t_{i-1})$ for $t \in (t_{i-1}, t_i)$, $i = 2, \ldots, J+1$, where $t_{J+1} = T$ and $v^{(i)}(x,t)$ is a mild solution of problem (2.5)–(2.7) in $Q_{t_i - t_{i-1}}$ with the initial data $v^{(i-1)}(x, t_{i-1} - t_{i-2})$ $(t_0 = 0)$ and with $K(x)$ replaced by $(-1)^{i-1}K(x)$.

Definition 2.4. A function $v(x,t)$ is called a (*strong*) *solution* of problem (2.1)–(2.3) in Q_T with the initial data $\varphi \in H^1$ if $v \in H^{2,1}(Q_T)$ and v satisfies Eq. (2.1) a.e. in Q_T and conditions (2.2), (2.3) in the sense of traces.

Remark 2.4. Theorem 2.2 below implies that, for any (strong) solution v, the function $\mathcal{H}(\hat{v})$ has finitely many switching moments. Thus, if one replaces the word "mild" by "strong" and the space $C([0,T]; L_2)$ by $H^{2,1}(Q_T)$ in Definition 2.3, then one obtains a definition of a strong solution equivalent to Definition 2.4.

Now we study the existence and uniqueness of mild solutions of the thermocontrol problem. To construct a mild (or strong, see Remark 2.4) solution, one should consecutively solve problem (2.5)–(2.7) (with $K(x)$ or $-K(x)$). It may however happen that the differences $t_i - t_{i-1}$ between the consecutive switching moments tend to zero as $i \to \infty$ and one never obtains a solution on a given time interval $[0,T]$.

The next theorem ensures the local existence of mild solutions of the thermocontrol problem.

Theorem 2.1. *Let $\varphi \in L_2$ and $\|\varphi\|_{L_2} \le R$ ($R > 0$ is arbitrary). Then the following assertions hold.*

1. *There exists $0 < T^* \le \infty$ such that, for any $T < T^*$, there is a unique mild solution $v \in C([0,T]; L_2)$ of problem (2.1)–(2.3) in Q_T.*
2. *If the set of the switching moments on the interval $[0,T]$ $(T < T^*)$ is not empty, then the consecutive switching moments $t_1 < t_2 < \ldots$ of $\mathcal{H}(\hat{v})$ satisfy*

$$t_i - t_{i-1} \le t^* + \frac{2(\beta - \alpha)}{m_0 K_0}, \quad i = 1, 2, \ldots, \tag{2.33}$$

where $t_0 = 0$ and $t^* = t^*(R)$ is defined in part 4 of Lemma 2.2. The number t^* depends on R but does not depend on $\varphi, T, \alpha, \beta$.

Proof. 1. Without loss of generality, we assume that

$$\hat{\varphi} = \int_Q m(x)\varphi(x)\,dx = \alpha.$$

By Lemma 2.2, there is a unique mild solution $v^{(1)}$ of problem (2.5)–(2.7) in Q_T with the initial data φ.

Since $\widehat{v^{(1)}}(t)$ is continuous on $[0,T]$, either $\widehat{v^{(1)}}(t) < \beta$ for $t < T$ or there is the first switching moment of $\mathcal{H}(\widehat{v^{(1)}})$, i.e., a number $t_1 \in (0,T)$ such that $\widehat{v^{(1)}}(t) < \beta$ for $t < t_1$ and $\widehat{v^{(1)}}(t_1) = \beta$.

In the first case, we obtain a unique mild solution of problem (2.1)–(2.3) in Q_T by setting $v = v^{(1)}$.

Consider the second case. Let us estimate $t_1 - t_0$, where $t_0 = 0$. Due to part 4 of Lemma 2.2, $\hat{v}(t)$ is continuously differentiable for $t > 0$ and $d\hat{v}(t)/dt \ge m_0K_0/2$

for $t \geq t^*$. Therefore,

$$t_1 - t_0 \leq t^* + \frac{2(\beta - \alpha)}{m_0 K_0}, \tag{2.34}$$

where $t_0 = 0$.

2. Now we consider problem (2.1)–(2.3) with $\mathcal{H}(\hat{v})(t)$ replaced by -1 and the initial value $v^{(1)}(x, t_1)$, i.e.,

$$v_t^{(2)}(x,t) = \Delta v^{(2)}(x,t) \quad ((x,t) \in Q_T), \tag{2.35}$$

$$v^{(2)}(x,0) = v^{(1)}(x,t_1) \quad (x \in Q), \tag{2.36}$$

$$\frac{\partial v^{(2)}}{\partial \nu} = -K(x) \quad (x \in \partial Q,\ t \in (0,T)). \tag{2.37}$$

Similarly to part 1 of the proof, we see that problem (2.35)–(2.37) has a unique mild solution $v^{(2)}$ in Q_T.

As in part 1, $\widehat{v^{(2)}}(t)$ is continuous on $[0,T]$ and either $\widehat{v^{(2)}}(t) > \alpha$ for $t < T - t_1$ or there is a first switching moment of $\mathcal{H}(\widehat{v^{(2)}})$, i.e., a number $\tau_2 \in (0, T - t_1)$ such that $\widehat{v^{(2)}}(t) > \alpha$ for $t < \tau_2$ and $\widehat{v^{(2)}}(\tau_2) = \alpha$. We set $t_2 = t_1 + \tau_2$.

In the first case, we obtain a unique mild solution of problem (2.1)–(2.3) in Q_T by setting $v(x,t) = v^{(1)}(x,t)$ for $(x,t) \in Q \times (0, t_1)$ and $v(x,t) = v^{(2)}(x, t - t_1)$ for $(x,t) \in Q \times (t_1, T)$.

Consider the second case. Similarly to (2.34), we obtain

$$t_2 - t_1 \leq t^* + \frac{2(\beta - \alpha)}{m_0 K_0}. \tag{2.38}$$

One can continue the above procedure and obtain a series of switching moments $t_1, t_2, \dots$ satisfying (2.33).

Setting

$$T^* = \sum_{i=1}^{\infty} (t_i - t_{i-1}),$$

we complete the proof. □

The next theorem shows that if the initial data φ belongs to H^1, then the strong solution of the thermocontrol problem exists globally.

Theorem 2.2. *Let $\varphi \in H^1$ and $\|\varphi\|_{H^1} \leq R$ ($R > 0$ is arbitrary). Then the following holds for any $T > 0$.*

1. *There exists a unique solution $v \in H^{2,1}(Q_T)$ of problem (2.1)–(2.3) in Q_T and*

$$\|v\|_{H^{2,1}(Q_T)} \leq c_2(\|\varphi\|_{H^1} + \|K\|_{H^{1/2}}), \tag{2.39}$$

where $c_2 > 0$ depends on T but does not depend on φ and R.

2. *The interval $(0,T]$ contains no more than finitely many switching moments $t_1 < t_2 < \dots < t_J$ of $\mathcal{H}(v)$ and*

$$t_i - t_{i-1} \geq \tau^*, \quad i = \begin{cases} 1, 2, \dots, J & \text{if } \hat{\varphi} \leq \alpha \text{ or } \hat{\varphi} \geq \beta, \\ 2, 3, \dots, J & \text{if } \alpha < \hat{\varphi} < \beta, \end{cases} \tag{2.40}$$

where $t_0 = 0$ and

$$\tau^* = \text{const} \frac{(\beta - \alpha)^2}{\|m\|_{L_2}^2} \tag{2.41}$$

with const > 0 depending on R rather than on m, α, β, φ, and T.

Proof. 1. Without loss of generality, we assume that

$$\hat{\varphi} = \int_Q m(x)\varphi(x)\,dx = \alpha$$

and modify problem (2.1)–(2.3) by replacing $\mathcal{H}(\hat{v})(t)$ by 1 in it, i.e., consider problem (2.5)–(2.7).

By Lemma 2.3, there is a unique solution $v^{(1)} \in H^{2,1}(Q_T)$ of problem (2.5)–(2.7) and

$$\|v^{(1)}\|_{H^{2,1}(Q_T)} \le k_1(\|\varphi\|_{H^1} + \|K\|_{H^{1/2}}), \tag{2.42}$$

where $k_1 > 0$ depends on T but does not depend on R and φ.

As in the proof of Theorem 2.1, either $\widehat{v^{(1)}}(t) < \beta$ for $t < T$ or there is a first switching moment of $\mathcal{H}(\widehat{v^{(1)}})$, i.e., a number $t_1 \in (0, T)$ such that $\widehat{v^{(1)}}(t) < \beta$ for $t < t_1$ and $\widehat{v^{(1)}}(t_1) = \beta$.

In the first case, we obtain a solution by setting $v = v^{(1)}$.

Consider the second case. Inequalities (2.32) and (2.33) imply that

$$t_1 - t_0 \ge \left(\frac{\beta - \alpha}{k_1'\|m\|_{L_2}(\|\varphi\|_{H^1} + \|K\|_{H^{1/2}})}\right)^2, \tag{2.43}$$

where $t_0 = 0$ and $k_1' = c_1(T_1) > 0$ $\left(T_1 = t^* + \dfrac{2(\beta-\alpha)}{m_0K_0}\right)$ depends on R but does not depend on T and φ.

2. Now we consider problem (2.1)–(2.3) with $\mathcal{H}(\hat{v})(t)$ replaced by -1 and the initial value $v^{(1)}(x, t_1)$, i.e.,

$$v_t^{(2)}(x,t) = \Delta v^{(1)}(x,t) \quad ((x,t) \in Q_T), \tag{2.44}$$

$$v^{(2)}(x,0) = v^{(1)}(x,t_1) \quad (x \in Q), \tag{2.45}$$

$$\frac{\partial v^{(2)}}{\partial \nu} = -K(x) \quad (x \in \partial Q,\ t \in (0,T)). \tag{2.46}$$

Similarly to part 1 of the proof, we see that problem (2.44)–(2.46) has a unique solution $v^{(2)} \in H^{2,1}(Q_T)$ and

$$\|v^{(2)}\|_{H^{2,1}(Q_T)} \le k_1(\|v^{(1)}(\cdot,t_1)\|_{H^1} + \|K\|_{H^{1/2}}) \le k_2(\|\varphi\|_{H^1} + \|K\|_{H^{1/2}}), \tag{2.47}$$

where $k_2, k_3, \dots > 0$ depend on T but do not depend on R and φ.

As in part 1, $\widehat{v^{(2)}}(t)$ is continuous on $[0,T]$ and either $\widehat{v^{(2)}}(t) > \alpha$ for $t < T - t_1$ or there is a first switching moment of $\mathcal{H}(\widehat{v^{(2)}})$, i.e., a number $\tau_2 \in (0, T - t_1)$ such that $\widehat{v^{(2)}}(t) > \alpha$ for $t < \tau_2$ and $\widehat{v^{(2)}}(\tau_2) = \alpha$. We set $t_2 = t_1 + \tau_2$.

In the first case, we obtain a solution by setting $v(x,t) = v^{(1)}(x,t)$ for $(x,t) \in Q \times (0, t_1)$ and $v(x,t) = v^{(2)}(x, t - t_1)$ for $(x,t) \in Q \times (t_1, T)$.

Consider the second case. Similarly to (2.43) we obtain

$$t_2 - t_1 = \tau_2 \ge \left(\frac{\beta - \alpha}{k_1'\|m\|_{L_2}(\|v^{(1)}(\cdot,t_1)\|_{H^1} + \|K\|_{H^{1/2}})}\right)^2. \tag{2.48}$$

3. One can continue the above procedure and obtain a series of switching moments $t_1, t_2, \dots$ ($t_1, \tau_2, \tau_3, \dots$, respectively) and a series of the corresponding solutions $v^{(i)}(\cdot, t)$ on the intervals $(0, \tau_i)$ satisfying

$$\|v^{(i)}\|_{H^{2,1}(Q_T)} \le k_i(\|\varphi\|_{H^1} + \|K\|_{H^{1/2}}), \tag{2.49}$$

$$t_i - t_{i-1} = \tau_i \geq \left(\frac{\beta - \alpha}{k_1' \|m\|_{L_2} (\|v^{(i-1)}(\cdot, \tau_{i-1})\|_{H^1} + \|K\|_{H^{1/2}})} \right)^2. \tag{2.50}$$

To complete the proof, we have to show that the differences between the consecutive switching moments $t_1, t_2, \dots$ are separated from zero, i.e. the norms $\|v^{(i)}(\cdot, \tau_i)\|_{H^1}$ are bounded uniformly with respect to φ and i.

Indeed, let $v_j^{(i)}(t)$ be the Fourier coefficient of $v^{(i)}(\cdot, t)$. Using (2.23), we have for $t \in [0, \tau_i]$

$$\begin{aligned} |v_j^{(i)}(t)| &\leq \max \left(|v_j^{(i)}(0)|, \frac{|K_j|}{\lambda_j} \right) \\ &= \max \left(|v_j^{(i-1)}(\tau_{i-1})|, \frac{|K_j|}{\lambda_j} \right) \leq \dots \leq \max \left(|\varphi_j|, \frac{|K_j|}{\lambda_j} \right). \end{aligned}$$

Together with Lemma 2.1, this yields

$$\sum_{j=1}^{\infty} (1 + \lambda_j) \left| v_j^{(i)}(t) \right|^2 \leq R^2 + c^2 \|K\|_{H^{1/2}}^2, \tag{2.51}$$

where $c > 0$ does not depend on T and φ.

To estimate $v_0^{(i)}(\tau_i)$, we note that τ_i is a switching moment, which means that $\widehat{v^{(i)}}(\tau_i) = \alpha$ or β. Therefore, using (2.24), we see that

$$\sum_{j=0}^{\infty} m_j v_j^{(i)}(\tau_i) = \alpha \text{ or } \beta.$$

Hence,

$$\begin{aligned} |v_0^{(i)}(\tau_i)|^2 &\leq m_0^{-2} \left(\max(|\alpha|, |\beta|) + \sum_{j=1}^{\infty} |m_j| \cdot \left| v_j^{(i)}(\tau_i) \right| \right)^2 \\ &\leq 2m_0^{-2} \left(\max(\alpha^2, \beta^2) + \|m\|_{L_2}^2 \sum_{j=1}^{\infty} \left| v_j^{(i)}(\tau_i) \right|^2 \right), \end{aligned} \tag{2.52}$$

where $m_0 > 0$ due to Condition 2.1.

Combining inequalities (2.51) and (2.52), we obtain

$$\|v^{(i)}(\cdot, t_i)\|_{H^1}^2 = \sum_{j=0}^{\infty} (1 + \lambda_j) \left| v_j^{(i)}(t) \right|^2 \leq k'' \left(R^2 + c^2 \|K\|_{H^{1/2}}^2 \right) + 2m_0^{-2} \max(\alpha^2, \beta^2),$$

where $k'' > 0$ does not depend on T, R and φ. □

Definition 2.5. We say that $v(x, t)$ $(t \geq 0)$ is a *solution of problem* (2.1)–(2.3) *in* Q_∞ if it is a solution in Q_T for all $T > 0$.

Theorem 2.2 implies the following result.

Corollary 2.1. *For any* $\varphi \in H^1$*, there is a unique solution of problem* (2.1)–(2.3) *in* Q_∞*.*

2.4. **Attracting set.** In this subsection, we make some remarks on the large-time behavior of solutions of problem (2.1)–(2.3) and specify an attracting set, in which periodic solutions may lie.

We begin with the following remark.

Remark 2.5. Relations (2.22) imply the following.

1. If $K_j \ne 0$ ($j \ne 0$), then there is a time moment $\theta_j = \theta_j(\varphi_j) \ge 0$ such that
$$|v_j(t)| < \frac{|K_j|}{\lambda_j} \quad \text{for } t \ge \theta_j.$$
2. If $K_j = 0$ ($j \ne 0$), then $v_j(t) \to 0$ exponentially as $t \to \infty$.

The above properties give rise to the consideration of the set
$$B_0 = \left\{ \psi \in H^1 : |\psi_j| \le \frac{|K_j|}{\lambda_j},\ j = 1, 2, \dots \right\} \tag{2.53}$$
Note that B_0 is a closed set in H^1 positively invariant for the solutions of problem (2.1)–(2.3) in Q_∞. Denote by $\operatorname{dist}(\varphi, B_0)$ the distance in H^1 between the element $\varphi \in H^1$ and the set B_0. It is easy to see that
$$\operatorname{dist}^2(\varphi, B_0) = \sum_{j=1}^{\infty} (1 + \lambda_j) \operatorname{dist}^2(\varphi_j, S_j), \tag{2.54}$$
where $S_j = [-|K_j|/\lambda_j, |K_j|/\lambda_j]$ is the jth side of the "box" B_0. If $K_j = 0$, we set $S_j = \{0\}$.

The following theorem shows that the set B_0 is a so-called B-attracting set, i.e., it uniformly attracts elements from bounded sets.

Theorem 2.3. *Let $R > 0$, $\|\varphi\|_{H^1} \le R$, and $v(x, t)$ be a solution of problem (2.1)–(2.3) in Q_∞. Then $\operatorname{dist}(v(\cdot, t), B_0)$ monotonically decreases and tends to zero as $t \to \infty$, uniformly with respect to φ, $\|\varphi\|_{H^1} \le R$.*

Proof. 1. Since $\operatorname{dist}(v_j(t), S_j)$ monotonically decreases (and vanishes within finite time if $S_j \ne \{0\}$), it follows from (2.54) that $\operatorname{dist}(v(\cdot, t), B_0)$ monotonically decreases.

2. Without loss of generality, we assume that $\hat\varphi = \alpha$. Denote by t_i the switching moments of $\mathcal{H}(\hat v)$. Consider the set of indices
$$\mathbb{D}(i) = \left\{ j \in \mathbb{N} : |v_j(t_i)| > \frac{|K_j|}{\lambda_j} \right\}.$$
Then, due to (2.54),
$$\operatorname{dist}^2(v(\cdot, t_i), B_0) = \sum_{j \in \mathbb{D}(i)} (1 + \lambda_j) \operatorname{dist}^2(v_j(t_i), S_i) \le \sum_{j \in \mathbb{D}(i)} (1 + \lambda_j) |v_j(t_i)|^2. \tag{2.55}$$

Let us estimate $v_j(t_i)$ for $j \in \mathbb{D}(i)$. Note that $t_i \ge \tau^*(i - 1)$ by (2.40), where $\tau^* > 0$ does not depend on φ. Thus, if $K_j = 0$, then
$$|v_j(t_i)| = |\varphi_j| e^{-\lambda_j t_i} \le |\varphi_j| e^{-\lambda_1 \tau^*(i-1)}. \tag{2.56}$$

Consider the case $K_j \ne 0$.

Let $v_j(t_i) > K_j/\lambda_j > 0$ (the other cases are similar). Then
$$\varphi_j > v_j(t_1) > v_j(t_2) > \cdots > v_j(t_i) > K_j/\lambda_j.$$

Consider even i. Using part 2 of Lemma 2.2 and (2.40), we have

$$v_j(t_i) = e^{-\lambda_j(t_i - t_{i-1})}\left(v_j(t_{i-1}) + \frac{K_j}{\lambda_j}\right) - \frac{K_j}{\lambda_j} \le e^{-\lambda_1 \tau^*}\left(v_j(t_{i-2}) + \frac{K_j}{\lambda_j}\right) - \frac{K_j}{\lambda_j}.$$

Therefore, by induction,

$$v_j(t_i) + \frac{K_j}{\lambda_j} \le e^{-\lambda_1 \tau^* i/2}\left(\varphi_j + \frac{K_j}{\lambda_j}\right).$$

Thus,

$$0 < v_j(t_i) \le e^{-\lambda_1 \tau^* i/2}\left(\varphi_j + \frac{K_j}{\lambda_j}\right). \tag{2.57}$$

Combining (2.56) and (2.57) with estimate (2.55) and using Lemma 2.1 yields

$$\begin{aligned} \operatorname{dist}^2(v(\cdot, t_i), B_0) \le e^{-2\lambda_1 \tau^*(i-1)} &\sum_{j \in \mathbb{D}(i), K_j = 0} (1+\lambda_j)|\varphi_j|^2 \\ &+ 2e^{-\lambda_1 \tau^* i} \sum_{j \in \mathbb{D}(i), K_j \ne 0} (1+\lambda_j)\left(|\varphi_j|^2 + \left|\frac{K_j}{\lambda_j}\right|^2\right) \le k_1 e^{-\lambda_1 \tau^* i}, \end{aligned}$$

where $k_1 = k_1(R) > 0$.

Therefore, $\operatorname{dist}(v(\cdot, t_i), B_0) \to 0$ as $i \to \infty$ uniformly with respect to φ, $\|\varphi\|_{H^1} \le R$. Taking into account the monotonicity of $\operatorname{dist}(v(\cdot, t), B_0)$ for all $t \ge 0$, we complete the proof. □

3. **Existence of periodic solution.**

3.1. **Preliminary considerations.**

In this section, we establish the existence of periodic solutions of the thermocontrol problem.

Definition 3.1. A function $v(x, t)$ is called a *T-periodic solution of problem* (2.1), (2.3) if there is a function $\varphi \in H^1$ such that

1. v is a solution of problem (2.1)–(2.3) in Q_∞ with the initial data φ,
2. $v(\cdot, t)$ and $\mathcal{H}(\hat{v})(t)$ are T-periodic in t for $t \ge 0$.

Due to the uniqueness part in Theorem 2.2, one can give the following equivalent definition of a periodic solution.

Definition 3.2. A function $v(x, t)$ is called a *T-periodic solution of problem* (2.1), (2.3) if there is a function $\varphi \in H^1$ such that the following holds:

1. v is a solution of problem (2.1)–(2.3) in Q_T with the initial data φ,
2. $v(x, T) = v(x, 0)$ $(= \varphi(x))$ and $\mathcal{H}(\hat{v})(T) = \mathcal{H}(\hat{v})(0)$.

Remark 3.1. Throughout the paper, we are dealing with periodic solutions of a special form only. Namely, if a T-periodic solution $v(x, t)$ satisfies $\hat{v}(\theta) = \hat{v}(\theta + T) = \alpha$ for some θ, then there is exactly one switching moment θ_1 on the interval $(\theta, \theta + T)$ and $\hat{v}(\theta_1) = \beta$.

Remark 3.2. Similarly, one can define a *mild T-periodic solution of problem* (2.1), (2.3). However, Lemma 2.2 implies that $\varphi = v(\cdot, T) \in H^2$. Therefore, by Theorem 2.2, any mild periodic solution is a strong periodic solution.

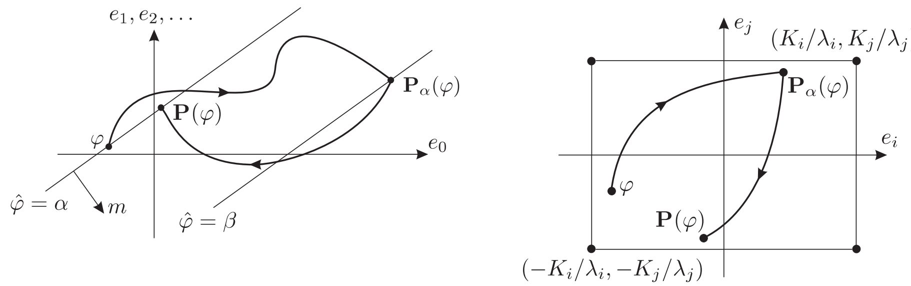

FIGURE 3.1. The operators $\mathbf{P}_\alpha$ and $\mathbf{P} = \mathbf{P}_\beta \mathbf{P}_\alpha$ on the planes (e_0, m) and (e_i, e_j), $i \neq j$

We consider nonlinear operators (see Fig. 3.1)

$$\mathbf{P}_\alpha : \{\varphi \in H^1 : \hat\varphi < \beta\} \to \{\varphi \in H^1 : \hat\varphi = \beta\},$$
$$\mathbf{P}_\beta : \{\varphi \in H^1 : \hat\varphi > \alpha\} \to \{\varphi \in H^1 : \hat\varphi = \alpha\}.$$

The operator $\mathbf{P}_\alpha$ is defined as follows. Let $\varphi \in H^1$ and $\hat\varphi < \beta$. Due to Lemma 2.3, for any $t_1 > 0$, there is a unique solution $v^\alpha(x,t)$ of the problem

$$v_t(x,t) = \Delta v(x,t) \quad ((x,t) \in Q_{t_1}), \tag{3.1}$$
$$v(x,0) = \varphi(x) \quad (x \in Q), \tag{3.2}$$
$$\frac{\partial v}{\partial \nu} = K(x) \quad (x \in \partial Q,\ t \in (0,t_1)). \tag{3.3}$$

By Lemma 2.2 (part 4), there exists $t_1 > 0$ such that $\widehat{v^\alpha}(t) < \beta$ for $t \in (0,t_1)$ and $\widehat{v^\alpha}(t_1) = \beta$. We set $\mathbf{P}_\alpha(\varphi) = v(\cdot, t_1)$.

The operator $\mathbf{P}_\beta$ is defined in a similar way. Let $\varphi \in H^1$ and $\hat\varphi > \alpha$. Due to Lemma 2.3, for any $\tau_2 > 0$, there is a unique solution $v^\beta(x,t)$ of the problem

$$v_t(x,t) = \Delta v(x,t) \quad ((x,t) \in Q_{\tau_2}), \tag{3.4}$$
$$v(x,0) = \varphi(x) \quad (x \in Q), \tag{3.5}$$
$$\frac{\partial v}{\partial \nu} = -K(x) \quad (x \in \partial Q,\ t \in (0,\tau_2)). \tag{3.6}$$

By Lemma 2.2 (part 4), there exists $\tau_2 > 0$ such that $\widehat{v^\beta}(t) > \alpha$ for $t \in (0,\tau_2)$ and $\widehat{v^\beta}(\tau_2) = \alpha$. We set $\mathbf{P}_\beta(\varphi) = v(\cdot, \tau_2)$.

Since the set B_0 given by (2.53) is attracting in H^1, it is natural to seek for periodic solutions in this set. To prove that the periodic solution is attracting itself, we will also need to study the behavior of trajectories in a neighborhood of B_0. Let B_ε be the closed ε-neighborhood of B_0 in the H^1-topology:

$$B_\varepsilon = \bigcup_{\varphi \in B_0} \{\psi \in H^1 : \|\psi - \varphi\|_{H^1} \le \varepsilon\}, \quad 0 \le \varepsilon \le \varepsilon_0, \tag{3.7}$$

where $0 < \varepsilon_0 \le 1$ is specified in Remark 3.4 below.

Denote

$$B_{\alpha,\varepsilon} = B_\varepsilon \cap \{\varphi \in H^1 : \hat\varphi = \alpha\}, \qquad B_{\beta,\varepsilon} = B_\varepsilon \cap \{\varphi \in H^1 : \hat\varphi = \beta\}.$$

We note that, once the components $\varphi_1, \varphi_2, \ldots$ of any element $\varphi \in B_{\alpha,\varepsilon}$ $(B_{\beta,\varepsilon})$ are fixed, the component φ_0 is uniquely determined by the equality $\hat\varphi = \alpha$ (β), cf. (2.24).

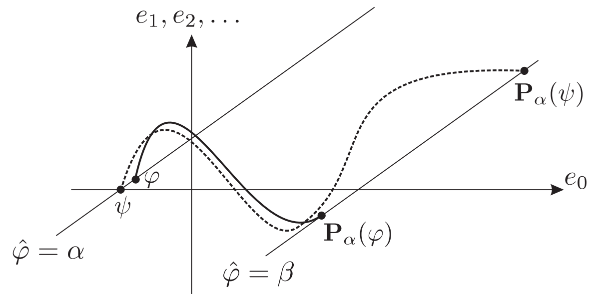

FIGURE 3.2. Discontinuity of $\mathbf{P}_\alpha$ if $d\hat{v}(t_1)/dt = 0$

We shall show that $B_{\alpha,\varepsilon}$ and $B_{\beta,\varepsilon}$ are closed bounded convex sets and the composition $\mathbf{P}(\varphi) = \mathbf{P}_\beta \mathbf{P}_\alpha(\varphi)$ is a compact continuous operator on $B_{\alpha,\varepsilon}$. Then the Schauder fixed point theorem ensures the existence of a fixed point $\varphi \in H^1$ of the operator $\mathbf{P}$. The corresponding periodic solution of period $T = t_1 + \tau_2$ is then given by $v(x,t) = v^\alpha(x,t)$ for $x \in Q$, $t \in (0,t_1)$ and $v(x,t) = v^\beta(x,t-t_1)$ for $x \in Q$, $t \in (t_1, T)$.

The following results from the definition of $B_{\alpha,\varepsilon}$ and $B_{\beta,\varepsilon}$ and from Lemma 2.1 (part 2).

Lemma 3.1. *The sets $B_{\alpha,\varepsilon}$ and $B_{\beta,\varepsilon}$ are closed bounded convex sets in H^1.*

Lemma 3.2. $\mathbf{P}_\alpha(B_{\alpha,\varepsilon}) \subset B_{\beta,\varepsilon}$ *and* $\mathbf{P}_\beta(B_{\beta,\varepsilon}) \subset B_{\alpha,\varepsilon}$.

Proof. The result follows from the fact that $\operatorname{dist}(v(\cdot,t), B_0)$ monotonically decreases (see Theorem 2.3). □

Lemma 3.3. *The operators $\mathbf{P}_\alpha : B_{\alpha,\varepsilon} \to B_{\beta,\varepsilon}$ and $\mathbf{P}_\beta : B_{\beta,\varepsilon} \to B_{\alpha,\varepsilon}$ are compact.*

Proof. It follows from (2.33) and (2.40) that, for any $\varphi \in B_{\alpha,\varepsilon} \subset H^1$, the first switching moment t_1 satisfies

$$0 < \tau^* \le t_1 \le t^* + \frac{2(\beta - \alpha)}{m_0 K_0},$$

where τ^* and t^* do not depend on φ from the bounded set $B_{\alpha,\varepsilon}$. Now the compactness of $\mathbf{P}_\alpha$ is a consequence of part 1 of Lemma 2.2 and the compactness of the embedding $H^2 \subset H^1$. The compactness of $\mathbf{P}_\beta$ is proved in the similar way. □

To apply the Schauder fixed point theorem, it remains to prove that the operator $\mathbf{P}_\alpha$ ($\mathbf{P}_\beta$) is continuous. The continuity may fail if, for some $\varphi \in B_{\alpha,\varepsilon}$, we have $d\hat{v}(t_1)/dt = 0$, where $v(x,t) = v^\alpha(x,t)$ is the solution of problem (3.1)–(3.3) (see Fig 3.2). We note that this difficulty is inherent not only in parabolic problems but in systems of ordinary differential equations with hysteresis, too (see, e.g., [3]).

Lemma 2.2 ensures that the mean temperature $\hat{v}(t)$ is increasing for sufficiently large t. We will show that the first switching occurs for such t, provided that $\beta - \alpha$ is not too small and the trajectories originate from $B_{\alpha,\varepsilon}$ (and even from a wider set). This will guarantee that $d\hat{v}(t_1)/dt > 0$.

Lemma 3.4. *There is a number $t_{\min} > 0$ which does not depend on $\varphi, T, \alpha, \beta$ such that, for any $\varphi \in H^1$, the solution $v(x,t) = v^\alpha(x,t)$ $(v^\beta(x,t))$ of problem* (3.1)–(3.3)

((3.4)–(3.6)) *satisfies*

$$\frac{m_0 K_0}{2} \le \frac{d\hat{v}(t)}{dt} \le \frac{3m_0 K_0}{2}, \quad t \ge t_{\min}, \tag{3.8}$$

provided that

$$\left(\sum_{j=1}^{\infty}(1+\lambda_j)|\varphi_j|^2\right)^{1/2} \le 2c\|K\|_{H^{1/2}}. \tag{3.9}$$

Proof. The result follows from part 4 of Lemma 2.2 by setting $t_{\min} = t^*(R^*)$, where $R^* = 2c\|K\|_{H^{1/2}}$. □

Now we formulate a sufficient condition in terms of the difference of the "temperature thresholds" $\beta - \alpha$ under which $d\hat{v}(t_1)/dt \ne 0$. The continuity of $\mathbf{P}_\alpha$ will follow from the implicit function theorem.

Set

$$C = 3c\tilde{m}\|K\|_{H^{1/2}}, \qquad \tilde{m} = \left(\sum_{j=1}^{\infty}|m_j|^2\right)^{1/2}, \tag{3.10}$$

where $c > 0$ is the constant from Lemma 2.1. Note that $\tilde{m} \equiv 0$ if and only if $m(x) \equiv \text{const}$. Let $t_{\min}$ be the number from Lemma 3.4 (which does not depend on $\varphi, T, \alpha, \beta$).

Condition 3.1. $\beta - \alpha > m_0 K_0 t_{\min} + C$.

The following lemma plays a fundamental role in the proof of the continuity of the operators $\mathbf{P}_\alpha$ and $\mathbf{P}_\beta$.

Lemma 3.5. *Let Condition 3.1 hold. Then there is $\delta > 0$ such that, for any $\varphi \in H^1$ satisfying*

$$\left(\sum_{j=1}^{\infty}(1+\lambda_j)|\varphi_j|^2\right)^{1/2} < 2c\|K\|_{H^{1/2}}, \qquad |\hat{\varphi} - \alpha| < \delta \quad (|\hat{\varphi} - \beta| < \delta), \tag{3.11}$$

the solution $v^\alpha(x,t)$ $(v^\beta(x,t))$ of problem (3.1)–(3.3) ((3.4)–(3.6)) satisfies

$$\frac{m_0 K_0}{2} \le \frac{d\widehat{v^\alpha}(t_1)}{dt}, \frac{d\widehat{v^\beta}(\tau_2)}{dt} \le \frac{3m_0 K_0}{2}.$$

Proof. Using representation (2.24), the formula for the solutions of (2.22), and Lemma 2.1, we have for $t \le t_{\min}$ and, e.g., $v(x,t) = v^\alpha(x,t)$:

$$\begin{aligned}\hat{v}(t) &= \hat{v}(0) + m_0 K_0 t + \sum_{j=1}^{\infty} m_j \left(\varphi_j - \frac{K_j}{\lambda_j}\right)\left(e^{-\lambda_j t} - 1\right) \\ &\le \alpha + \delta + m_0 K_0 t_{\min} + \left(\sum_{j=1}^{\infty}|m_j \varphi_j| + \sum_{j=1}^{\infty}\left|m_j \frac{K_j}{\lambda_j}\right|\right) \\ &\le \alpha + \delta + m_0 K_0 t_{\min} + C < \beta,\end{aligned}$$

provided that $\delta > 0$ is small enough. This implies that the switching moment t_1 is greater than $t_{\min}$. Now the desired result follows from Lemma 3.4. □

Remark 3.3. In what follows, we will write $(3.11)_\alpha$ or $(3.11)_\beta$ depending on whether we mean the neighborhood for which $|\hat{\varphi}-\alpha|<\delta$ or $|\hat{\varphi}-\beta|<\delta$, respectively.

Remark 3.4. Now we choose the number ε_0 which bounds the numbers ε in the definition of the set $B_{\alpha,\varepsilon}$ $(B_{\beta,\varepsilon})$. First we fix $\delta>0$ from Lemma 3.5. Then we choose $\varepsilon_0=\varepsilon_0(\delta)>0$ so that $B_{\alpha,\varepsilon}$ $(B_{\beta,\varepsilon})$ is contained in the open neighborhood $(3.11)_\alpha$ $((3.11)_\beta)$.

3.2. **Existence of a periodic solution.** In this subsection, we establish the existence of a periodic solution of the thermocontrol problem (2.1), (2.3).

Theorem 3.1. *Let Condition 3.1 hold. Then there is a periodic solution of problem* (2.1)*,* (2.3)*.*

Due to Lemmas 3.1, 3.2, and 3.3 and the Schauder fixed point theorem, it remains to prove the continuity of the operator $\mathbf{P}_\beta\mathbf{P}_\alpha: B_{\alpha,\varepsilon}\to B_{\alpha,\varepsilon}$. To do so, we prove the continuity (and even the Fréchet differentiability) of the operator

$$\mathbf{P}_\alpha: B_{\alpha,\varepsilon}\to B_{\beta,\varepsilon}.$$

The arguments for $\mathbf{P}_\beta$ are the same.

We introduce the linear continuous operator $\mathbf{M}: H^1\to\mathbb{R}$ given by

$$\mathbf{M}\varphi=\hat{\varphi}\quad\left(=\int_Q m(x)\varphi(x)\,dx\right).$$

We recall that $\mathbf{S}_t'$ $(t>0)$ denotes the derivative of the analytic semigroup $\mathbf{S}_t: L_2\to L_2$ (see Sec. 2). It is well known that the linear operators $\mathbf{S}_t,\mathbf{S}_t': L_2\to \mathrm{D}(P)\subset H^2$ are bounded for any fixed $t>0$, where P is given by (2.13). Therefore, they are bounded as the operators acting from H^1 into itself.

We consider the operator $\mathbf{v}: H^1\times\mathbb{R}\to H^1$ given by

$$\mathbf{v}(\varphi,t)=\mathbf{S}_t(\varphi-v_K)+ft+v_K,\quad t>0. \tag{3.12}$$

Clearly, the function $v(\cdot,t)=\mathbf{v}(\varphi,t)$ coincides with the solution of the thermocontrol problem (2.1)–(2.3) with the initial data φ for $t\le t_1$, where t_1 is the first switching moment of $\mathcal{H}(\hat{v})$.

We also consider the operator (functional) $\mathbf{t}_1: H^1\to\mathbb{R}$ given by

$$\mathbf{t}_1(\varphi)=\text{the first switching moment of }\mathcal{H}(\hat{v})$$

and defined in the neighborhood $(3.11)_\alpha$ (which does not intersect with the set $\{\hat{\varphi}=\beta\}$).

Due to the continuity of $\hat{v}(t)$, we have $\mathbf{t}_1(\varphi)>0$ for any φ from the above neighborhood of $B_{\alpha,\varepsilon}$.

Clearly,

$$\mathbf{P}_\alpha(\varphi)=\mathbf{v}(\varphi,\mathbf{t}_1(\varphi)).$$

Since $\mathbf{v}(\varphi,t)$ is continuous with respect to (φ,t), the operator $\mathbf{P}_\alpha(\varphi)$ is also continuous, provided that $\mathbf{t}_1(\varphi)$ is continuous. Thus, we will prove that $\mathbf{t}_1(\varphi)$ is continuous (and even continuously Fréchet-differentiable).

Lemma 3.6. *Let Condition 3.1 be satisfied. Then the following assertions hold for φ from the neighborhood $(3.11)_\alpha$.*

1. *The operator $\mathbf{v}(\varphi,t)$ is continuously differentiable with respect to φ, t, and (φ,t), where $t>0$. Moreover,*

(a) $D_\varphi \mathbf{v}(\varphi,t) : H^1 \to H^1$ *is given by*

$$D_\varphi \mathbf{v}(\varphi,t)\psi = \mathbf{S}_t\psi \qquad \forall \psi \in H^1,\ t>0, \tag{3.13}$$

(b) $D_t\mathbf{v}(\varphi,t) : \mathbb{R} \to H^1$ *is given by*

$$D_t\mathbf{v}(\varphi,t)\theta = (\mathbf{S}_t'(\varphi - v_K) + f)\theta \qquad \forall \theta \in \mathbb{R},\ t>0, \tag{3.14}$$

(c) $D\mathbf{v}(\varphi,t) : H^1 \times \mathbb{R} \to H^1$ *is given by*

$$D\mathbf{v}(\varphi,t)(\psi,\theta) = D_\varphi\mathbf{v}(\varphi,t)\psi + D_t\mathbf{v}(\varphi,t)\theta \qquad \forall (\psi,\theta) \in H^1\times\mathbb{R},\ t>0. \tag{3.15}$$

2. *The operator* $\mathbf{t}_1(\varphi)$ *is continuously differentiable in the neighborhood* $(3.11)_\alpha$. *The linear bounded operator* $D_\varphi\mathbf{t}_1(\varphi) : H^1 \to \mathbb{R}$ *is given by*

$$D_\varphi\mathbf{t}_1(\varphi)\psi = -\left(\frac{d\hat{v}(t_1)}{dt}\right)^{-1}\mathbf{M}\mathbf{S}_{t_1}\psi \qquad \forall\psi\in H^1. \tag{3.16}$$

Here $v(x,t) = v^\alpha(x,t)$ *is a solution of problem* (3.1)–(3.3) *with the initial data* φ *and* $t_1 = \mathbf{t}_1(\varphi)$ *is the first switching moment of* $\mathcal{H}(\hat{v})$.

Proof. Part 1 follows from (3.12) and the fact that $\mathbf{S}_t, \mathbf{S}_t' : H^1 \to H^1$ are bounded for $t>0$.

Let us prove part 2. Clearly, $\mathbf{t}_1(\varphi)$ is the least (positive) solution of the equation

$$\hat{\mathbf{v}}(\varphi,t_1) := \mathbf{M}\mathbf{v}(\varphi,t_1) = \beta,$$

which exists for any $\varphi \in H^1$ (and even from L_2) by Lemma 2.2.

Since $\mathbf{M}$ is a linear continuous operator, it follows that $\hat{\mathbf{v}} = \mathbf{M}\mathbf{v} : H^1\times\mathbb{R}\to\mathbb{R}$ is also continuously differentiable with respect to φ, t, and (φ,t). Moreover, using (3.14), the Fourier representation of the semigroup (2.17) and of the solution (2.21), we see that $D_t\hat{\mathbf{v}}(\varphi,t) : \mathbb{R}\to\mathbb{R}$ is given by

$$D_t\hat{\mathbf{v}}(\varphi,t)\theta = \mathbf{M}(\mathbf{S}_t'(\varphi - v_K) + f)\theta = \frac{d\hat{v}(t)}{dt}\theta \qquad \forall\theta\in\mathbb{R},\ t>0. \tag{3.17}$$

Now Lemma 3.5 and Remark 3.4 imply that there is a bounded inverse operator $[D_t\hat{\mathbf{v}}(\varphi,t)]^{-1} = \left(\frac{d\hat{v}(t)}{dt}\right)^{-1}$ continuous in a neighborhood of any $(\varphi,\mathbf{t}_1(\varphi))$, φ in the neighborhood $(3.11)_\alpha$.

By the implicit function theorem, $\mathbf{t}_1(\varphi)$ is continuously differentiable and the derivative $D_\varphi\mathbf{t}_1(\varphi) : H^1\to\mathbb{R}$ is given by (see (3.13) and (3.17))

$$D_\varphi\mathbf{t}_1(\varphi)\psi = -(D_t\hat{\mathbf{v}}(\varphi,t_1))^{-1}D_\varphi\hat{\mathbf{v}}(\varphi,t_1)\psi = -\left(\frac{d\hat{v}(t_1)}{dt}\right)^{-1}\mathbf{M}\mathbf{S}_{t_1}\psi \qquad \forall\psi\in H^1.$$

□

Corollary 3.1. *Let Condition 3.1 hold. Then the operator* $\mathbf{t}_1 : H^1\to\mathbb{R}$ *is uniformly Lipschitz continuous on* $B_{\alpha,\varepsilon}$.

Proof. Formula (3.16), Remark 3.4, and estimates (2.33) and (2.40) imply that, for any $\varphi,\psi\in B_{\alpha,\varepsilon}$,

$$|\mathbf{t}_1(\varphi) - \mathbf{t}_1(\psi)| \le \sup_{\xi\in B_{\alpha,\varepsilon}}\|D_\xi\mathbf{t}_1(\xi)\|\cdot\|\varphi-\psi\|_{H^1} \le \frac{2}{m_0K_0}\|\mathbf{M}\|\sup_\tau\|\mathbf{S}_\tau\|\cdot\|\varphi-\psi\|_{H^1},$$

where the last supremum is taken over $\tau\in\left[\tau^*, t_{\min} + \frac{2(\beta-\alpha)}{m_0K_0}\right]$. The latter fact ensures that $\sup_\tau\|\mathbf{S}_\tau\| < \infty$, and the corollary is proved. □

Lemma 3.7. *Let Condition 3.1 hold. Then the operator $\mathbf{P}_\alpha(\varphi)$ is continuously differentiable in the neighborhood $(3.11)_\alpha$ of $B_{\alpha,\varepsilon}$. The linear bounded operator $D_\varphi \mathbf{P}_\alpha(\varphi) : H^1 \to H^1$ is given by*

$$D_\varphi \mathbf{P}_\alpha(\varphi) = \mathbf{S}_{t_1} - \left(\frac{d\hat{v}(t_1)}{dt}\right)^{-1} \cdot \left(\mathbf{S}'_{t_1}(\varphi - v_K) + f\right) \cdot \mathbf{M}\mathbf{S}_{t_1}, \tag{3.18}$$

where $v(x,t) = v^\alpha(x,t)$ is the solution of problem (3.1)–(3.3) with the initial data φ and $t_1 = \mathbf{t}_1(\varphi)$ is the first switching moment of $\mathcal{H}(\hat{v})$.

Proof. As we have mentioned before, $\mathbf{P}_\alpha(\varphi) = \mathbf{v}(\varphi, \mathbf{t}_1(\varphi))$. Therefore, using the chain rule and Lemma 3.6, we obtain

$$\begin{aligned} D_\varphi \mathbf{P}_\alpha(\varphi) &= D_\varphi \mathbf{v}(\varphi, t_1) + D_t \mathbf{v}(\varphi, t_1) D_\varphi \mathbf{t}_1(\varphi) \\ &= \mathbf{S}_{t_1} - \left(\frac{d\hat{v}(t_1)}{dt}\right)^{-1} \cdot \left(\mathbf{S}'_{t_1}(\varphi - v_K) + f\right) \cdot \mathbf{M}\mathbf{S}_{t_1}. \end{aligned}$$

□

Now we can prove Theorem 3.1.

Proof of Theorem 3.1 follows from Lemmas 3.1, 3.2, 3.3, 3.7, and the Schauder fixed point theorem. □

Note that if $\hat{z}(0) = \alpha$, then $z(\cdot, 0) \in B_{\alpha,0}$. To conclude this subsection, we prove the following "symmetry" result (see Figures 3.3 and 3.4).

Theorem 3.2. *Let $v(x,t)$ be a T-periodic solution of problem (2.1), (2.3), and let $v(\cdot,0) = \varphi^\alpha \in B_{\alpha,0}$, t_1 be the first switching moment of $\mathcal{H}(\hat{v})$, and $v(\cdot, t_1) = \varphi^\beta$ $(\in B_{\beta,0})$. Then*

1. *the solution $z(x,t)$ of problem (2.1)–(2.3) with the initial data $z(\cdot,0) = \psi^\alpha$ $(\in B_{\alpha,0})$ given by*

$$\psi_j^\alpha = -\varphi_j^\beta \quad (j = 1, 2, \dots), \qquad \psi_0^\alpha = \frac{\alpha + \beta}{m_0} - \varphi_0^\beta$$

is a T-periodic solution of problem (2.1), (2.3) with the first switching moment of $\mathcal{H}(\hat{z})$ equal to $T - t_1$;

2. *$z(x,t) = \dfrac{\alpha+\beta}{m_0} e_0 - v(x, t + t_1)$;*
3. *if $\varphi_j^\alpha = -\varphi_j^\beta$ for at least one $j \geq 1$ with $K_j \neq 0$, then $t_1 = T/2$; if $t_1 = T/2$, then $z(x,t) \equiv v(x,t)$ and $\varphi_j^\alpha = -\varphi_j^\beta$ for all $j = 1, 2, \dots$.*

Proof. 1. First of all, we note that ψ^α indeed belongs to $B_{\alpha,0}$ because

$$\sum_{j=0}^\infty m_j \psi_j^\alpha = \alpha + \beta - \varphi_0^\beta m_0 - \sum_{j=1}^\infty m_j \varphi_j^\beta = \alpha + \beta - \beta = \alpha.$$

Let $t_2 = T - t_1$. We show that $z(\cdot, t_2) \in B_{\beta,0}$. Using Lemma 2.2, part 2, we have for $j = 1, 2$

$$z_j(t_2) = \left(\psi_j^\alpha - \frac{K_j}{\lambda_j}\right) e^{-\lambda_j t_2} + \frac{K_j}{\lambda_j} = -\left[\left(\varphi_j^\beta + \frac{K_j}{\lambda_j}\right) e^{-\lambda_j t_2} - \frac{K_j}{\lambda_j}\right] = -\varphi_j^\alpha, \tag{3.19}$$

$$z_0(t_2) = \psi_0^\alpha + K_0 t_2 = \frac{\alpha+\beta}{m_0} - \varphi_0^\beta + K_0 t_2 = \frac{\alpha+\beta}{m_0} - \varphi_0^\alpha. \tag{3.20}$$

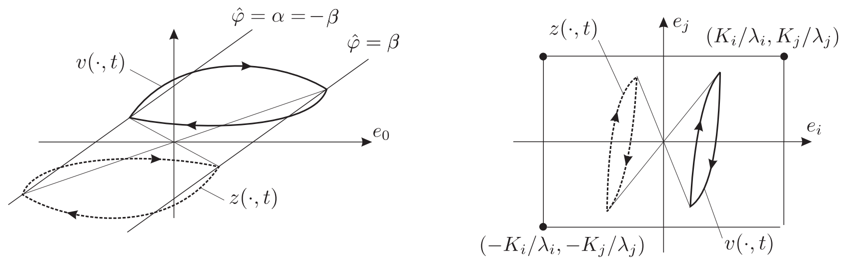

FIGURE 3.3. Periodic solutions $v(x,t)$ and $z(x,t) = \dfrac{\alpha+\beta}{m_0}e_0 - v(x,t+t_1)$ (parts 1 and 2 in Theorem 3.2)

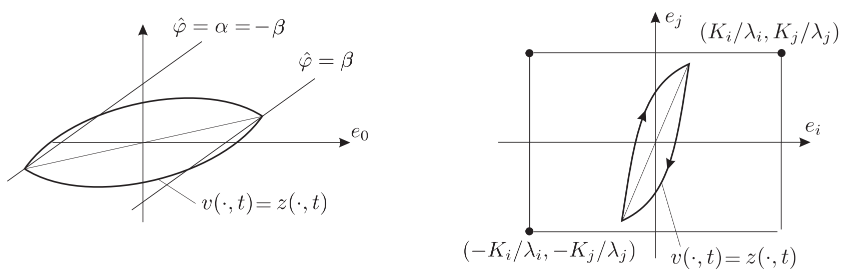

FIGURE 3.4. Periodic solution $v(x,t) = z(x,t)$, $t_1 = t_2 = T/2$ (part 3 in Theorem 3.2)

Therefore,

$$\sum_{j=0}^{\infty} m_j z_j(t_2) = \beta,$$

which proves that $z(\cdot,t_2) \in B_{\beta,0}$, i.e., the first switching moment of $\mathcal{H}(\hat{z})$ is equal to $t_2 = T - t_1$.

Similarly to (3.19) and (3.20), one can check that the second switching moment of $\mathcal{H}(\hat{z})$ is equal to $t_2 + t_1 = T$ and

$$z_j(T) = -\varphi_j^\beta = \psi_j^\alpha \quad (j = 1, 2, \dots), \qquad z_0(T) = \frac{\alpha+\beta}{m_0} - \varphi_0^\beta = \psi_0^\alpha,$$

which implies the first part of the theorem.

2. One can easily see that $\dfrac{\alpha+\beta}{m_0}e_0 - v(x,t_1) = \dfrac{\alpha+\beta}{m_0}e_0 - \varphi^\beta(x) = \psi^\alpha(x)$. On the other hand, $z(x,t)$ is a solution of problem (2.1)–(2.3) with the initial data $z(x,0) = \psi^\alpha(x)$. Therefore, part 2 of the theorem follows from the uniqueness part in Theorem 2.2.

3. The third part of the theorem follows from the relations (see part 2 of Lemma 2.2)

$$\varphi_j^\beta = v_j(t_1) = \left(\varphi_j^\alpha - \frac{K_j}{\lambda_j}\right)e^{-\lambda_j t_1} + \frac{K_j}{\lambda_j},$$

$$\varphi_j^\alpha = v_j(t_1+t_2) = \left(\varphi_j^\beta + \frac{K_j}{\lambda_j}\right)e^{-\lambda_j t_2} - \frac{K_j}{\lambda_j},$$

where $j = 1, 2, \dots$, which yield $\varphi_j^\alpha = -\varphi_j^\beta = \psi_j^\alpha$ $(j = 1, 2, \dots)$ if and only if $t_1 = t_2$ $(\neq 0)$. In particular, if $t_1 = t_2$, then $\varphi^\alpha = \psi^\alpha$ and, hence, $z(x,t) \equiv v(x,t)$ by the uniqueness part in Theorem 2.2. $\square$

4. **Global and phase-shifted attraction to periodic solution.** Under a condition that is slightly stronger than Condition 3.1, we prove that there is a unique periodic solution of problem (2.1), (2.3). To do so, we show that the Poincaré map is a contraction on the corresponding set. Then we show that any other solution is attracted to $z(x, t-\delta)$ with some δ (depending on the solution) exponentially fast.

Our plan is as follows.

1. We consider the set $B_{\alpha,\varepsilon}$ as the Poincaré cross-section at the point $z(\cdot, 0)$. Then we define the space $\mathcal{W}$ by the equality $H^1 = \mathrm{Span}(e_0) \oplus \mathcal{W}$ and show that $B_{\alpha,\varepsilon}$ is represented as the graph of an operator-valued function $\varphi = \mathbf{R}_\alpha(\Phi)$ for Φ in a subset of $\mathcal{W}$, where $\mathbf{R}_\alpha$ is an affine function (see Fig. 4.1).
2. We consider the projection $\mathcal{B}_\varepsilon$ of the sets $B_{\alpha,\varepsilon}$ and $B_{\beta,\varepsilon}$ onto $\mathcal{W}$ and the corresponding projection Π of the Poincaré map $\mathbf{P}$. We prove that Π is a contraction map on $\mathcal{B}_\varepsilon \subset \mathcal{W}$. This will imply the uniqueness of a periodic solution $z(x,t)$.
3. We show that $z(x,t)$ is a global B-attractor.

4.1. **Poincaré cross-section.** We consider the orthogonal complement $\mathcal{W}$ to e_0 in H^1 and the orthogonal projector

$$\mathbf{E} : H^1 \to \mathcal{W}$$

onto $\mathcal{W}$ given by $\mathbf{E}\varphi = \Phi$, where

$$\varphi(x) = \sum_{j=0}^{\infty} \varphi_j e_j(x), \qquad \Phi(x) = \sum_{j=1}^{\infty} \varphi_j e_j(x).$$

We also introduce the lifting operator

$$\mathbf{R}_\alpha : \mathcal{W} \to H^1$$

given by

$$\mathbf{R}_\alpha(\Phi) = \frac{1}{m_0}\left(\alpha - \sum_{j=1}^{\infty} m_j \Phi_j\right) e_0 + \sum_{j=1}^{\infty} \Phi_j e_j.$$

Thus, $\mathbf{R}_\alpha \mathbf{E}(\varphi) = \varphi$ for $\varphi \in H^1$, $\hat{\varphi} = \alpha$, and $\mathbf{E}\mathbf{R}_\alpha(\Phi) = \Phi$ for $\Phi \in \mathcal{W}$ (see Fig. 4.1).

Consider the set

$$\mathcal{B}_\varepsilon = \mathbf{E}(B_{\alpha,\varepsilon}) = \mathbf{E}(B_{\beta,\varepsilon}).$$

Thus, $\mathcal{B}_\varepsilon$ is the closed ε-neighborhood of the set

$$\mathcal{B}_0 = \mathbf{E}(B_0) = \left\{\Phi \in \mathcal{W} : |\Phi_j| \le \frac{|K_j|}{\lambda_j},\ j = 1, 2, \dots\right\}$$

in the $\mathcal{W}$-topology (see Fig. 4.2).

Clearly, $B_{\alpha,\varepsilon} = \mathbf{R}_\alpha(\mathcal{B}_\varepsilon)$.

Denote by $\Pi_\alpha : \mathcal{W} \to \mathcal{W}$ the "projection" of $\mathbf{P}_\alpha$ onto $\mathcal{W}$ given by

$$\Pi_\alpha(\Phi) = \mathbf{E}\mathbf{P}_\alpha \mathbf{R}_\alpha(\Phi).$$

Due to Theorem 2.3, $\Pi_\alpha(\mathcal{B}_\varepsilon) \subset \mathcal{B}_\varepsilon$. Similarly, one can define the operators $\mathbf{R}_\beta$ and Π_β.

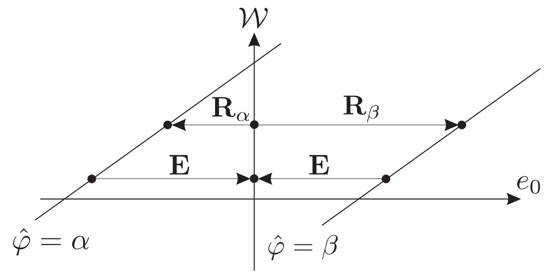

FIGURE 4.1. The projection operator $\mathbf{E}$ and the lifting operators $\mathbf{R}_\alpha$ and $\mathbf{R}_\beta$

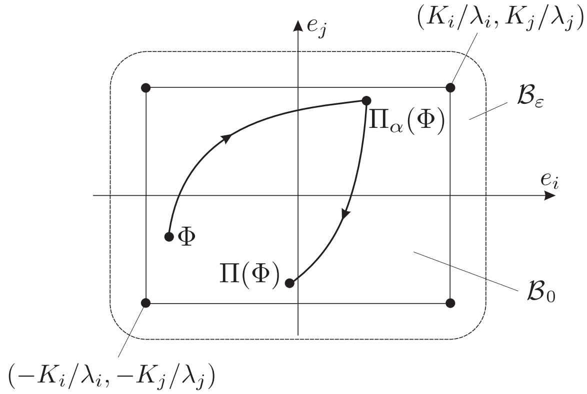

FIGURE 4.2. The operators Π_α and $\Pi = \Pi_\beta \Pi_\alpha$ in the space $\mathcal{W} = \mathrm{Span}(e_1, e_2, \dots)$

The operators $\mathbf{E}$, $\mathbf{R}_\alpha$, and $\mathbf{R}_\beta$ are continuously (and even infinitely) differentiable. Therefore, the operators Π_α and Π_β are also continuously differentiable, provided so are $\mathbf{P}_\alpha$ and $\mathbf{P}_\beta$.

4.2. **Projection of the Poincaré map.** We introduce the operator $\Pi : \mathcal{W} \to \mathcal{W}$ by the formula

$$\Pi(\Phi) = \mathbf{EPR}_\alpha(\Phi).$$

The following property of Π is straightforward (see Fig. 4.2).

Lemma 4.1. $\Pi = \Pi_\beta \Pi_\alpha$.

Now we will prove that $\|D_\Phi \Pi(\Phi)\| \le 1/4$ for $\Phi \in \mathcal{B}_\varepsilon$, provided that $\beta - \alpha$ is not too small. (Here and further the norm of $D_\Phi \Pi(\Phi)$ is the norm of the operator on $\mathcal{W}$.)

First, we calculate the derivative $D_\Phi \Pi_\alpha(\Phi) : \mathcal{W} \to \mathcal{W}$. It follows from the definitions of the operators $\mathbf{E}$, $\mathbf{R}_\alpha$, and Π_α and from the chain rule that

$$D_\Phi \Pi_\alpha(\Phi)\Psi = \mathbf{E}[D_\varphi \mathbf{P}_\alpha(\mathbf{R}_\alpha(\Phi))]\psi, \tag{4.1}$$

where

$$\psi = -\frac{\mathbf{M}\Psi}{m_0} e_0 + \sum_{j=1}^{\infty} \Psi_j e_j = -\frac{1}{m_0}\left(\sum_{j=1}^{\infty} m_j \Psi_j\right) e_0 + \sum_{j=1}^{\infty} \Psi_j e_j \qquad \forall \Psi \in \mathcal{W}. \tag{4.2}$$

Now we assume that Condition 3.1 holds (we will somewhat strengthen it below). Denote by $v(x,t) = v^\alpha(x,t)$ the solution of problem (3.1)–(3.3) with the initial data $v(\cdot,0) = \mathbf{R}_\alpha(\Phi)$ and by t_1 the first switching moment of $\mathcal{H}(\hat{v})$. Then using (4.1), Lemma 3.7, and the relation $\mathbf{E}f = 0$ for $f = \text{const}$ yields

$$D_\Phi \Pi_\alpha(\Phi)\Psi = \mathbf{ES}_{t_1}\psi - \left(\frac{d\hat{v}(t_1)}{dt}\right)^{-1} \mathbf{ES}'_{t_1}(\mathbf{R}_\alpha(\Phi) - v_K)\mathbf{MS}_{t_1}\psi. \tag{4.3}$$

We rewrite (4.3) in terms of the Fourier series.

Lemma 4.2. *Let Condition 3.1 hold. Then the derivative of $\Pi_\alpha : \mathcal{W} \to \mathcal{W}$ is given by*

$$\begin{aligned} D_\Phi \Pi_\alpha(\Phi)\Psi = &\sum_{j=1}^\infty e^{-\lambda_j t_1}\Psi_j e_j(x) \\ &+ \left(\frac{d\hat{v}(t_1)}{dt}\right)^{-1} \left(\sum_{j=1}^\infty m_j \Psi_j \left(e^{-\lambda_j t_1} - 1\right)\right) \sum_{j=1}^\infty \lambda_j e^{-\lambda_j t_1} \left(\Phi_j - \frac{K_j}{\lambda_j}\right) e_j(x) \end{aligned} \tag{4.4}$$

for any $\Phi \in \mathcal{B}_\varepsilon$ and $\Psi \in \mathcal{W}$, where Φ_j and Ψ_j ($j = 1, 2, \dots$) are the Fourier coefficients of Φ and Ψ, respectively.

Proof. Using the representation in (2.17), we rewrite the terms

$$\mathbf{ES}_{t_1}\psi, \quad \mathbf{ES}'_{t_1}(\mathbf{R}_\alpha(\Phi) - v_K), \quad \mathbf{MS}_{t_1}\psi.$$

We have

$$\mathbf{ES}_{t_1}\psi = \sum_{j=1}^\infty e^{-\lambda_j t_1}\Psi_j e_j(x). \tag{4.5}$$

Further, since $v_{Kj} = K_j/\lambda_j$ for $j = 1, 2, \dots$ (see Lemma 2.1), we obtain

$$\mathbf{ES}'_{t_1}(\mathbf{R}_\alpha(\Phi) - v_K) = -\sum_{j=1}^\infty \lambda_j e^{-\lambda_j t_1}\left(\Phi_j - \frac{K_j}{\lambda_j}\right) e_j(x). \tag{4.6}$$

Finally, taking into account (4.2), we have

$$\mathbf{MS}_{t_1}\psi = m_0\psi_0 + \sum_{j=1}^\infty m_j e^{-\lambda_j t_1}\psi_j = \sum_{j=1}^\infty m_j \Psi_j \left(e^{-\lambda_j t_1} - 1\right). \tag{4.7}$$

Combining (4.3) with (4.5)–(4.7) completes the proof. □

Now we can prove the following result.

Lemma 4.3. *There is $s_{\min} \ge t_{\min}$ such that if*

$$\beta - \alpha > m_0 K_0 s_{\min} + C, \tag{4.8}$$

then $\|D_\Phi \Pi_\alpha(\Phi)\| \le 1/2$ for all $\Phi \in \mathcal{B}_\varepsilon$. Here $t_{\min}$ and C are the same as in Condition 3.1.

Proof. Let $v(x,t)$ and t_1 be the same as in Lemma 4.2.

By assumption, Condition 3.1 holds. Therefore, as in the proof of Lemma 3.5, we have

$$t_1 \ge t_{\min} \ (> 0),$$

where $t_{\min}$ does not depend on Φ, T, α, β. Using this inequality, we estimate each term in (4.4).

First, we have

$$\left\| \sum_{j=1}^{\infty} e^{-\lambda_j t_1} \Psi_j e_j \right\|_{\mathcal{W}} \le e^{-\lambda_1 t_1} \|\Psi\|_{\mathcal{W}}. \tag{4.9}$$

Due to Lemma 3.4,

$$\left| \left(\frac{d\hat{v}(t_1)}{dt} \right)^{-1} \right| \le \frac{2}{m_0 K_0}. \tag{4.10}$$

By the Schwartz inequality,

$$\left| \sum_{j=1}^{\infty} m_j \Psi_j \left(e^{-\lambda_j t_1} - 1 \right) \right| \le \tilde{m} \|\Psi\|_{\mathcal{W}} \tag{4.11}$$

(recall that $\tilde{m}$ is defined in (3.10)).

Finally, using Lemma 2.1 and the inequality (cf. (2.30))

$$\lambda_j^2 e^{-2\lambda_j t} \le 4(et_1)^{-2} e^{-\lambda_1 t_1}, \tag{4.12}$$

we obtain for $\Phi \in \mathcal{B}_\varepsilon$ and $t_1 > t_{\min}$

$$\begin{aligned} &\left\| \sum_{j=1}^{\infty} \lambda_j e^{-\lambda_j t_1} \left(\Phi_j - \frac{K_j}{\lambda_j} \right) e_j \right\|_{\mathcal{W}} \\ &= \left(2 \sum_{j=1}^{\infty} (1+\lambda_j) \lambda_j^2 e^{-2\lambda_j t_1} \left(|\Phi_j|^2 + \left| \frac{K_j}{\lambda_j} \right|^2 \right) \right)^{1/2} \le k_1 e^{-\lambda_1 t_1/2}, \end{aligned} \tag{4.13}$$

where

$$k_1 = \frac{2\sqrt{10} c \|K\|_{H^{1/2}}}{e t_{\min}}.$$

Combining Lemma 4.2 and inequalities (4.9)–(4.13), we derive the estimate

$$\|D_\Phi \Pi_\alpha(\Phi)\| \le \left(1 + \frac{2k_1 \tilde{m}}{m_0 K_0} \right) e^{-\lambda_1 t_1/2}. \tag{4.14}$$

As in the proof of Lemma 3.5, we see that if condition (4.8) holds, then

$$t_1 > s_{\min}.$$

Therefore, (4.14) yields the desired result, provided that $s_{\min} \ge t_{\min}$ is sufficiently large. □

Thus, the sufficient conditions under which $\|D_\Phi \Pi_\alpha(\Phi)\| \le 1/2$ is as follows (cf. Condition 3.1).

Condition 4.1. *$\beta - \alpha > m_0 K_0 s_{\min} + C$, where $s_{\min}$ is defined in Lemma 4.3 and C is the same as in Condition 3.1.*

Lemma 4.4. *Let Condition 4.1 hold. Then there is a unique fixed point $\psi \in B_{\alpha,0}$ of the operator $\mathbf{P} : B_{\alpha,\varepsilon} \to B_{\alpha,\varepsilon}$ and, for any $\varphi \in B_{\alpha,\varepsilon}$,*

$$\|\mathbf{P}^k(\varphi) - \psi\|_{H^1} \le \mathrm{const}\, 4^{-k} \|\varphi - \psi\|_{H^1}, \quad k = 1, 2, \dots,$$

where const > 0 *does not depend on φ, ψ, and k.*

Proof. Using Lemmas 4.1 and 4.3 and the chain rule, we see that $\|D_\Phi \Pi(\Phi)\| \le 1/4$ for all $\Phi \in \mathcal{B}_\varepsilon$. Therefore, for any $\Phi, \Psi \in \mathcal{B}_\varepsilon$ and $\sigma \in [0,1]$, we have $\Phi + \sigma(\Psi - \Phi) \in \mathcal{B}_\varepsilon$ and

$$\|\Pi(\Phi) - \Pi(\Psi)\|_{\mathcal{W}} = \left\| \int_0^1 D_\sigma[\Pi(\Phi + \sigma(\Psi - \Phi))]\, d\sigma \right\|_{\mathcal{W}}$$
$$= \left\| \int_0^1 D\Pi(\Phi + \sigma(\Psi - \Phi))(\Psi - \Phi)\, d\sigma \right\|_{\mathcal{W}} \le 4^{-1}\|\Psi - \Phi\|_{\mathcal{W}},$$

where $D\Pi(\Phi + \sigma(\Psi - \Phi))$ is the derivative of the operator Π calculated at the point $(\Phi + \sigma(\Psi - \Phi))$. Therefore, $\Pi : \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$ is a contraction map for all $0 \le \varepsilon \le \varepsilon_0$. Since $\mathcal{B}_0 \subset \mathcal{B}_\varepsilon$, it follows that the unique fixed point of Π belongs to $\mathcal{B}_0$. This fact and the formula $\mathbf{P}^k(\varphi) = (\mathbf{R}_\alpha \Pi^k \mathbf{E})(\varphi)$ for $\varphi \in B_{\alpha,\varepsilon}$ imply the conclusion of the lemma. □

Lemma 4.4 and Theorems 2.3 and 3.2 imply the first main result of this section.

Theorem 4.1. *Let Condition 4.1 hold. Then there exists a unique periodic solution $z(x,t)$ of problem (2.1), (2.3) (up to time translations). If $z(\cdot,0) \in B_{\alpha,0}$, then the period T of $z(x,t)$ satisfies $T = 2s_1$, where s_1 is the first switching moment of $\mathcal{H}(\hat{z})$.*

Remark 4.1. Lemma 4.4 and Theorem 2.3 also imply that if $z(x,t)$ is a periodic solution with the initial data $\psi = z(\cdot,0) \in B_{\alpha,0}$ and $v(x,t)$ another solution with $\hat{v}(t_2), \hat{v}(t_4), \hat{v}(t_6), \dots = \alpha$ $(t_2 < t_4 < t_6 < \dots)$, then the distances

$$\|v(\cdot, t_{2i}) - \psi\|_{H^1}$$

tend to zero monotonically decreasing as $i \to \infty$.

In the next subsection, we prove that the periodic solution itself is a global B-attractor.

4.3. **Global B-attractor and phase-shifted attraction.** Throughout this subsection, we assume that Condition 4.1 holds and that $z(x,t)$ is a (unique) T-periodic solution of problem (2.1), (2.3). Denote $z(x,0) = \psi(x)$. Since $z(\cdot,t) \in B_{\alpha,0}$ for some $t \in [0,T]$, we assume without loss of generality that this holds for $t = 0$, i.e., $\psi \in B_{\alpha,0}$.

Definition 4.1. We say that a T-periodic solution $z(x,t)$ of problem (2.1), (2.3) is a *global B-attractor* if the set $\Gamma = \{z(\cdot,s) \in H^1 : s \in [0,T]\}$ is a global B-attractor, i.e., if, for any $R > 0$ and $\varphi \in H^1$, $\|\varphi\|_{H^1} \le R$, the solution $v(x,t)$ of problem (2.1)–(2.3) in Q_∞ with the initial data φ satisfies

$$\operatorname{dist}(v(\cdot,t), \Gamma) = \min_{s\in[0,T]} \|v(\cdot,t) - z(\cdot,s)\|_{H^1} \to 0 \quad \text{as } t \to \infty, \tag{4.15}$$

where the convergence to zero is uniform with respect to φ, $\|\varphi\|_{H^1} \le R$.

We shall prove that $z(x,t)$ is a global B-attractor.

Let $v(x,t)$ be an arbitrary solution of problem (2.1)–(2.3) in Q_∞ with the initial data $v(x,0) = \varphi(x)$. We assume throughout that $\|\varphi\|_{H^1} \le R$.

First, we note the following.

Remark 4.2. 1. If $\varphi \in B_\varepsilon$, then, by Theorem 2.3, $v(\cdot,t) \in B_\varepsilon$ for all $t \ge 0$.

2. Estimate (2.33) and Theorem 2.3 imply that, for any $R > 0$, there is $\theta^* = \theta^*(R) > 0$ with the following property. For any φ, $\|\varphi\|_{H^1} \le R$, there exists a time moment $\theta \le \theta^*$ such that $v(\cdot,\theta) \in B_{\alpha,\varepsilon}$.

Due to Theorem 4.1, $s_1 = T/2$, $s_2 = 2s_1 = T$, $s_3 = 3s_1 = 3T/2$, $s_4 = 4s_1 = 2T, \dots$ are the switching moments of $\mathcal{H}(\hat{z})$. Denote by $t_1, t_2, t_3, \dots$ the switching moments of $\mathcal{H}(\hat{v})$ and by $\tau_1 = t_1$, $\tau_2 = t_2 - t_1$, $\tau_3 = t_3 - t_2, \dots$ the corresponding differences between the switching moments.

First, we show that these differences converge to s_1.

Lemma 4.5. *Let Condition 4.1 hold. Then*

$$|\tau_{i+1} - s_1| \le k2^{-i}, \tag{4.16}$$

where $k = k(R) > 0$ does not depend on φ.

Proof. We prove the lemma for even i. The case of odd i is similar.

1. First, we assume that $\varphi \in B_{\alpha,\varepsilon}$. By using Corollary 3.1 and Lemma 4.4, we have

$$|\tau_{i+1} - s_1| = |\mathbf{t}_1(\mathbf{P}^{i/2}(\varphi)) - \mathbf{t}_1(\psi)| \le k_1 \|\mathbf{P}^{i/2}(\varphi) - \psi\|_{H^1} \le k_2 2^{-i},$$

where $k_1, k_2 > 0$ depend on $B_{\alpha,\varepsilon}$ but do not depend on φ and ψ.

2. Now we consider an arbitrary φ with $\|\varphi\|_{H^1} \le R$ and the solution $v(x,t)$ on the interval $[0,\theta]$ (cf. Remark 4.2, part 2). Estimates (2.33) and (2.40) imply that the differences τ_{i+1} and the number of switchings of $\mathcal{H}(\hat{v})$ on the interval $[0,\theta] \subset [0,\theta^*]$ are bounded by a constant depending only on R. This fact combined with part 1 of the proof completes the argument. □

We will also need to estimate the solution $v(x,t)$ on the time intervals between the corresponding switchings of $\mathcal{H}(\hat{v})$ and $\mathcal{H}(\hat{z})$. By Lemma 4.5, these intervals are small for large t.

Lemma 4.6. *1. Let $v(x,t)$ be a solution of problem (2.1)–(2.3) in Q_∞ with initial data $\varphi \in B_{\alpha,\varepsilon}$, and let t_i and τ_i be the same as above. Then $v(\cdot,t)$ is Lipschitz-continuous in a left neighborhood of t_i uniformly with respect to i and φ, i.e.,*

$$\|v(\cdot,t) - v(\cdot,s)\|_{H^1} \le k_1|t-s| \quad \textit{for } t,s \in [\theta_i, t_{i+1}],\ i = 0,1,2,\dots, \tag{4.17}$$

where $\theta_i = t_i + \tau_{i+1}/2$ and $k_1 > 0$ does not depend on t, s, i, and φ.

2. Let $z(x,t)$ be a T-periodic solution of problem (2.1), (2.3). Then $z(\cdot,t)$ is uniformly Lipschitz-continuous for $t \in [0,T]$, i.e.,

$$\|z(\cdot,t) - z(\cdot,s)\|_{H^1} \le k_2|t-s| \quad \textit{for } t,s \in [0,T], \tag{4.18}$$

where $k_2 > 0$ does not depend on t and s.

Proof. 1. We prove the first part of the lemma for even i. The case of odd i is similar. Let $t,s \in [\theta_i, t_{i+1}]$ and $t > s$. Then, using Lemma 2.2 (part 2), we have

$$\begin{aligned} v(x,t) - v(x,s) &= \sum_{j=0}^{\infty} (v_j(t) - v_j(s)) e_j(x) \\ &= K_0 e_0 (t-s) + \sum_{j=1}^{\infty} \left((\mathbf{P}^{i/2}(\varphi))_j - \frac{K_j}{\lambda_j} \right) e^{-\lambda_j (s - t_i)} \left(e^{-\lambda_j (t-s)} - 1 \right) e_j(x), \end{aligned}$$

where the series converge in H^1. Therefore, taking into account that $s - t_i \ge \tau_{i+1}/2 \ge \tau^*/2$ (τ^* does not depend on φ from the bounded set $B_{\alpha,\varepsilon}$, see (2.40)), we have

$$\|v(\cdot,t) - v(\cdot,s)\|_{H^1}^2 \le K_0^2 (t-s)^2 + 2F(t-s), \tag{4.19}$$

where

$$F(t-s)=\sum_{j=1}^{\infty}(1+\lambda_j)\left(\left|(\mathbf{P}^{i/2}(\varphi))_j\right|^2+\left|\frac{K_j}{\lambda_j}\right|^2\right)e^{-\lambda_j\tau^*}\left(e^{-\lambda_j(t-s)}-1\right)^2.$$

Taking into account the fact that $\mathbf{P}^{i/2}(\varphi)$ lies in the bounded set $B_{\alpha,\varepsilon}$, Lemma 2.1, and the asymptotics of λ_j (Remark 2.3), we see that $F(a)$ is infinitely differentiable for $a\ge 0$ and $F(0)=F'(0)=0$. Therefore,

$$F(a)\le\frac{1}{2}\max_{\sigma\in[0,\tau_{i+1}/2]}|F''(\sigma)|a^2\le\tilde{k}a^2,$$

where $\tilde{k}>0$ does not depend on i (because τ_i are uniformly bounded with respect to i) and φ. Combining the estimate of $F(a)$ with (4.19), we obtain (4.17).

2. Let $z(x,t)$ be a periodic solution and $z(\cdot,0)\in B_{\alpha,0}$ ($\subset H^1$). By using the representations (2.14) and (2.17), one can see that $z(\cdot,t)$ is a continuously differentiable H^1-valued function on $[T,T+s_1]$ and on $[T+s_1,2T]$. Therefore, it is uniformly Lipschitz-continuous on $[T,2T]$. By periodicity, it is also uniformly Lipschitz-continuous on $[0,T]$. □

Now, using the above two lemmas, we can prove the first result of this section concerning a global B-attractor and the phase-shifted attraction.

Theorem 4.2. *Let Condition 4.1 hold. Then the (unique) T-periodic solution $z(x,t)$ of problem (2.1), (2.3) is a global B-attractor.*

Moreover, any solution $v(x,t)$ of problem (2.1)–(2.3) in Q_∞ with initial data $\varphi\in H^1$, $\|\varphi\|_{H^1}\le R$ ($R>0$ is arbitrary), satisfies

$$\|v(\cdot,t)-z(\cdot,t-\delta)\|_{H^1}\le k4^{-t/T},\quad t\ge 0, \tag{4.20}$$

where $\delta=\delta(\varphi)$ and the constant $k=k(R)>0$ does not depend on φ.

Furthermore, $|\delta|\le\delta^*$, where $\delta^*=\delta^*(R)$ does not depend on φ.

Proof. 1. Denote

$$\delta=\sum_{l=1}^{\infty}(\tau_l-s_1),$$

where the series is absolutely convergent due to Lemma 4.5 and

$$|\delta|\le\sum_{l=1}^{\infty}|\tau_l-s_1|\le\delta^*,$$

where $\delta^*=\delta^*(R)$. To be definite, we assume that

$$\delta>0.$$

Since $t_i=\sum\limits_{l=1}^{i}\tau_l$ and $s_i=\sum\limits_{l=1}^{i}s_1$, it follows from Lemma 4.5 that

$$|t_i-\delta-s_i|\le\sum_{l=i+1}^{\infty}|\tau_l-s_1|\le k_1 2^{-i}, \tag{4.21}$$

where $k_1,k_2,\ldots>0$ depend on R, but do not depend on i and φ.

Thus, the switching moment t_i asymptotically "delays" by δ with respect to the switching moment s_i.

2. Let us estimate $v(x,t+\delta)-z(x,t)$ for $t\in[s_i,s_{i+1}]$. Due to (4.21) and (2.40), there is $i^*=i^*(R)$ such that, for $i\ge i^*$, neither $\mathcal{H}(\hat v(\cdot+\delta))$ nor $\mathcal{H}(\hat z)$ switches on the (small) intervals between s_i and $t_i-\delta$ and between $t_{i+1}-\delta$ and s_{i+1}.

Let

$$s_i<t_i-\delta<t_{i+1}-\delta<s_{i+1}.$$

All the other cases are treated similarly.

First, we assume that $\varphi\in B_{\alpha,\varepsilon}$, $i\ge i^*$, and i is even (the case of odd i is analogous) and consider the following three cases.

Case 1. $t\in[s_i,t_i-\delta]$ (small interval),
Case 2. $t\in[t_i-\delta,t_{i+1}-\delta]$,
Case 3. $t\in[t_{i+1}-\delta,s_{i+1}]$ (small interval).

2.1. Let $t\in[s_i,t_i-\delta]$ (small interval). Then $\mathcal{H}(\hat v(\cdot+\delta))(t)=-1$ and $\mathcal{H}(\hat z)(t)=1$. Using Lemmas 4.4 and 4.6 and inequality (4.21), we have

$$\begin{aligned}\|v(\cdot,t+\delta)-z(\cdot,t)\|_{H^1}\\ &\le\|v(\cdot,t+\delta)-v(\cdot,t_i)\|_{H^1}+\|v(\cdot,t_i)-z(\cdot,s_i)\|_{H^1}+\|z(\cdot,s_i)-z(\cdot,t)\|_{H^1}\\ &\le k_2 2^{-i}.\end{aligned}\tag{4.22}$$

2.2. Let $t\in[t_i-\delta,t_{i+1}-\delta]$. Then $\mathcal{H}(\hat v(\cdot+\delta))(t)=\mathcal{H}(\hat z)(t)=1$. Applying Lemma 2.3 to the function $v(x,t+\delta)-z(x,t)$ and taking into account Remark 2.1, we have

$$\begin{aligned}\|v(\cdot,t+\delta)-z(\cdot,t)\|_{H^1}&\le k_3\|v(\cdot,t_i)-z(\cdot,t_i-\delta)\|_{H^1}\\ &\le k_3\left(\|v(\cdot,t_i)-z(\cdot,s_i)\|_{H^1}+\|z(\cdot,s_i)-z(\cdot,t_i-\delta)\|_{H^1}\right).\end{aligned}$$

Therefore, by Lemmas 4.4 and 4.6 and by inequality (4.21),

$$\|v(\cdot,t+\delta)-z(\cdot,t)\|_{H^1}\le k_4 2^{-i}.\tag{4.23}$$

2.3. Let $t\in[t_{i+1}-\delta,s_{i+1}]$ (small interval). Then $\mathcal{H}(\hat v(\cdot+\delta))(t)=-1$ and $\mathcal{H}(\hat z)(t)=1$. We have

$$\begin{aligned}\|v(\cdot,t+\delta)-z(\cdot,t)\|_{H^1}&\le\|v(\cdot,t+\delta)-z(\cdot,t-t_{i+1}+\delta+s_{i+1})\|_{H^1}\\ &+\|z(\cdot,t)-z(\cdot,s_{i+1})\|_{H^1}+\|z(\cdot,s_{i+1})-z(\cdot,t-t_{i+1}+\delta+s_{i+1})\|_{H^1}\end{aligned}$$

The hysteresis operator $\mathcal{H}$ is equal to -1 for both functions in the first norm; hence it is estimated via 2^{-i} as in part 2.2. The second and third norms are estimated via 2^{-i} with the help of Lemma 4.6 and inequality (4.21). Therefore,

$$\|v(\cdot,t+\delta)-z(\cdot,t)\|_{H^1}\le k_5 2^{-i}.\tag{4.24}$$

Inequalities (4.22)–(4.24) (and the analogous inequalities for odd i) yield

$$\|v(\cdot,t)-z(\cdot,t-\delta)\|_{H^1}\le k_6 2^{-i}\le k_7 2^{-t/s_1}\tag{4.25}$$

for $t\in[s_i+\delta,s_{i+1}+\delta]$, $i\ge i^*$, $\varphi\in B_{\alpha,\varepsilon}$.

3. Now consider arbitrary φ, $\|\varphi\|_{H^1}\le R$ and $i\ge 0$. Theorem 2.2 implies that

$$\max_{t\in[0,\theta^*+s_{i^*}+\delta^*]}\|v(\cdot,t)\|_{H^1}\le k_7(\theta^*,i^*,\delta^*,R)=k_8(R),$$

where θ^* is defined in Remark 4.2. Therefore, the inequality in (4.25) holds for any φ, $\|\varphi\|_{H^1}\le R$, and $t\ge 0$. □

Now we complement Theorem 4.2 by showing that any solution $v(x,t)$ not only converges point-wise to the T-periodic solution $z(x,t)$ (after appropriately shifting the time argument), but also converges to the *corresponding part* of the T-periodic trajectory. Namely, it converges to the part

$$\Gamma_1 = \{z(\cdot, s),\ \ s \in [0, T/2]\} \tag{4.26}$$

if $t \in [t_i, t_{i+1}]$ and to the part

$$\Gamma_2 = \{z(\cdot, s),\ \ s \in [T/2, T]\} \tag{4.27}$$

if $t \in [t_{i+1}, t_{i+2}]$ as $i \to \infty$ and i is even. Thus, the next theorem is a mathematical formulation of the intuitively expectable observation: the parts of the v-trajectory with $\mathcal{H}(\hat{v}) = 1$ ($\mathcal{H}(\hat{v}) = -1$) become close to the part of the periodic z-trajectory with $\mathcal{H}(\hat{z}) = 1$ ($\mathcal{H}(\hat{z}) = -1$).

Theorem 4.3. *Let Condition 4.1 hold. Then, given $R > 0$, there is $k = k(R) > 0$ such that any solution $v(x,t)$ of problem (2.1)–(2.3) in Q_∞ with initial data $\varphi \in H^1$, $\|\varphi\|_{H^1} \le R$, satisfies*

$$\begin{aligned} \operatorname{dist}(v(\cdot,t),\Gamma_1) &= \min_{s\in[0,T/2]} \|v(\cdot,t) - z(\cdot,s)\|_{H^1} \le k2^{-i} \quad \text{for } t \in [t_i, t_{i+1}], \\ \operatorname{dist}(v(\cdot,t),\Gamma_2) &= \min_{s\in[T/2,T]} \|v(\cdot,t) - z(\cdot,s)\|_{H^1} \le k2^{-i} \quad \text{for } t \in [t_{i+1}, t_{i+2}], \end{aligned} \tag{4.28}$$

where $i = 0, 2, 4, \dots,$ t_i are the switching moments of $\mathcal{H}(\hat{v})$, T is the period of $z(x,t)$, and $T/2$ and T are the first and the second switching moments of $\mathcal{H}(\hat{z})$.

Proof. We prove the first inequality in (4.28). The second inequality can be proved in the same way.

1. Let $\varphi \in B_{\alpha,\varepsilon}$. We take an arbitrary $i = 0, 2, 4, \dots$ and assume that $\tau_{i+1} = t_{i+1} - t_i > s_1 = T/2$. The case $\tau_{i+1} \le s_1$ is analogous and simpler.

First, we consider $t \in [t_i, t_i + s_1]$. The function $w(x,s) = v(x, s+t_i) - z(x,s)$ is a solution of the parabolic problem

$$\begin{gathered} w_s(x,s) = \Delta w(x,s) \quad ((x,s) \in Q_{s_1}), \\ w(x,0) = \mathbf{P}^{i/2}(\varphi) - \psi \quad (x \in Q), \\ \frac{\partial w}{\partial \nu} = 0 \quad ((x,s) \in \Gamma_{s_1}). \end{gathered}$$

Therefore, setting $s = t - t_i \in [0, s_1]$ for $t \in [t_i, t_i + s_1]$ and using Lemma 2.3 and Remark 2.1, we have

$$\|v(\cdot,t) - z(\cdot,s)\|_{H^1} = \|w(\cdot,s)\|_{H^1} \le k_1 \|\mathbf{P}^{i/2}(\varphi) - \psi\|_{H^1},$$

where $k_1, k_2, \dots > 0$ do not depend on φ and i. This inequality and Lemma 4.4 yield

$$\|v(\cdot,t) - z(\cdot,s)\|_{H^1} \le k_2 2^{-i}. \tag{4.29}$$

2. It remains to consider $t \in [t_i + s_1, t_{i+1}] = [t_i + s_1, t_i + \tau_{i+1}]$. We estimate the difference $v(x,t) - v(x, t_i + s_1)$, assuming without loss of generality that $s_1 \ge \tau_{i+1}/2$. (Otherwise, one can first consider sufficiently large i, for which this inequality holds by Lemma 4.5, and then argue as in part 3 of the proof of Theorem 4.2.) Then, by using Lemma 4.6 (part 1) and Lemma 4.5, we have

$$\|v(\cdot,t) - v(\cdot, t_i + s_1)\|_{H^1} \le k_3(t - t_i - s_1) \le k_3(\tau_{i+1} - s_1) \le k_4 2^{-i}. \tag{4.30}$$

Estimates (4.29) and (4.30) prove the first inequality in (4.28) for $\varphi \in B_{\alpha,\varepsilon}$.

3. Now, for arbitrary φ, $\|\varphi\|_{H^1} \le R$, we take into account Remark 4.2 and the estimate (see Theorem 2.2)

$$\max_{t\in[0,\theta^*]} \|v(\cdot,t)\|_{H^1} \le k_6(\theta^*, R) = k_7(R),$$

which complete the proof. □

4.4. **Stability of periodic solution.** When studying the stability of the periodic solution, one considers its small neighborhood. When doing so, one has to take into account the initial state of the hysteresis operator. For any periodic solution $z(x,t)$, let Γ_1 and Γ_2 be given by (4.26) and (4.27), and let $\Gamma = \Gamma_1 \cup \Gamma_2$ (cf. Remark 3.1).

Definition 4.2. A T-periodic solution $z(x,t)$ of problem (2.1), (2.3) is *orbitally uniformly asymptotically stable* if

1. for any neighborhood U of Γ in H^1, there exist neighborhoods V_1 of Γ_1 and V_2 of Γ_2 in H^1 such that if

$$\varphi \in V_1,\ \hat\varphi < \beta \quad \text{or} \quad \varphi \in V_2,\ \hat\varphi \ge \beta,$$

then the solution $v(x,t)$ of problem (2.1)–(2.3) in Q_∞ with the initial data φ belongs to U for all $t \ge 0$;
2. there exist neighborhoods W_1 of Γ_1 and W_2 of Γ_2 in H^1 such that, for all

$$\varphi \in W_1,\ \hat\varphi < \beta \quad \text{or} \quad \varphi \in W_2,\ \hat\varphi \ge \beta,$$

the solutions $v(x,t)$ of problem (2.1)–(2.3) in Q_∞ with initial data φ satisfy

$$\operatorname{dist}(v(\cdot,t),\Gamma) \to 0 \quad \text{as} \quad t \to +\infty$$

uniformly with respect to φ.

We underline that the phenomena in Definitions 4.1 (global attractivity) and 4.2 (asymptotic orbital stability) do not follow from each other. Clearly, orbital asymptotic stability does not imply global attractivity. On the other hand, the fact that a periodic solution is a global B-attractor does not generally imply that it is stable. In principle, all trajectories originated arbitrarily close to the periodic trajectory Γ may leave a given small neighborhood of Γ and then converge to Γ as $t \to +\infty$.

However, we will show that, in our case, the periodic solution is not only a global B-attractor, but is also stable in the above sense.

Let $z(x,t)$ be a T-periodic solution of problem (2.1), (2.3) such that $z(\cdot,0) = \psi \in B_{\alpha,0}$ ($B_{\beta,0}$). Let $v(x,t)$ be an arbitrary solution of problem (2.1)–(2.3) in Q_∞ with initial data $v(\cdot,t) = \varphi \in B_{\alpha,\varepsilon}$ ($B_{\beta,\varepsilon}$).

Lemma 4.7. *Let Condition 4.1 hold, and let $z(x,t)$ and $v(x,t)$ be as above. Then, for any $\sigma > 0$, there is $\delta > 0$ such that*

$$\operatorname{dist}(v(\cdot,t),\Gamma) = \min_{s\in[0,T]} \|v(\cdot,t) - z(\cdot,s)\|_{H^1} \le \sigma \quad \forall t \ge 0$$

whenever $\|\varphi - \psi\|_{H^1} \le \delta$.

Proof. To be definite, we assume that $\psi \in B_{\alpha,0}$. Fix $\sigma > 0$. Let $\delta > 0$ be so small that the δ-neighborhood of ψ lies in $B_{\alpha,\varepsilon}$. Let $s_1, s_2, \dots$ and $t_1, t_2, \dots$ be the successive switching moments of $\mathcal{H}(\hat z)$ and $\mathcal{H}(\hat v)$, respectively. Due to (2.40), $s_1, t_1 \ge \tau^*$, where τ^* does not depend on φ.

Let $s_1 \le t_1$ (the case $s_1 > t_1$ is similar). Then the difference $z - v$ satisfies for $t \le s_1$ the following relations:

$$(z-v)_t = \Delta(z-v) \quad ((x,t) \in Q_{s_1}),$$
$$(z-v)(x,0) = \psi(x) - \varphi(x) \quad (x \in Q),$$
$$\frac{\partial(z-v)}{\partial\nu} = 0 \quad ((x,t) \in \Gamma_{s_1}).$$

Therefore, due to Lemma 2.3 and Remark 2.1, we have for $t \le s_1$

$$\|z(\cdot,t) - v(\cdot,t)\|_{H^1} \le k_1\|\psi - \varphi\|_{H^1} \le k_1\delta, \tag{4.31}$$

where $k_1 > 0$ is equal to the constant $c_1(\tau^*)$ from estimate (2.31), i.e., does not depend on φ and δ.

Further, due to Corollary 3.1,

$$|s_1 - t_1| \le k_2\|\psi - \varphi\|_{H^1} \le k_2\delta,$$

where $k_2, k_3, \ldots > 0$ do not depend on φ and δ. Therefore, taking into account Lemma 4.6 and inequality (4.31), we have for $t \in [s_1, t_1]$

$$\begin{aligned}\|z(\cdot,s_1) - v(\cdot,t)\|_{H^1} &\le \|z(\cdot,s_1) - v(\cdot,s_1)\|_{H^1} + \|v(\cdot,s_1) - v(\cdot,t)\|_{H^1}\\ &\le k_1\delta + k_3|s_1 - t_1| \le k_4\delta.\end{aligned} \tag{4.32}$$

Estimates (4.31) and (4.32) yield

$$\operatorname{dist}(v(\cdot,t),\Gamma_1) \le k_4\delta \quad \forall t \in [0,t_1],$$
$$\|z(\cdot,s_1) - v(\cdot,t_1)\|_{H^1} \le k_4\delta,$$

where Γ_1 is the part of the periodic trajectory given by (4.26).

Repeating the above arguments, we obtain

$$\operatorname{dist}(v(\cdot,t),\Gamma) \le k_4^i\delta \quad \forall t \in [0,t_i], \quad i = 1,2,\ldots.$$

However, by Lemma 4.4, there is a number i^* such that

$$\|v(\cdot,t_{2i^*}) - \psi\|_{H^1} \le \|\varphi - \psi\|_{H^1} \le \delta.$$

Thus, the conclusion of the lemma follows by taking $\delta = \sigma/k_4^{2i^*}$ (or $\delta = \sigma$ if $k_4 \le 1$). ☐

Now, to prove the stability of the periodic solution, we have to consider arbitrary solutions with initial data not necessarily from $B_{\alpha,\varepsilon}$, but rather in a neighborhood of Γ_1 or Γ_2. To be definite, we will consider a neighborhood of Γ_1.

Theorem 4.4. *Let Condition 4.1 hold, and let $m \in H^1$. Then the T-periodic solution $z(x,t)$ of problem (2.1), (2.3) is orbitally uniformly asymptotically stable.*

Proof. 1. By Theorem 4.3, it suffices to check only the first part of Definition 4.2.

Due to Lemma 4.7, we have to prove the following. For any $\delta > 0$, there is $\mu > 0$ such that if $\|\varphi - \psi\|_{H^1} \le \mu$ for some $\psi \in \Gamma_1$ and $\hat\varphi < \beta$, then

$$\begin{aligned}&\operatorname{dist}(v(\cdot,t),\Gamma_1) \le \delta \qquad \forall t \le t_1,\\ &\|z(\cdot,s_1) - v(\cdot,t_1)\|_{H^1} \le \delta,\end{aligned} \tag{4.33}$$

where $z(x,0) = \psi(x)$, $v(x,0) = \varphi(x)$, and s_1 and t_1 are the first switching moments of $\mathcal{H}(\hat z)$ and $\mathcal{H}(\hat v)$, respectively. The number μ does not depend on $\psi \in \Gamma_1$.

2. The difficulty in the proof of (4.33) is that φ and ψ need not be in $B_{\alpha,\varepsilon}$; hence, Lemma 3.6 and Corollary 3.1 do not directly apply. Moreover, $\hat\varphi$ need not

be close to α, and Lemma 3.5 does not apply either. For this reason, we will use the assumption $m \in H^1$ to "control" the derivative $d\hat{v}/dt$ at the switching moment t_1.

If we show that

$$\frac{d\hat{v}(t_1)}{dt} \geq \frac{m_0 K_0}{8} \tag{4.34}$$

for any φ in the μ-neighborhood of ψ, then, as in Lemmas 3.6 and 3.7, it will follow that $\mathbf{t}_1(\varphi)$ and $\mathbf{P}_\alpha(\varphi)$ are continuously differentiable in the μ-neighborhood of ψ.

3. Denote by $z^\alpha(x,t)$ and $v^\alpha(x,t)$ the solutions of problem (2.5)–(2.7) with the initial data ψ and φ, respectively.

We know from Lemma 3.5 that

$$\frac{d\widehat{z^\alpha}(s_1)}{dt} \geq \frac{m_0 K_0}{2}.$$

By the continuity, there is $\theta > 0$ such that

$$\frac{d\widehat{z^\alpha}(t)}{dt} \geq \frac{m_0 K_0}{4} \quad \forall t \in [s_1, s_1 + \theta]. \tag{4.35}$$

On the other hand, using Lemma 2.2 (see, in particular, (2.26) and (2.27)) and the relation $m \in H^1$, we obtain for $t \leq \min(s_1, t_1)$

$$\left|\widehat{z^\alpha}(t) - \widehat{v^\alpha}(t)\right| = \left| m_0(\psi_0 - \varphi_0) + \sum_{j=1}^{\infty} m_j(\psi_j - \varphi_j)e^{-\lambda_j t}\right| \leq k_1\mu, \tag{4.36}$$

$$\left|\frac{d\widehat{z^\alpha}(t)}{dt} - \frac{d\widehat{v^\alpha}(t)}{dt}\right| = \left|\sum_{j=1}^{\infty} m_j\lambda_j(\varphi_j - \psi_j)e^{-\lambda_j t}\right| \leq k_2\|m\|_{H^1}\|\varphi - \psi\|_{H^1} \leq k_3\mu, \tag{4.37}$$

where $k_1, k_2, \ldots > 0$ do not depend on μ, ψ, and φ.

4. First, we assume that $s_1 \leq t_1$. As in the proof of Lemma 4.7 (see (4.31)), we have for $t \leq s_1$

$$\|z(\cdot,t) - v(\cdot,t)\|_{H^1} \leq k_4\mu. \tag{4.38}$$

Assuming that μ is small enough, we deduce from (4.35) and (4.37) that

$$\frac{d\hat{v}(t)}{dt} \geq \frac{m_0 K_0}{8} \quad \forall t \in [s_1, s_1 + \theta].$$

On the other hand, due to (4.36), $\beta - \hat{v}(s_1) \leq k_1\mu$. Therefore, if μ is small enough, the switching of $\mathcal{H}(\hat{v})$ occurs before the moment $s_1 + \theta$, i.e.,

$$\frac{d\hat{v}(t)}{dt} \geq \frac{m_0 K_0}{8} \quad \forall t \in [s_1, t_1].$$

Hence,

$$t_1 - s_1 \leq \frac{8k_1}{m_0 K_0}\mu. \tag{4.39}$$

Now, using the estimate analogous to (4.31) and the Lipschitz continuity of $z^\alpha(\cdot,t)$ for $t \geq t_1$ combined with (4.39), we have for $t \in [s_1, t_1]$

$$\|v(\cdot,t) - z(\cdot,s_1)\|_{H^1} \leq \|v^\alpha(\cdot,t) - z^\alpha(\cdot,t)\|_{H^1} + \|z^\alpha(\cdot,t) - z^\alpha(\cdot,s_1)\|_{H^1} \leq k_5\mu. \tag{4.40}$$

Estimates (4.38) and (4.40) yield the first inequality in (4.33). The second inequality follows from the continuous differentiability of $\mathbf{P}_\alpha(\varphi)$:

$$\|v(\cdot,s_1) - z(\cdot,t_1)\|_{H^1} = \|\mathbf{P}_\alpha(\varphi) - \mathbf{P}_\alpha(\psi)\|_{H^1} \leq k_6\mu.$$

5. Finally, we assume that $t_1 \le s_1$. As before, for $t \le t_1$,

$$\|z(\cdot,t) - v(\cdot,t)\|_{H^1} \le k_7\mu, \tag{4.41}$$

which implies the first inequality in (4.33).

Further, due to (4.36),

$$\beta - \hat{z}(t_1) \le k_8\mu,$$

while $dz(t)/dt \ge m_0K_0/2$ at the switching moment s_1, i.e., when $z(t) = \beta$. Therefore, one can choose μ so small that

$$s_1 - t_1 \le k_9\mu, \qquad \frac{d\hat{z}(t)}{dt} \ge \frac{m_0K_0}{4} \quad \forall t \in [t_1, s_1].$$

(Combined with (4.37), this implies $\dfrac{d\hat{v}(t_1)}{dt} \ge \dfrac{m_0K_0}{8}$, provided that μ is small enough.)

Taking into account the Lipschitz continuity of $z(\cdot,t)$ for $t \in [t_1, s_1]$, the fact that $s_1 - t_1 \le k_9\mu$, and estimate (4.41) for $t = t_1$, we have

$$\|z(\cdot,s_1) - v(\cdot,t_1)\|_{H^1} \le \|z(\cdot,s_1) - z(\cdot,t_1)\|_{H^1} + \|z(\cdot,t_1) - v(\cdot,t_1)\|_{H^1} \le k_{10}\mu.$$

This yields the second inequality in (4.33). □

Remark 4.3. In the proof of Theorem 4.4, we showed the following important property of the periodic solution $z(x,t)$. If $m \in H^1$ and

$$\frac{d\hat{z}}{dt} \ne 0 \quad \text{at the switching moments,} \tag{4.42}$$

then the operators $\mathbf{P}_\alpha(\varphi)$, $\mathbf{P}(\varphi)$, and $\mathbf{t}_1(\varphi)$ are continuously differentiable in a neighborhood of $\Gamma_1 \cap \{\varphi \in H^1 : \hat{\varphi} < \beta\}$. The operator $\mathbf{P}_\beta(\varphi)$ is continuously differentiable in a neighborhood of $\Gamma_2 \cap \{\varphi \in H^1 : \hat{\varphi} > \alpha\}$. Note that this property holds for any α and β, irrespectively of Condition 3.1 or 4.1. However, the transversality condition (4.42) is crucial.

5. **An alternative condition for the existence of periodic solution.** In this section, we formulate a theorem on the existence and uniqueness of a periodic solution of problem (2.1), (2.3) as well as a theorem on attraction to it under the assumption that the weight function $m(x)$ is close to a constant.

Theorem 5.1. *Let $m \in H^1$. There is a number $M > 0$ such that if*

$$\|\nabla m\|_{L_2} \le Mm_0, \tag{5.1}$$

then there is a periodic solution of problem (2.1), (2.3). The number M depends on $K(x)$ but does not depend on $m(x)$, α, and β.

Proof. The main point in the preceding results was to show that

$$\frac{m_0K_0}{2} \le \frac{d\hat{v}(t)}{dt} \le \frac{3m_0K_0}{2}. \tag{5.2}$$

for sufficiently large t. Under assumption (5.1), we will show that (5.2) holds for all $t > 0$ and all φ from the neighborhoods $(3.11)_\alpha$ and $(3.11)_\beta$. This will guarantee the continuity of the Poincaré map $\mathbf{P}$, hence the assertion of the theorem.

Let φ belongs, e.g., to the neighborhood $(3.11)_\alpha$. It suffices to show that

$$\sum_{j=1}^{\infty} \left| m_j(K_j - \lambda_j\varphi_j)e^{-\lambda_j t} \right| \le \frac{m_0K_0}{2} \qquad \forall t \ge 0 \tag{5.3}$$

(cf. the proof of Lemma 2.2). Indeed, using Lemma 2.1, we obtain

$$\begin{aligned}&\sum_{j=1}^{\infty}\left|m_j(K_j-\lambda_j\varphi_j)e^{-\lambda_j t}\right|\\&\quad\le\left(\sum_{j=1}^{\infty}\lambda_j|m_j|^2\right)^{1/2}\left[\left(\sum_{j=1}^{\infty}\frac{|K_j|^2}{\lambda_j}\right)^{1/2}+\left(\sum_{j=1}^{\infty}\lambda_j|\varphi_j|^2\right)^{1/2}\right]\\&\quad\le 3c\|K\|_{H^{1/2}}\|\nabla m\|_{L_2}.\end{aligned}$$

Thus, (5.3) follows, provided that

$$\|\nabla m\|_{L_2}\le\frac{K_0}{6c\|K\|_{H^{1/2}}}m_0.$$

□

Theorem 5.2. *Let $m\in H^1$ and inequality (5.1) hold. There is a number $\tilde M$ such that if*

$$\tilde m\le\tilde M, \tag{5.4}$$

then the following are true:

1. *the conclusions of Theorems 4.1–4.3 with 2 and 4 in (4.28) and (4.20) replaced by constants A and $2A$,*
2. *the conclusion of Theorem 4.4.*

Here $\tilde m$ is given by (3.10), $\tilde M>0$ and $A>1$ depend on m_0, K, and $\beta-\alpha$.

Proof. 1. The crucial point in the proof of Theorems 4.1–4.3 was to show that $\|D_\Phi\Pi_\alpha(\Phi)\|\le 1/2$ for all $\Phi\in\mathcal{B}_\varepsilon$. We will show that, under assumptions (5.1) and (5.4),

$$\|D_\Phi\Pi_\alpha(\Phi)\|\le\sigma<1\quad\forall\Phi\in\mathcal{B}_\varepsilon \tag{5.5}$$

(analogously for Π_β). The rest will follow from the contraction mapping principle as before.

2. Thus, let us prove (5.5) with appropriate σ. It follows from the proof of Theorem 5.1 that the operator $\Pi_\alpha:\mathcal{W}\to\mathcal{W}$ is continuously differentiable on $\mathcal{B}_\varepsilon$. Further, as in the proof of Lemma 4.3, we have for $t_1\ge\tau^*$ (cf. (2.40))

$$\|D_\Phi\Pi_\alpha(\Phi)\|\le\left(1+\frac{2k_1\tilde m}{m_0K_0}\right)e^{-\lambda_1\tau^*/2}, \tag{5.6}$$

where

$$k_1=\frac{2\sqrt{10}c\|K\|_{H^{1/2}}}{e\tau^*}. \tag{5.7}$$

Note that, for $\tilde m\le m_0$, one has

$$\|m\|_{L_2}^2=m_0^2+\tilde m^2\le 2m_0^2.$$

Therefore, taking into account (2.41), we obtain

$$\tau^*\ge k_2\frac{(\beta-\alpha)^2}{2m_0^2}, \tag{5.8}$$

where $k_2>0$ depends only on $\mathcal{B}_\varepsilon$.

Combining inequalities (5.6)–(5.8) yields

$$\|D_\Phi\Pi_\alpha(\Phi)\|\le(1+M_1\tilde m)M_2, \tag{5.9}$$

where $M_1 > 0$ and $0 < M_2 < 1$ depend on m_0, K, and $\beta - \alpha$ but do not depend on $\tilde{m}$. The desired estimate (5.5) follows from (5.6) by choosing $\tilde{m} \le \tilde{M}$ with a sufficiently small $\tilde{M}$. □

6. **Outlook.** In this section, we point out possible directions of further research for parabolic problems with discontinuous hysteresis. We formulate some open questions and discuss a possibility of extending the results and methods of the present paper.

6.1. **Small $\beta - \alpha$.** The understanding of how the thresholds α and β influence the properties of periodic solutions is far from complete. We showed that there are two positive numbers $\delta_1 \le \delta_2$ with the following properties.

1. If $\beta - \alpha > \delta_1$, then (the projection of) the Poincaré map is continuous; hence, there is a periodic solution.
2. If $\beta - \alpha > \delta_2$, then (the projection of) the Poincaré map is a contraction; hence, there is unique periodic solution, which is stable and is a global attractor.

The question remains whether the situation $\delta_1 \ne \delta_2$ is possible and, if so, what happens between δ_1 and δ_2. Can several periodic solutions co-exist? If there exists only one periodic solution, is it stable? Is it a global attractor?

The next question is how the system behaves for $\beta - \alpha \le \delta_1$. We expect that this situation is much more complicated because one need to seek for fixed points of a *discontinuous* Poincaré map, or some power of it. This gives rise to the study of *periodic solutions with several switchings on the period.*

Consider the following example. Let Q be a one-dimensional domain, e.g., $Q = (0, \pi)$. Consider the following problem

$$v_t(x,t) = v_{xx}(x,t) \quad (x \in (0,\pi),\ t > 0),$$
$$v_x(0,t) = 0, \qquad v_x(\pi,t) = \mathcal{H}(\hat{v})(t) \qquad (t > 0).$$

From the physical point of view, this models a thermocontrol process in a rod with heat-insulation at one end and heating (cooling) element at the other.

It is easy to find that

$$\lambda_0 = 0, \quad e_0 = \sqrt{1/\pi}, \quad K_0 = e_0(\pi) = \sqrt{1/\pi},$$
$$\lambda_j = j^2, \quad e_j(x) = \sqrt{2/\pi}\cos jx, \quad K_j = e_j(\pi) = (-1)^j\sqrt{2/\pi}, \quad j = 1, 2, \dots .$$

Let us consider the case where $m_0 = 0.8$, $m_1 = m_2 = 1$, $m_3 = m_4 = \dots = 0$, $\alpha = 0$, and $\beta = 0.05$. In Figures 6.1 and 6.2, a periodic solution and, respectively, a trajectory converging to it are shown. More precisely, their projections to the plane spanned by the vectors e_1 and e_2 are depicted. The graphs were obtained numerically with the software "Dynamical Systems Iterations" (see [23]).

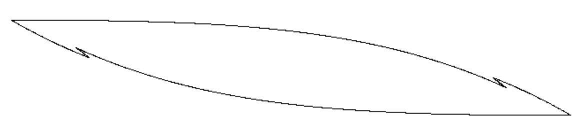

FIGURE 6.1. Periodic solution; projection to the axis e_1, e_2; the parameters are $m_0 = 0.8$, $m_1 = m_2 = 1$, $m_3 = m_4 = \dots = 0$; $\lambda_j = j^2$; $K_0 = \sqrt{1/\pi}$, $K_j = (-1)^j\sqrt{2/\pi}$; $\alpha = 0$ and $\beta = 0.05$

FIGURE 6.2. A trajectory converging to the periodic solution. The parameters are as in Fig. 6.1

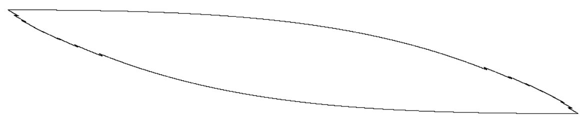

FIGURE 6.3. Periodic solution; $\alpha = 0$, $\beta = 0.01$, the other parameters are as in Fig. 6.1

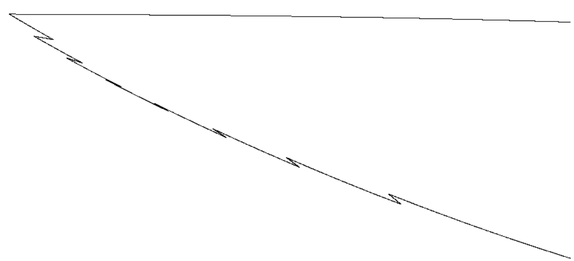

FIGURE 6.4. Zoomed upper-left corner of Fig. 6.3

What we see in the figures indicates the existence of periodic solutions with more than one switching on the period. The periodic trajectory consists of six parts: three parts correspond to the excursion from α to β and three parts to the excursion in the opposite direction.

Furthermore, the smaller the difference $\beta - \alpha$ is, the more switchings may occur on the period (see Fig. 6.3 and its zoomed upper-left corner in Fig. 6.4). Moreover, due to numerical simulations, these solutions can be stable and can be global attractors.

6.2. **Continuous temperature control on the boundary.** In this paper, we assume that the thermal elements on the boundary change their values by jump. A more general setting including both continuous and discontinuous temperature control is as follows (cf. [10, 11, 26, 14]):

$$v_t(x,t) = \Delta v(x,t) \quad (x \in Q,\ t > 0),$$

$$\frac{\partial v}{\partial \nu} = K(x)u(t) \quad (x \in \partial Q,\ t > 0),$$

where the control function $u(t)$ satisfies the ordinary differential equation

$$a\dot{u}(t) + u(t) = \mathcal{H}(\hat{v})(t)$$

with $a \geq 0$. The right-hand side of the latter equation contains the hysteresis operator $\mathcal{H}$ which depends on the mean temperature $\hat{v}$, as before.

The “discontinuous” case $a = 0$ is considered in the present paper. Similarly, if $a > 0$, one can use the Fourier method to obtain an infinite-dimensional dynamical system for the unknowns $(u(t), v_1(t), v_2(t), \dots)$. If $\beta - \alpha$ is large enough, we expect the results analogous to the case $a = 0$. However, the dynamics for small $\beta - \alpha$ remains an open question.

6.3. **Parabolic equations with variable coefficients and other boundary conditions.** Throughout the paper, the parabolic equation under consideration was the heat equation. A more general linear parabolic equation would be

$$v_t(x,t) = Pv(x,t) \quad (x \in Q,\ t > 0)$$

with a general boundary condition of the form

$$Bv = K(x)\mathcal{H}(\hat{v})(t) \quad (x \in \partial Q,\ t > 0).$$

Here B defines the Dirichlet, Neumann, oblique-derivative or more general (e.g., nonlocal) boundary condition;

$$Pw(x) = \sum_{i,j=1}^{n} a_{ij}(x) w_{x_i x_j}(x) + \sum_{i=1}^{n} a_i(x) w_{x_i}(x) + a_0(x) w(x)$$

is a second-order elliptic operator with variable coefficients, whose domain is given by the corresponding homogeneous boundary condition (cf. (2.13)):

$$\mathrm{D}(P) = \{w \in H^2 : Bw = 0\ (x \in \partial Q)\}.$$

In applications, the operator P is not necessarily selfadjoint, but often turns out to be sectorial. Hence, it generates an analytic semigroup (see, e.g., [24]). Furthermore, under suitable assumptions, all the eigenvalues of the operator P are of finite algebraic multiplicity, while the solution v can be expanded into the Fourier series with respect to the root vectors (i.e., eigen- and associated vectors), which is Abel-summable to v [19, 1]; see also [25] for more general boundary operators B (in particular, nonlocal operators) and operators P with nonselfadjoint principal part.

Therefore, the original parabolic problem with hysteresis can be reduced, at least formally, to an infinite-dimensional dynamical system by projecting onto corresponding eigenspaces. However, the justification of such a reduction is a subtle issue because the root vectors of the operator P do not, in general, form a basis in the classical sense.

On the other hand, the resulting dynamical system may be more complicated than that in the present paper. If the operator P is selfadjoint, we still have a system of infinitely many ODEs of the form

$$\dot{v}_j = -\lambda_j v_j + K_j \mathcal{H}(\hat{v}),$$

where $-\lambda_j$ are the eigenvalues of the operator P and K_j are constants that are defined by the function $K(x)$ and the operators P and B. But if P is not selfadjoint, its eigenvectors may have associated vectors, which form Jordan chains. For example, let $-\lambda_j = -\lambda_{j+1}$ be the eigenvalue of P of algebraic multiplicity 2; we denote it by $-\lambda$. Let e_j and e_{j+1} be its eigen- and associated vectors. They satisfy

$$Pe_j = -\lambda e_j \quad (x \in Q), \qquad Be_j = 0 \quad (x \in \partial Q),$$

$$Pe_{j+1} = -\lambda e_{j+1} + e_j \quad (x \in Q), \qquad Be_{j+1} = 0 \quad (x \in \partial Q).$$

In this case, the infinite system of ODEs will contain a coupled two-dimensional subsystem of the form

$$\begin{pmatrix} \dot{v}_j \\ \dot{v}_{j+1} \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} v_j \\ v_{j+1} \end{pmatrix} + \begin{pmatrix} K_j \\ K_{j+1} \end{pmatrix} \mathcal{H}(\hat{v}).$$

6.4. **Nonlinear reaction-diffusion equations.** Another generalization is a nonlinear reaction-diffusion equation instead of the linear heat equation:

$$v_t = \Delta v + f(v).$$

In this case, one can try to use the Galerkin method instead of the Fourier method to reduce the problem to a dynamical system. The difficulty here is that, on the kth step, we obtain a k-dimensional system, which is *nonlinear* even on the time intervals between the switchings of $\mathcal{H}$. Moreover, the nonlinearity changes when passing from the k-dimensional system to the $(k+1)$-dimensional system.

Thus, the first questions to answer are as follows:

1. Are there periodic solutions in the finite-dimensional systems?
2. Can the dynamics of the original problem be approximated by the dynamics of the k-dimensional nonlinear systems as $k \to \infty$?

Acknowledgments. The author is grateful to B. Fiedler, A. V. Fursikov, W. Jäger, and A. L. Skubachevskii for their attention to this work. Special thanks to S. B. Tikhomirov for numerous fruitful discussions and thoroughly reading the manuscript. The author would also like to thank S. B. Tikhomirov for providing his software "Dynamical Systems Iterations" (see [23]), which was used for numerical simulations.

REFERENCES

[1] M. S. Agranovich, *On series in root vectors of operators defined by forms with a selfadjoint principal part*, Funktsional. Anal. i Prilozhen., **28** (1994), 1–21; English transl. in Funct. Anal. Appl., **28** (1994), 151–167.

[2] H. W. Alt, *On the thermostat problem*, Control Cyb., **14** (1985), 171–193.

[3] P.-A. Bliman and A. M. Krasnosel'skii, *Periodic solutions of linear systems coupled with relay*, in "Proceedings of the Second World Congress of Nonlinear Analysts, Part 2 (Athens, 1996)," Nonlinear Anal., **30** (1997), 687–696.

[4] M. Brokate and A. Friedman, *Optimal design for heat conduction problems with hysteresis*, SIAM J. Control Opt., **27** (1989), 697–717.

[5] M. Brokate and J. Sprekels, "Hysteresis and Phase Transitions," Springer, Berlin, 1996.

[6] P. Colli, M. Grasselli and J. Sprekels, *Automatic control via thermostats of a hyperbolic Stefan problem with memory*, Appl. Math. Optim., **39** (1999), 229–255.

[7] M. Fečkan, *Periodic solutions in systems at resonances with small relay hysteresis*, Math. Slovaca, **49** (1999), 41–52.

[8] A. Friedman and K.-H. Hoffmann, *Control of free boundary problems with hysteresis*, SIAM J. Control. Optim., **26** (1988), 42–55.

[9] A. Friedman and L.-S. Jiang *Periodic solutions for a thermostat control problem*, Commun. Partial Differential Equations, **13** (1988), 515–550.

[10] K. Glashoff and J. Sprekels, *An application of Glicksberg's theorem to set-valued integral equations arising in the theory of thermostats*, SIAM J. Math. Anal., **12** (1981), 477–486.

[11] K. Glashoff and J. Sprekels, *The regulation of temperature by thermostats and set-valued integral equations*, J. Integral Equ., **4** (1982), 95–112.

[12] I. G. Götz, K.-H. Hoffmann and A. M. Meirmanov, *Periodic solutions of the Stefan problem with hysteresis-type boundary conditions*, Manuscripta Math., **78** (1983), 179–199.

[13] P. L. Gurevich and W. Jäger, *Parabolic problems with the Preisach hysteresis operator in boundary conditions*, J. Differential Equations, **47** (2009), 2966–3010.

[14] P. L. Gurevich, W. Jäger and A. L. Skubachevskii, *On periodicity of solutions for thermo-control problems with hysteresis-type switches*, SIAM J. Math. Anal., **41** (2009), 733–752.

[15] K.-H. Hoffmann, M. Niezgódka and J. Sprekels, *Feedback control via thermostats of multidimensional two-phase Stefan problems*, Nonlinear Anal., **15** (1990), 955–976.

[16] N. Kenmochi and A. Visintin, *Asymptotic stability for nonlinear PDEs with hysteresis*, European J. Appl. Math., **5** (1994), 39–56.

[17] M. A. Krasnosel'skii and A. V. Pokrovskii, "Systems with Hysteresis," Springer-Verlag, Berlin–Heidelberg–New York, 1989; (Translated from Russian: "Sistemy s Gisterezisom," Nauka, Moscow, 1983).

[18] P. Krejči, J. Sprekels and U. Stefanelli, *Phase-field models with hysteresis in one-dimensional thermo-visco-plasticity*, SIAM J. Math. Anal., **34** (2002), 409–434.

[19] V. B. Lidskii, *Summability of series in terms of the principal vectors of non-selfadjoint operators*, Trudy Moskov. Mat. Obsc., **11** (1962), 3–35.

[20] J. L. Lions and E. Magenes, "Non-Homogeneous Boundary Value Problems and Applications, Vol. **I**," Springer, Berlin–Heidelberg–New York, 1972.

[21] J. L. Lions and E. Magenes, "Non-Homogeneous Boundary Value Problems and Applications, Vol. **II**," Springer, Berlin–Heidelberg–New York, 1972.

[22] J. Macki, P. Nistri and P. Zecca, *Mathematical models for hysteresis*, SIAM Rev., **35** (1993), 94–123.

[23] G. S. Osipenko, M. V. Senkov and S. B. Tikhomirov, *Algorithms of construction of invariant manifolds and attractors*, Abstracts Intern. Conf. "Fundamental Research in Technical Universities," **101** (2005).

[24] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Appl. Math. Sci., **44**, Springer, New York, 1983.

[25] V. V. Pod"yapol'skii, *Completeness of a system of root functions of a nonlocal problem in* L_p, Mat. Zametki, **71** (2002), 878–889; English transl. in Math. Notes, **71** (2002), 804–814.

[26] J. Prüss, *Periodic solutions of the thermostat problem*, in Proc. Conf. "Differential Equations in Banach Spaces" (Bologna, July 1985), Lecture Notes Math., **1223**, Springer-Verlag, Berlin – New York, (1986), 216–226.

[27] T. I. Seidman, *Switching systems and periodicity*, in Proc. Conf. "Nonlinear Semigroups, Partial Differential Equations and Attractors" (Washington, DC, 1987), Lecture Notes Math., **1394**, Springer-Verlag, Berlin – New York, (1989), 199–210.

[28] S. Varigonda and T. Georgiou, *Dynamics of relay relaxation oscillators*, IEEE Trans. Automat. Control, **46** (2001), 65–77.

[29] A. Visintin, "Differential Models of Hysteresis," Springer-Verlag, Berlin – Heidelberg, 1994.

[30] A. Visintin, *Quasilinear parabolic P.D.E.s with discontinuous hysteresis*, Annali di Matematica, **185** (2006), 487–519.

[31] L. F. Xu, *Two parabolic equations with hysteresis*, J. Partial Differential Equations, **4** (1991), 51–65.

Received January 2010; revised June 2010.

E-mail address: gurevichp@gmail.com