The symmetric moment curve and centrally symmetric polytopes with many faces

Alexander Barvinok

Univertsity of Michigan

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Joint work with Isabella Novik and Seung Jin Lee. The papers are available at
`http://www.math.lsa.umich.edu/~barvinok/papers.html`

The classical moment curve

C. Carathéodory (1911), T.S. Motzkin (1957), D. Gale (1963):

$$M_d(t) = \left(t,\ t^2,\ t^3,\ \ldots,\ t^d\right), \qquad M_d : \mathbb{R} \longrightarrow \mathbb{R}^d$$

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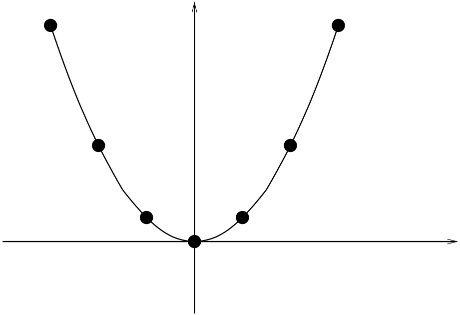

Properties:

- Every affine hyperplane in $\mathbb{R}^d$ intersects $M_d(t)$ in at most d points
- Pick any $t_1 < \ldots < t_m$ and let

$$P = \operatorname{conv}\Big(M_d(t_1), \ldots, M_d(t_m)\Big).$$

Then, for any $I \subset \{1, \ldots, m\}$ such that $|I| \leq d/2$ the set

$$\operatorname{conv}\Big(M_d(t_i) : \quad i \in I\Big)$$

is a face of P.
For example, if $d = 4$ then every two vertices of P span an edge.

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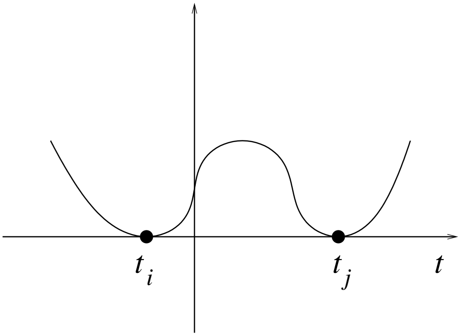

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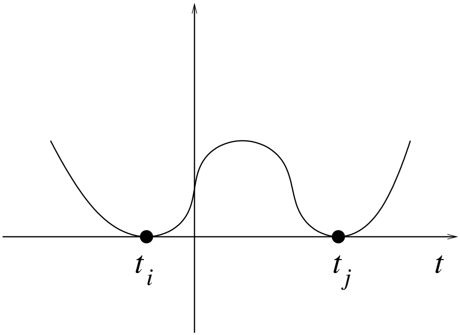

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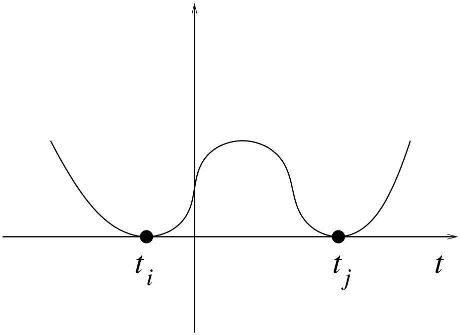

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The trigonometric version

If $d = 2k$, there is a trigonometric version

$$T_k(t) = (\cos t,\ \sin t,\ \cos 2t,\ \sin 2t, \ldots, \cos kt,\ \sin kt),$$
$$T_k : \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}^{2k}.$$

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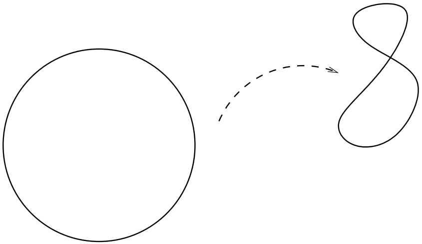

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P. McMullen (1970):

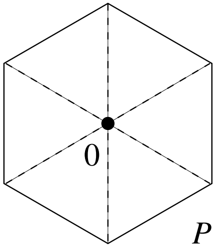
cyclic polytopes maximize the number of faces of every dimension among all polytopes of a given dimension and with a given number of vertices.

Symmetric polytopes

What if we want

$$P = -P$$

symmetric polytope with many faces?



Symmetric polytopes

Some results in the negative direction:

N. Linial and I. Novik (2006):

Let $P \subset \mathbb{R}^d$ be a symmetric polytope with N vertices such that $[v, u]$ is an edge of P for any two vertices u, v such that $u \neq v, -v$. Then

$$N \leq 2^d.$$

A. Barvinok and I. Novik (2008):

Let $P \subset \mathbb{R}^d$ be a symmetric polytope with N vertices. Then

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Note

$$U_k(t+\pi) = -U_k(t).$$

Properties:

- Every affine hyperplane in $\mathbb{R}^{2k}$ intersects $U_k(t)$ in not more than $4k-2$ points (cannot have fewer for a centrally symmetric curve);

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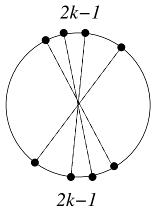

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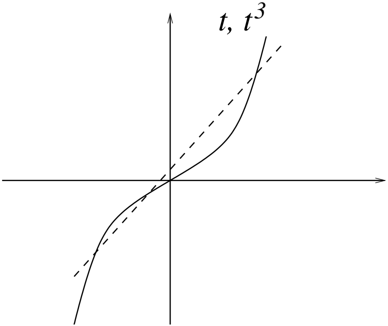
Interestingly, for the curve

$$\left(t,\ t^3,\ \ldots,\ t^{2k+1}\right)$$

there are affine hyperplanes intersecting it in $4k-1$ points.



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Further properties:

- There is an angle

$$\frac{\pi}{2} < \phi_k < \pi$$

such that if $t_1, \ldots, t_k$ lie in an open arc of $\mathbb{S}^1$ of length ϕ_k then

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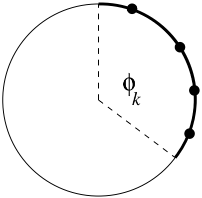

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Moreover,

$$\lim_{k \longrightarrow +\infty} \phi_k = \frac{\pi}{2} \approx 1.57.$$

Some values of ϕ_k:

$$\phi_2 = \frac{2\pi}{3} \approx 2.09, \quad \phi_3 = \pi - \arccos \frac{3-\sqrt{5}}{2} \approx 1.96,$$

$$\phi_4 = 2 \arccos \left(-\frac{1}{48} \left(91 + 336\sqrt{15}\right)^{1/3} + \frac{119}{48\left(91 + 336\sqrt{15}\right)^{1/3}} + \frac{29}{48} \right) \approx 1.87.$$

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We have

$$U_2(t) = (\cos t,\ \sin t,\ \cos 3t,\ \sin 3t), \quad U_2 : \mathbb{S}^1 \longrightarrow \mathbb{R}^4.$$

The faces of

$$\operatorname{conv}\Big(U_2(t) : \quad t \in \mathbb{S}^1\Big)$$

classified by Z. Smilansky (1985):

- Vertices:

$$U_2(t) \text{ where } t \in \mathbb{S}^1;$$

- Edges:

$$[U_2(t_1),\ U_2(t_2)] \text{ where } t_1 \neq t_2 \text{ lie in an open arc of length } < \frac{2\pi}{3};$$

- 2-faces:

$$\operatorname{conv}\Big(U_2(t_1),\ U_2(t_2),\ U_2(t_3)\Big) \text{ where } t_1, t_2, t_3 \in \mathbb{S}^1$$

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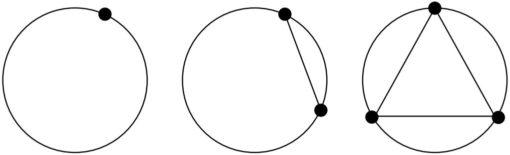

vertices edges 2–faces

An interesting phenomenon

Consider

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Pick a finite set $T \subset \mathbb{S}^1$ of points so that $T + \pi = T$ and consider a polytope P in $\mathbb{R}^4$

$$P = \operatorname{conv}\Big(U_2(t) : \quad t \in T\Big).$$

- If the points of T are uniformly distributed in $\mathbb{S}^1$ then about $2/3$ of all pairs $\{U_2(t_1), U_2(t_2)\}$ span an edge of P.
- If the points of T cluster around the corners of a square in $\mathbb{S}^1$ then about $3/4$ of all pairs $\{U_2(t_1), U_2(t_2)\}$ span an edge of P.

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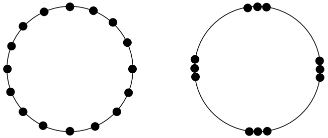

Constructing 2-neighborly polytopes

For $m \geq 1$, consider the curve

$$\Phi_m(t) = (\cos t,\ \sin t,\ \cos 3t,\ \sin 3t,\ \ldots, \cos 3^m t,\ \sin 3^m t),$$
$$\Phi_m : \quad \mathbb{S}^1 \longrightarrow \mathbb{R}^{2(m+1)}.$$

Let

$$A_m = \left\{ \frac{\pi(j-1)}{3^m - 1}, \quad j = 1, \ldots, 2\left(3^m - 1\right) \right\}$$

be the set of $2\left(3^m - 1\right)$ equally spaced points in $\mathbb{S}^1$ and

$$P_m = \operatorname{conv}\Big(\Phi_m(t) : \quad t \in A_m\Big).$$

P_m is a centrally symmetric 2-neighborly polytope of dimension $2(m+1)$ with $2(3^m - 1)$ vertices.

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P_m is a centrally symmetric 2-neighborly polytope of dimension $2(m+1)$ with $2(3^m - 1)$ vertices.

Constructing 2-neighborly polytopes

For $m \geq 1$, consider the curve

$$\Phi_m(t) = (\cos t,\ \sin t,\ \cos 3t,\ \sin 3t,\ \ldots, \cos 3^m t,\ \sin 3^m t),$$
$$\Phi_m : \quad \mathbb{S}^1 \longrightarrow \mathbb{R}^{2(m+1)}.$$

Let

$$A_m = \left\{ \frac{\pi(j-1)}{3^m - 1}, \quad j = 1, \ldots, 2(3^m - 1) \right\}$$

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For $N = 2(3^m - 1)$ and $d = 2(m+1)$ observe that $N \approx \left(\sqrt{3}\right)^d \approx (1.73)^d$. Note that the upper bound is $N \leq 2^d$.

The proof is based on the following lemma:

Lemma

Let $C \subset \mathbb{S}^1$ be a finite set and let

$$P(C) = \operatorname{conv}\Big(\Phi_m(t): \quad t \in C\Big).$$

Suppose that

$$3^k t_1 \not\equiv 3^k t_2 \mod 2\pi \quad \text{for all} \quad t_1, t_2 \in C$$

such that $t_1 \neq t_2$ and all $k = 1, \ldots, m-1$. Then for every pair of distinct $t_1, t_2 \in C$ which lie in an open arc of length $\pi\left(1 - \frac{1}{3^m}\right)$ vertices $\Phi_m(t_1)$ and $\Phi_m(t_2)$ span an edge of P.

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Consider the map

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and use the induction on m. □

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Replacing each point in A_m by a cluster of points, we obtain a polytope of dimension $d = 2(m+1)$ with arbitrarily many N vertices and about

$$\frac{N^2}{2}\left(1 - 3^{-d/2}\right) \approx \frac{N^2}{2}\left(1 - 0.58^d\right)$$

edges. Note that the upper bound on the number of edges is

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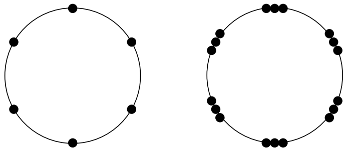

Constructing polytopes with many faces

For a non-negative integer m, consider the map

$$\Psi_{k,m}(t) = \left(U_{3k}(t),\ U_{3k}(5t),\ \ldots,\ U_{3k}\left(5^m t\right)\right)$$
$$\Psi_{k,m}:\ \mathbb{S}^1 \longrightarrow \mathbb{R}^{6k(m+1)}.$$

Let $A_{m,n}$ be the set of $n5^m$ equally spaced points,

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Ramifications

So far, we managed to construct, for a fixed k, centrally symmetric polytopes of an arbitrarily high dimension d and with an arbitrarily large number of vertices N, so that the fraction of "non-k-faces" decreases exponentially in d.

Fix $\alpha > 0$. Letting k vary, we construct centrally symmetric polytopes of dimension $d = k^{1+o(1)}$ with an arbitrarily large number of vertices N, so that the fraction of "non-k-faces" decreases as $d^{-\alpha}$.

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