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Nonlocal Elliptic Problems in Dihedral Angles and the Green Formula

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The most difficult case in studying elliptic problems with nonlocal conditions is that where the support of nonlocal data intersects the boundary (see [1–4]). In this case, the solutions have power singularities near some set. Therefore, it is natural to consider nonlocal elliptic problems in weight spaces (see [5, 6]). In obtaining *a priori* estimates of solutions and constructing right regularizers of nonlocal problems in bounded domains, model nonlocal boundary value problems in planar and dihedral angles arise (see [3, 4]). In this paper, an approach to studying nonlocal problems based on the use of the Green formula and conjugate nonlocal problems is suggested. This approach makes it possible to remove additional constraints imposed in [3] on the corresponding local model problem and obtain necessary and sufficient conditions for the Fredholm property of nonlocal problems in planar angles and the unique solvability of nonlocal problems in dihedral angles. As the conjugate problems, nonlocal transmission problems arise; they were earlier considered for bounded domains with smooth boundaries in $\mathbb{R}^n$ [7, 8] and in the one-dimensional case [9].

1. Consider the dihedral angle $\Omega = \{x = (y, z)\colon b_1 < \varphi < b_2, z \in \mathbb{R}^{n-2}\}$ with the faces $\Gamma_j = \{x = (y, z)\colon \varphi = b_j, z \in \mathbb{R}^{n-2}\}$ $(j = 1, 2)$ and the edge $M = \{x = (y, z)\colon y = 0, z \in \mathbb{R}^{n-2}\}$. Here, $x = (y, z) \in \mathbb{R}^n$, $y \in \mathbb{R}^2$, and $z \in \mathbb{R}^{n-2}$; r, and φ are the polar coordinates of the point y; and $0 < b_1 < b_2 < 2\pi$.

Let $\mathcal{P}(D_y, D_z)$, $B_{j\mu}(D_y, D_z)$, and $B^{\mathcal{G}}_{j\mu}(D_y, D_z)$ denote homogeneous differential operators with constant complex coefficients of orders $2m$, $m_{j\mu} \le 2m - 1$, and $m_{j\mu} \le 2m - 1$, respectively ($j = 1, 2$ and $\mu = 1, 2, \dots, m$). We assume that the operator $\mathcal{P}(D_y, D_z)$ is properly elliptic and the system of operators $\{B_{j\mu}(D_y, D_z)\}_{\mu=1}^m$ is normal and covers $\mathcal{P}(D_y, D_z)$ on Γ_j for $j = 1, 2$ (see [10, Chapter 2]).

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Consider the following nonlocal boundary value problem in the dihedral angle Ω:

$$\mathcal{P}(D_y, D_z)U = f(x), \quad x \in \Omega, \tag{1}$$

$$\begin{aligned} \mathcal{B}_{j\mu}(D_y, D_z)U &\equiv B_{j\mu}(D_y, D_z)U|_{\Gamma_j} \\ &+ (B^{\mathcal{G}}_{j\mu}(D_y, D_z)U)(\mathcal{G}_j y, z)\big|_{\Gamma_j} = g_{j\mu}(x), \quad x \in \Gamma_j. \end{aligned} \tag{2}$$

Here and in what follows, the subscripts j and μ take the values $j = 1, 2$ and $\mu = 1, 2, \dots, m$; we write $(B^{\mathcal{G}}_{j\mu}(D_y, D_z)U)(\mathcal{G}_j y, z)$ to denote that the expression $(B^{\mathcal{G}}_{j\mu}(D_{y'}, D_{z'})U)(x')$ is taken at $x' = (\mathcal{G}_j y, z)$, where $\mathcal{G}_j$ is the operator of rotation through the angle φ_j and dilation by a factor of χ_j in the plane $\{y\}$ such that $b_1 < b_1 + \varphi_1 = b_2 + \varphi_2 = b < b_2$ and $0 < \chi_j$. Note that we impose no constraints on the nonlocal operators $B^{\mathcal{G}}_{j\mu}(D_y, D_z)$ (except the constraint on their order).

Let us introduce the space $H^l_a(\Omega)$ being the completion of the set $C_0^\infty(\overline{\Omega}\backslash M)$ in the norm $\|w\|_{H^l_a(\Omega)} = \left(\sum_{|\alpha| \le l} \int_\Omega r^{2(a - l + |\alpha|)} |D_x^\alpha w(x)|^2 dx\right)^{\frac{1}{2}}$, where $C_0^\infty(\overline{\Omega}\backslash M)$ is the set of functions infinitely differentiable in $\overline{\Omega}$ and compactly supported on $\overline{\Omega}\backslash M$; $a \in \mathbb{R}$; and $l \ge 0$ is an integer. By $H_a^{l - \frac{1}{2}}(\Gamma)$ $(l \ge 1)$, we denote the space of traces on an $(n-1)$-dimensional half-plane $\Gamma \subset \overline{\Omega}$ with the norm $\|\psi\|_{H_a^{l-1/2}(\Gamma)} = \inf \|w\|_{H^l_a(\Omega)}$ over $w \in H^l_a(\Omega)$ such that $w|_\Gamma = \psi$.

Consider the bounded operator

$$\mathcal{L} = \{\mathcal{P}(D_y, D_z), \mathcal{B}_{j\mu}(D_y, D_z)\}\colon$$

$$H_a^{2m}(\Omega) \to H_a^0(\Omega) \times \prod_{j,\mu} H_a^{2m - m_{j\mu} - \frac{1}{2}}(\Gamma_j)$$

corresponding to problem (1), (2).

2. Consider the auxiliary nonlocal boundary value problem

$$\mathcal{P}(D_y, \theta)u = f(y), \quad y \in K, \tag{3}$$

$$\mathcal{B}_{j\mu}(D_y, \theta)u \equiv B_{j\mu}(D_y, \theta)u|_{\gamma_j} + (\mathcal{B}^{\mathcal{G}}_{j\mu}(D_y, \theta)u)(\mathcal{G}_j y)\big|_{\gamma_j} = g_{j\mu}(y), \quad y \in \gamma_j, \tag{4}$$

where $K = \{y: b_1 < \varphi < b_2\}$; $\gamma_j = \{y: \varphi = b_j\}$; and θ is an arbitrary point of the unit sphere $S^{n-3} = \{z \in \mathbb{R}^{n-2}: |z| = 1\}$.

Let us introduce the space $E^l_a(K)$ being the completion of the set $C_0^\infty(\bar{K}\setminus\{0\})$ in the norm $\|w\|_{E^l_a(K)} = \left(\sum_{|\alpha| \le l} \int_K r^{2a}(r^{2(|\alpha|-l)} + 1)\left|D^\alpha_y w(y)\right|^2 dy\right)^{\frac{1}{2}}$. By $E_a^{l-\frac{1}{2}}(\gamma)$ for $l \ge 1$, we denote the space of traces on a $\gamma \subset \bar{K}$ with the norm $\|\psi\|_{E_a^{l-\frac{1}{2}}(\gamma)} = \inf\|w\|_{E^l_a(K)}$ over $w \in E^l_a(K)$ such that $w|_\gamma = \psi$.

Put $E^0_a(K, \gamma) = E^0_a(K) \times \prod_{j,\mu} E_a^{2m - m_{j\mu} - \frac{1}{2}}(\gamma_j)$. Consider the bounded operator $\mathcal{L}(\theta)$: $E^{2m}_a(K) \to E^0_a(K, \gamma)$ defined by

$$\mathcal{L}(\theta)u = \{\mathcal{P}(D_y, \theta)u, \mathcal{B}_{j\mu}(D_y, \theta)u\}.$$

The operator $\mathcal{L}(\theta)$ corresponds to nonlocal boundary value problem (3), (4). The solvability of problems (1), (2) and (3), (4) is closely related to the arrangement of the eigenvalues of some model nonlocal problem for an ordinary differential equation. To obtain this problem, we write the operators $\mathcal{P}(D_y, 0)$, $B_{j\mu}(D_y, 0)$, and $B^{\mathcal{G}}_{j\mu}(D_y, 0)$ in the polar coordinates: $\mathcal{P}(D_y, 0) = r^{-2m}\mathbf{P}(\varphi, D_\varphi, rD_r)$, $B_{j\mu}(D_y, 0) = r^{-m_{j\mu}}\mathbf{B}_{j\mu}(\varphi, D_\varphi, rD_r)$, and $B^{\mathcal{G}}_{j\mu}(D_y, 0) = r^{-m_{j\mu}}\mathbf{B}^{\mathcal{G}}_{j\mu}(\varphi, D_\varphi, rD_r)$, where $D_\varphi = -i\frac{\partial}{\partial\varphi}$ and $D_r = -i\frac{\partial}{\partial r}$. Let us set $\theta = 0$ and $\{f, g_{j\mu}\} = 0$ in (3) and (4), introduce the new variable $\tau = \ln r$, and apply the Fourier transform with respect to τ. We obtain the problem

$$\mathbf{P}(\varphi, D_\varphi, \lambda)w(\varphi, \lambda) = 0, \quad b_1 < \varphi < b_2, \tag{5}$$

$$\mathbf{B}_{j\mu}(\varphi, D_\varphi, \lambda)w(\varphi, \lambda)|_{\varphi = b_j} + e^{(i\lambda - m_{j\mu})\ln\chi_j}\mathbf{B}^{\mathcal{G}}_{j\mu}(\varphi, D_\varphi, \lambda)w(\varphi + \varphi_j, \lambda)\big|_{\varphi = b_j} = 0, \tag{6}$$

where

$$w(\varphi, \lambda) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} u(\varphi, \tau)e^{-i\lambda\tau}d\tau.$$

This problem is the ordinary differential equation of form (5) for a function $w \in W^{2m}_2(b_1, b_2)$ with nonlocal conditions (6), which relate the values of the function w and its derivatives at the point $\varphi = b_j$ to the values of the function w and its derivatives at an interior point $\varphi = b$ of the interval (b_1, b_2).

Lemma 1. *Suppose that the line* $\operatorname{Im}\lambda = a + 1 - 2m$ *contains no eigenvalues of model problem* (5), (6). *Then, for all* $u \in E^{2m}_a(K)$ *and all* $\theta \in S^{n-3}$, *we have*

$$\|u\|_{E^{2m}_a(K)} \le c_1(\|\mathcal{L}(\theta)u\|_{E^0_a(K, \gamma)} + \|u\|_{L_2(K \cap S)}), \tag{7}$$

where $S = \{y: 0 < R_1 < r < R_2\}$ *and* $c_1 > 0$ *does not depend on* θ *and* u. *If there exists* $\theta \in S^{n-3}$ *such that estimate* (7) *holds for all* $u \in E^{2m}_a(K)$, *then the line* $\operatorname{Im}\lambda = a + 1 - 2m$ *contains no eigenvalues of problem* (5), (6).

Lemma 1 is proved in [3, Section 3]. It implies that the operator $\mathcal{L}(\theta)$ has a finite-dimensional kernel and closed image. To prove that the cokernel of $\mathcal{L}(\theta)$ is finite-dimensional, we apply the Green formula to problem (3), (4).

3. Consider $\Gamma = \{x = (y, z): \varphi = b, z \in \mathbb{R}^{n-2}\}$ and $\gamma = \{y: \varphi = b\}$. The sets Γ and γ are the carriers of nonlocal data in problems (1), (2) and (3), (4), respectively. Set $K_1 = \{y: b_1 < \varphi < b\}$ and $K_2 = \{y: b < \varphi < b_2\}$. Let $\mathcal{Q}(D_y, D_z)$ be the operator formally conjugate to $\mathcal{P}(D_y, D_z)$. In the statement of the following theorem, $\mathcal{P} = \mathcal{P}(D_y, D_z)$, $\mathcal{Q} = \mathcal{Q}(D_y, D_z)$, etc.

Theorem 1. *For the operators* $\mathcal{P}$, $B_{j\mu}$, *and* $B^{\mathcal{G}}_{j\mu}$ *defined in Section* 1, *there exist* (*but are not unique*) (i) *a system of operators* $\{B'_{j\mu}\}^m_{\mu=1}$ *of orders* $2m - 1 - m'_{j\mu}$ *normal on* Γ_j *with constant coefficients that complements* $\{B_{j\mu}\}^m_{\mu=1}$ *to a Dirichlet system of order* $2m$ *on* Γ_j[1]; (ii) *a system* $\{B_\mu, B'_\mu\}^m_{\mu=1}$ *being a Dirichlet system of order* $2m$ *on* Γ *such that the orders of the operators* B_μ *and* B'_μ *are* $2m - \mu$ *and* $m - \mu$, *respectively.*

If a choice of such systems is made, then there exist operators $C_{j\mu}$, $C'_{j\mu}$, T_ν, *and* $T^{\mathcal{G}}_{j\nu}$ ($j = 1, 2$; $\mu = 1, 2, \ldots, m$; $\nu = 1, 2, \ldots, 2m$) *with constant coefficients possessing the following properties*: (i) *the orders of the operators* $C_{j\mu}$, $C'_{j\mu}$, T_ν, *and* $T^{\mathcal{G}}_{j\nu}$ *are* $m'_{j\mu}$, $2m - 1 - m_{j\mu}$, $\nu - 1$, *and* $\nu - 1$, *respectively*; (ii) *the system* $\{C_{j\mu}\}^m_{\mu=1}$ *covers the*

[1] The definition of a Dirichlet system is given in [10, Chapter 2, Section 2.2].

operator $\mathcal{Q}$ *on* Γ_j *and complements* $\{C'_{j\mu}\}_{\mu=1}^{m}$ *to a Dirichlet system of order* $2m$ *on* Γ_j, *and the system* $\{T_\nu\}_{\nu=1}^{2m}$ *is a Dirichlet system of order* $2m$ *on* Γ;

(iii) *for any functions* $u \in E_a^{2m}(K)$ *and* $v_j \in E_{-a+2m}^{2m}(K_j)$, *the Green formula with parameter* θ *holds*:

$$\sum_j (\mathcal{P}(D_y, \theta)u, v_j)_{K_j} + \sum_{j,\mu} (\mathcal{B}_{j\mu}(D_y, \theta)u, C'_{j\mu}(D_y, \theta)v_j|_{\gamma_j})_{\gamma_j} + \sum_\mu (B_\mu(D_y, \theta)u|_\gamma, \mathcal{T}_\mu(D_y, \theta)v)_\gamma$$
$$= \sum_j (u, \mathcal{Q}(D_y, \theta)v_j)_{K_j} + \sum_{j,\mu} (B'_{j\mu}(D_y, \theta)u|_{\gamma_j}, C_{j\mu}(D_y, \theta)v_j|_{\gamma_j})_{\gamma_j} + \sum_\mu (B'_\mu(D_y, \theta)u|_\gamma, \mathcal{T}_{m+\mu}(D_y, \theta)v)_\gamma. \quad (8)$$

Here, $(\cdot,\cdot)_{K_j}$, $(\cdot,\cdot)_{\gamma_j}$, *and* $(\cdot,\cdot)_\gamma$ *are scalar products in* $L_2(K_j)$, $L_2(\gamma_j)$, *and* $L_2(\gamma)$, *respectively*; $\mathcal{T}_\nu(D_y, \theta)v \equiv T_\nu(D_y, \theta)v_1|_\gamma - T_\nu(D_y, \theta)v_2|_\gamma + \sum_k (T_{k\nu}^{\mathcal{G}}(D_y, \theta)v_k)(\mathcal{G}_k^{-1}y)\big|_\gamma$; $\mathcal{G}_k^{-1}$ *is the operator of rotation through the angle* $-\varphi_k$ *and dilation by a factor of* $\frac{1}{\chi_k}$ *in the plane* $\{y\}$; $k = 1, 2$; *and* $\nu = 1, 2, \ldots, 2m$.

Formula (8) generates the following problem formally conjugate to problem (3), (4):

$$\mathcal{Q}(D_y, \theta)v_j = f_j(y), \quad y \in K_j, \quad (9)$$

$$\mathcal{C}_{j\mu}(D_y, \theta)v \equiv C_{j\mu}(D_y, \theta)v_j|_{\gamma_j} = g_{j\mu}(y), \quad y \in \gamma_j, \quad (10)$$

$$\mathcal{T}_\nu(D_y, \theta)v \equiv T_\nu(D_y, \theta)v_1|_\gamma - T_\nu(D_y, \theta)v_2|_\gamma + \sum_k (T_{k\nu}^{\mathcal{G}}(D_y, \theta)v_k)(\mathcal{G}_k^{-1}y)\big|_\gamma = h_\nu(y), \quad y \in \gamma, \quad (11)$$

where, as above, $j = 1, 2$; $\mu = 1, 2, \ldots, m$; $\theta \in S^{n-3}$; and $v(y) \equiv v_j(y)$ for $y \in K_j$. Here and in what follows, the subscripts ν and k take the values $\nu = 1, 2, \ldots, 2m$, $k = 1, 2$. We call problem (9)–(11) a nonlocal transmission problem.

Let us introduce the notations $\mathcal{E}_{-a+2m}^{0}(K, \gamma) = E_{-a+2m}^{0}(K) \times \prod_{j,\mu} E_{-a+2m}^{2m-m'_{j\mu}-\frac{1}{2}}(\gamma_j) \times \prod_\nu E_{-a+2m}^{2m-\nu+\frac{1}{2}}(\gamma)$ and $\mathcal{E}_{-a+2m}^{2m}(K) = \bigoplus_j E_{-a+2m}^{2m}(K_j)$. Consider the bounded operator $\mathcal{M}(\theta)$: $\mathcal{E}_{-a+2m}^{2m}(K) \to \mathcal{E}_{-a+2m}^{0}(K, \gamma)$ defined as

$$\mathcal{M}(\theta)v = \{w, \mathcal{C}_{j\mu}(D_y, \theta)v, \mathcal{T}_\nu(D_y, \theta)v\}.$$

Here, $w_j(y) \equiv \mathcal{Q}(D_y, \theta)v_j(y)$ for $y \in K_j$; v_j is the restriction of the function v to K_j. The operator $\mathcal{M}(\theta)$ corresponds to nonlocal transmission problem (9)–(11).

Lemma 2. *Suppose that the line* $\operatorname{Im}\lambda = a + 1 - 2m$ *contains no eigenvalues of model problem* (5), (6). *Then, for all* $v \in \mathcal{E}_{-a+2m}^{2m}(K)$ *and* $\theta \in S^{n-3}$, *we have*

$$\|v\|_{\mathcal{E}_{-a+2m}^{2m}(K)} \le c_2(\|\mathcal{M}(\theta)v\|_{\mathcal{E}_{-a+2m}^{0}(K,\gamma)} + \|v\|_{L_2(K \cap S')}), \quad (12)$$

where $S' = \{y: 0 < R'_1 < r < R'_2\}$ *and* $c_2 > 0$ *does not depend on* θ *and* v. *If there exists* $\theta \in S^{n-3}$ *such that* (12) *holds for all* $v \in \mathcal{E}_{-a+2m}^{2m}(K)$, *then the line* $\operatorname{Im}\lambda = a + 1 - 2m$ *contains no eigenvalues of problem* (5), (6).

Lemma 2 implies that $\mathcal{M}(\theta)$ has a finite-dimensional kernel and closed image.

4. For $\mathcal{L}(\theta)$, consider the conjugate operator $\mathcal{L}^*(\theta)$: $(E_a^0(K, \gamma))^* \to (E_a^{2m}(K))^*$ acting on $F = \{f, g_{j\mu}\} \in (E_a^0(K, \gamma))^*$ by the rule

$$\langle u, \mathcal{L}^*(\theta)F\rangle = \langle \mathcal{P}(D_y, \theta)u, f\rangle + \sum_{j,\mu} \langle \mathcal{B}_{j\mu}(D_y, \theta)u, g_{j\mu}\rangle$$

for any $u \in E_a^{2m}(K)$. Here, $\langle\cdot,\cdot\rangle$ stands for the sesquilinear form on the corresponding dual pairs of spaces.

Let us establish the relation between the kernels of the operators $\mathcal{L}^*(\theta)$ and $\mathcal{M}(\theta)$.

Lemma 3. *The kernel of the operator* $\mathcal{L}^*(\theta)$ *coincides with the set of values of the element* $\{v, C'_{j\mu}(D_y, \theta)v_j|_{\gamma_j}\}$ *for all* $v \in \mathcal{E}_{-a+2m}^{2m}(K)$ *and* $v_j \in C^\infty(\bar{K}_j \setminus \{0\})$ *being solutions to problem* (9)–(11) *with* $\{f_j, g_{j\mu}, h_\nu\} = 0$.

Lemmas 1–3 imply the following result.

Theorem 2. *If the line* $\operatorname{Im}\lambda = a + 1 - 2m$ *contains no eigenvalues of model problem* (5), (6), *then the operator* $\mathcal{L}(\theta)$ *is Fredholm of index zero for all* $\theta \in S^{n-3}$. *If the operator* $\mathcal{L}(\theta)$ *is Fredholm of index zero at some* $\theta \in S^{n-3}$, *then the line* $\operatorname{Im}\lambda = a + 1 - 2m$ *contains no eigenvalues of problem* (5), (6).

5. Let us examine the solvability of nonlocal boundary-value problem (1), (2).

Theorem 3. *If the line* $\operatorname{Im}\lambda = a + 1 - 2m$ *contains no eigenvalues of model problem* (5), (6) *and* $\dim\ker(\mathscr{L}(\theta)) = \operatorname{codim}\mathscr{R}(\mathscr{L}(\theta)) = 0$ *for all* $\theta \in S^{n-3}$, *then the operator* $\mathscr{L}$ *is an isomorphism.*

Theorem 4. *If the operator* $\mathscr{L}$ *is Fredholm of index zero, then the operator* $\mathscr{L}(\theta)$ *is an isomorphism for all* $\theta \in S^{n-3}$.

Lemma 1 and Theorems 3 and 4 imply in particular that, if the operator $\mathscr{L}$ is Fredholm of index zero, then it is an isomorphism.

6. All results obtained in Sections 1–5 can be extended over systems of equations for N functions defined in N dihedral (planar) angles with nonlocal conditions containing finitely many nonlocal terms and relating the values of the functions and their derivatives on the faces of the angles to the values of the functions and their derivatives on some half-planes (rays) lying strictly inside the angles. Application of *a priori* estimates in weight spaces similar to the estimates obtained in [3, 6] makes it possible to study the Fredholm property of the operator $\mathscr{L}(\theta)$: $E_a^{l+2m}(K) \to E_a^l(K) \times \prod_{j,\mu} E_a^{l+2m-m_{j\mu}-\frac{1}{2}}(\gamma_j)$ and the invertibility of the operator $\mathscr{L}$: $H_a^{l+2m}(\Omega) \to H_a^l(\Omega) \times \prod_{j,\mu} H_a^{l+2m-m_{j\mu}-\frac{1}{2}}(\Gamma_j)$ for any $l = 1, 2, \ldots$.

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