Eigenvalue bounds for sets avoiding norm 1 in $\mathbb{R}^d$

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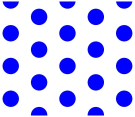
Sets avoiding norm 1

- A subset A of $\mathbb{R}^d$ avoids norm 1 if $\|x - y\| \neq 1$ for all $x, y \in A$.
- Example in dimension 2, Euclidean norm:



Disks of diameter 1, centers at distance at least 2 apart.

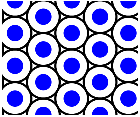
- The density $\delta(A)$ of a measurable subset A is defined as usual:

$$\delta(A) = \limsup_{r \to +\infty} \frac{\mathrm{vol}(A \cap B(r))}{\mathrm{vol}(B(r))}.$$

Question: How large can be $\delta(A)$ if A avoids norm 1 ?

Sets avoiding norm 1

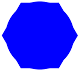
- $\delta(A) = \pi/8\sqrt{3} \approx 0.226$



- In general (arbitrary dimension and norm), a similar construction achieves

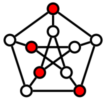
$$\delta(A) = (\text{density of an optimal packing of unit balls})/2^d.$$

- In dimension 2 for the Euclidean norm the best known construction is an hexagonal arrangement of tortoises, giving $\delta \approx 0.229$.

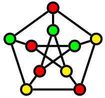


Finite graphs $G = (V, E)$

- A stable set or independent set is a subset S of V such that $S^2 \cap E = \emptyset$. The independence number $\alpha(G)$ is the maximal number of elements of an independent set.



- The chromatic number $\chi(G)$ is the least number of colors needed to color the vertices of G so that vertices connected by an edge receive different colors.



- Because the color classes are independent sets, we have

$$\chi(G) \geq \frac{|V|}{\alpha(G)}$$

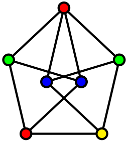
The unit distance graph

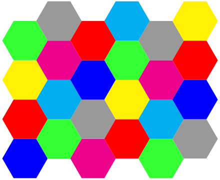
- It is the graph with vertex set $\mathbb{R}^d$ and edge set $\{xy \ : \ \|x - y\| = 1\}$.
- A set A avoiding norm 1 is an independent set of the unit distance graph. Its independence number (ratio) is

$$\overline{\alpha}(\mathbb{R}^d, \| \ \|) := \sup_{A \text{ avoids } 1} \delta(A)$$

- The determination of its chromatic number $\chi(\mathbb{R}^d)$ (Euclidean norm) is a widely open famous problem (introduced by Nelson 1950 for the plane).

The chromatic number of the plane





$4 \leq \chi(\mathbb{R}^2) \leq 7$ (Nelson and Isbell, 1950)

The chromatic number of $\mathbb{R}^d$

- Lower bounds based on

$$\chi(\mathbb{R}^d) \geq \chi(G)$$

for all finite induced subgraph of the unit distance graph $G \hookrightarrow \mathbb{R}^d$.

- De Bruijn and Erdös (1951):

$$\chi(\mathbb{R}^d) = \max_{\substack{G \text{ finite} \\ G \hookrightarrow \mathbb{R}^d}} \chi(G)$$

- Good sequences of graphs: Raiski (1970), Larman and Rogers (1972), Frankl and Wilson (1981), Székely and Wormald (1989).

$\chi(\mathbb{R}^d)$ for large d

$$(1.2 + o(1))^d \leq \chi(\mathbb{R}^d) \leq (3 + o(1))^d$$

- Lower bound : Frankl and Wilson (1981).
- FW 1.207^d is improved to 1.239^d by Raigorodskii (2000).
- Upper bound: Larman and Rogers (1972). They use Voronoï decomposition of lattice packings.

Frankl and Wilson graphs

- $p < d/4$ is a prime number.
- $\mathrm{FW}(d, p)$ is the graph with:

$$V = \{x \in \{0,1\}^d \ : \ \mathrm{wt}(x) = 2p - 1\} \quad E = \{xy \ : \ |x \cap y| = p - 1\}.$$

- Then

$$\alpha(\mathrm{FW}(d, p)) \leq \binom{d}{p-1}.$$

- Follows from Frankl and Wilson intersection theorems (1981).

Frankl and Wilson graphs

- If $p \sim ad$,

$$\chi(\mathrm{FW}(d,p)) \geq \frac{|V_d|}{\alpha(\mathrm{FW}(d,p))} \geq \frac{\binom{d}{2p-1}}{\binom{d}{p-1}} \approx e^{(H(2a)-H(a))d}$$

- Optimizing on a leads to $(1.207)^d$.
- Raigorodski uses vertices in $\{0,1,-1\}^d$ and a similar proof.

The measurable chromatic number of $\mathbb{R}^d$

- The measurable chromatic number $\chi_m(\mathbb{R}^d)$: the color classes are required to be measurable.
- Obviously $\chi_m(\mathbb{R}^d) \geq \chi(\mathbb{R}^d)$.
- Falconer (1981): $\chi_m(\mathbb{R}^d) \geq d + 3$. In particular

$$\chi_m(\mathbb{R}^2) \geq 5$$

- The color classes avoid norm 1, thus are independent sets of the unit distance graph, so:

$$\chi_m(\mathbb{R}^d) \geq \frac{1}{\overline{\alpha}(\mathbb{R}^d)}.$$

Upper bounds for $\overline{\alpha}(\mathbb{R}^d, \| \|)$

- Larman and Rogers (1972): if $G = (V, E)$ is a finite induced subgraph of the unit distance graph, and if $\alpha(G)$ denotes its independence number,

$$\overline{\alpha}(\mathbb{R}^d, \| \|) \le \overline{\alpha}(G) := \frac{\alpha(G)}{|V|}.$$

- Proof is easy: if A avoids norm 1,

$$(\mathbf{1}_A * \delta_V)(x) \le \alpha(G).$$

Indeed, if $\mathbf{1}_A * \delta_V$ reaches a value $m > \alpha(G)$, there exists x s.t. $x = a_1 - v_1 = \cdots = a_m - v_m$; then $\|v_i - v_j\| = \|a_i - a_j\| \neq 1$ so $\{v_1, \dots, v_m\}$ is an independent set of G, a contradiction.
Taking densities,

$$|V|\delta(A) \le \alpha(G).$$

Upper bounds for $\overline{\alpha}(\mathbb{R}^d, \| \|)$

- Example: for $\| \|_\infty$, $V = \{0, 1\}^d$ leads to the complete graph so $\overline{\alpha}(G) = 1/2^d$. It shows
$$\overline{\alpha}(\mathbb{R}^d, \| \|_\infty) = \frac{1}{2^d}.$$

- For $\| \|_p$, $1 \leq p < \infty$, the Frankl-Wilson graphs lead to the asymptotic
$$\overline{\alpha}(\mathbb{R}^d, \| \|_p) \lessapprox \frac{1}{1.207^d}.$$

- For small dimensions and $p = 2$ Szekely and Wormald (1989) give better bounds.

An upper bounds for $\overline{\alpha}(\mathbb{R}^d, \| \|)$ from Fourier analysis

Theorem [B., E. de Corte, F.M. de Oliveira Filho, F. Vallentin (2013)]

Let μ be a signed Borel measure centrally symmetric and supported on $S^{d-1}_{\| \|} := \{x \in \mathbb{R}^d \ : \ \|x\| = 1\}$, let

$$m_\mu := \min_{\xi \in \mathbb{R}^d} \widehat{\mu}(\xi).$$

Then,

$$\overline{\alpha}(\mathbb{R}^d, \| \|) \leq \frac{-m_\mu}{\widehat{\mu}(0^d) - m_\mu}.$$

B., E. de Corte, F.M. de Oliveira Filho, F. Vallentin, *Spectral bounds for the independence ratio and the chromatic number of an operator*, arxiv:1301.1054, to appear in Israel J. Math.

Sketch of proof:

We will pretend $\mathbb{R}^d$ *is a probability space (!!!)*.

- Let A avoids norm 1. Because μ is supported on $S^{d-1}_{\|\ \|}$,

$$(\mathbf{1}_A * \mu, \mathbf{1}_A) = 0.$$

  Indeed:

$$(\mathbf{1}_A * \mu, \mathbf{1}_A) = \int\int \mathbf{1}_A(x+y)\,\mathbf{1}_A(x) d\mu(y) dx = 0$$

- We decompose $\mathbf{1}_A$ orthogonally:

$$\mathbf{1}_A = \beta\,\mathbf{1} + g, \qquad (\mathbf{1}, g) = 0$$

  then replace in $(\mathbf{1}_A * \mu, \mathbf{1}_A) = 0$.

Sketch of proof:

- We obtain:

$$0 = (\mathbf{1}_A * \mu, \mathbf{1}_A) = ((\beta\,\mathbf{1} + g) * \mu, \beta\,\mathbf{1} + g) = \beta^2 + (g * \mu, g)$$

- Applying Parseval:

$$(g * \mu, g) = (\widehat{g}\widehat{\mu}, \widehat{g}) \geq m_\mu (g, g)$$

- Thus:

$$\beta^2 = -(g * \mu, g) \leq -m_\mu (g, g)$$

- To conclude we notice:

$$\beta = (\mathbf{1}_A, \mathbf{1}) = \delta(A) \quad \text{et} \quad (g, g) = (\mathbf{1}_A, \mathbf{1}_A) - \beta^2 = \delta(A) - \delta(A)^2$$

An analogy with finite graphs

- This upper bound is the analog of the so-called Delsarte bound for graphs:
- $G = (V, E)$ a finite graph. For all symmetric matrix $B \in \mathbb{R}^{V \times V}$ s.t. $B\mathbf{1} = d\,\mathbf{1}$ and $B_{x,y} = 0$ if $xy \notin E$,

$$\overline{\alpha}(G) = \frac{\alpha(G)}{|V|} \leq \frac{-\lambda_{\min}(B)}{d - \lambda_{\min}(B)}$$

  where $\lambda_{\min}(B)$ is the minimal eigenvalue of B.
- If G is regular of degree d, one can take for B the adjacency matrix of G. It leads to the Hoffman bound.
- If $\mathrm{Aut}(G)$ is transitive on E, the adjacency matrix is an optimal choice for B.

An analogy with finite graphs

- $G = (V, E)$ a finite graph, $B \in \mathbb{R}^{V\times V}$ s.t. $B_{x,y} = 0$ if $xy \notin E$ defines an operator:

$$B : \mathbb{R}^V \to \mathbb{R}^V$$
$$f \mapsto Bf, \quad (Bf)_x = \sum_{y\in V(x)} B_{x,y} f_y$$

- For the unit distance graph, the measure μ also defines an operator:

$$L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$
$$f \mapsto f * \mu, \quad (f*\mu)(x) = \int_{\|y\|=1} f(x+y)d\mu(y)$$

whose spectrum is $\{\widehat{\mu}(\xi), \xi \in \mathbb{R}^d\}$. So $m_\mu = \min \widehat{\mu}(\xi)$ replaces $\lambda_{\min}(B)$.

How to optimize over μ

- Recall the bound: if μ is supported on $S^{d-1}_{\|\ \|}$, let $m_\mu := \min \widehat{\mu}(\xi)$,

$$\overline{\alpha}(\mathbb{R}^d, \|\ \|) \leq \frac{-m_\mu}{\widehat{\mu}(0^d) - m_\mu}.$$

- Problem: how should we choose μ so that this bound is good ?
- Because the RHS is convex, μ can be assumed to be invariant under $\mathrm{Aut}(S^{d-1}_{\|\ \|})$.
- For the Euclidean norm, it means μ is invariant under $O(\mathbb{R}^d)$ so there is essentially one choice: the surface measure of the unit sphere.

The Fourier bound for the Euclidean norm

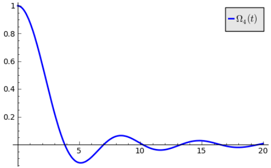
- Taking $\mu = \omega_d$ the normalized surface measure on S^{d-1},

$$\widehat{\omega_d}(\xi) = \int_{S^{d-1}} e^{2i\pi(x\cdot\xi)} d\omega_d(x) = \Omega_d(\|\xi\|)$$

where

$$\Omega_d(t) = \Gamma(d/2)(2/t)^{d/2-1} J_{d/2-1}(t)$$

$J_{d/2-1}(t)$ is the Bessel function of the first kind with parameter $d/2-1$.



$\min \Omega_d(t) = \Omega_d(j_{d/2,1})$ where $j_{d/2,1}$ is the first zero of $J_{d/2}$.

The Fourier bound for the Euclidean norm

Theorem [F.M. de Oliveira Filho, F. Vallentin 2010]

$$\overline{\alpha}(\mathbb{R}^d) \leq \frac{-\Omega_d(j_{d/2,1})}{1 - \Omega_d(j_{d/2,1})}$$

- Asymptotically,

$$\frac{-\Omega_d(j_{d/2,1})}{1 - \Omega_d(j_{d/2,1})} \approx \left(\sqrt{e/2}\right)^{-n} \approx 1.165^{-d}$$

  So it is not as good as the Frankl-Wilson 1.207^{-d} and Raigorodskii 1.239^{-d} bounds (although better for small dimensions).
- It is possible to improve the Fourier bound by additional graphical constraint.

The improved Fourier bound for the Euclidean norm

Let $G \hookrightarrow \mathbb{R}^d$, for $x_i \in V$, let $r_i := \|x_i\|$.

$$\vartheta_G(\mathbb{R}^d) := \inf\Big\{ z_0 + z_2 \frac{\alpha(G)}{|V|} : \quad \begin{array}{l} z_2 \geq 0 \\ z_0 + z_1 + z_2 = 1 \\ z_0 + z_1 \Omega_d(t) + z_2 \big(\frac{1}{|V|} \sum_{i=1}^{|V|} \Omega_d(r_i t)\big) \geq 0 \\ \text{for all } t > 0 \Big\}. \end{array}$$

Theorem [Filho, Vallentin 2010 for simplices]

$$\overline{\alpha}(\mathbb{R}^d) \leq \vartheta_G(\mathbb{R}^d)$$

Theorem [B., A. Thiery 2012]

$$\vartheta_R(\mathbb{R}^d) \lessapprox (1.268)^{-d} \qquad d \to +\infty$$

Numerical results: upper bounds for $\overline{\alpha}(\mathbb{R}^d)$

d	previous	Fourier bound	improved F. bound	G
2	0.279069	0.287120	0.2623	Moser Spindles
3	0.187500	0.178466	0.165609	simplices [OV 2010]
4	0.128000	0.116826	0.10006	600-cell
5	0.0953947	0.0793346	0.0752845	simplex [OV 2010]
6	0.0708129	0.0553734	0.04870	Schäfli/kissing
7	0.0531136	0.0394820	0.02764	kissing of E_8
8	0.0346096	0.0286356	0.01959	E_8
9	0.0288215	0.0210611	0.01678	J(10,5,2)
10	0.0223483	0.0156717	0.01269	J(11,5,2)
11	0.0178932	0.0117771	0.0088775	J(12,6,2)*
12	0.0143759	0.00892554	0.006111	J(13,6,2)*
13	0.0120332	0.00681436	0.00394332	J(14,7,3)*
14	0.00981770	0.00523614	0.00300286	J(15,7,3)*
15	0.00841374	0.00404638	0.00242256	J(16,8,3)*
16	0.00677838	0.00314283	0.00161645	J(17,8,3)*
17	0.00577854	0.00245212	0.00110487	J(18,9,4)*
18	0.00518111	0.00192105	0.00084949	J(19,9,4)*
19	0.00380311	0.00151057	0.00074601	J(20,9,3)*
20	0.00318213	0.001191806	0.00046909	J(21,10,4)*
21	0.00267706	0.000943209	0.00031431	J(22,11,5)*
22	0.00190205	0.000748582	0.00024621	J(23,11,5)*
23	0.00132755	0.000595665	0.0002122678	J(24,12,5)
24	0.00107286	0.000475128	0.00018437	orth. graph [KP 2008]

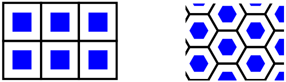
Numerical results : lower bounds for $\chi_m(\mathbb{R}^d)$

d	previous	$\chi_m(\mathbb{R}^d)$	G
2	5		
3	6	7	simplices
4	8	10	600-cells
5	11	14	simplex
6	15	21	Schläfli/kissing
7	19	37	kissing of E_8
8	30	52	E_8
9	35	60	J(10,5,2)
10	48	79	J(11,5,2)
11	64	113	J(12,6,2)*
12	85	164	J(13,6,20)*
13	113	254	J(14,7,3)*
14	147	334	J(15,7,3)*
15	191	413	J(16,8,3)*
16	248	619	J(17,8,3)*
17	319	906	J(18,9,4)*
18	408	1178	J(19,9,4)*
19	521	1341	J(20,9,3)*
20	662	2132	J(21,10,4)*
21	839	3182	J(22,11,5)*
22	1060	4062	J(23,11,5)*
23	1336	4712	J(24,12,5)*
24	1679	5424	orth. graph

Polytopal norms

- Joint ongoing work with D. Henrion, J.-B. Lasserre, S. Robins, F. Vallentin. Many open questions!
- For the hypercube ($\| \|_\infty$) we have seen $\overline{\alpha}(\mathbb{R}^d, \| \|_\infty) = 1/2^d$.

Question: does it hold for any polytope that tiles ?



It is open for the hexagon. The best we can prove: 0.28.

Polytopal norms

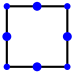
- In the Fourier bound,

$$\overline{\alpha}(\mathbb{R}^d, \| \ \|) \leq \frac{-m_\mu}{\widehat{\mu}(0^d) - m_\mu}$$

Question: what is the optimal measure μ ?

Symmetrization does not lead to a single measure.

- For the hypercube, a weighted sum of point measures at the center of all faces gives back the $1/2^d$ bound. Not the surface measure!



- In general, point measures lead to polynomial optimization problems.

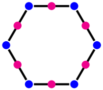
Point measures and polynomial optimization

- Let $\mathcal{Q} = \{Q_1, \dots, Q_m\}$ rational points in $\mathbb{R}^d$. Let $\mu = \sum_{j=1}^m w_j \delta_{Q_j}$. We assume (for simplicity) that $\mathcal{Q}$ is invariant by flipping the signs of coordinates. Then

$$\widehat{\mu}(\xi) = \sum_{j=1}^{m} w_j \cos(2\pi Q_{j1}\xi_1)\dots\cos(2\pi Q_{jd}\xi_d).$$

- If k is the lcm of denominators of coordinates of Q_j, the above is a polynomial in the variables $x_1 := \cos(2\pi\xi_1/k), \dots, x_d := \cos(2\pi\xi_1/k)$ (using Chebyshev polynomials) with $\deg_{x_j} \leq k$.

- Example: the hexagon



$$\widehat{\mu}_{\text{vertices}}(\xi) = P_v(x_1, x_2) := (8x_1^4 + 8x_1^2x_2^2 - 12x_1^2 - 4x_2^2 + 3)/3$$

$$\widehat{\mu}_{\text{edges}}(\xi) = P_e(x_1, x_2) := (8x_1^3x_2 - 6x_1x_2 + 2x_2^2 - 1)/3.$$

We have

$$6(P_v(x_1, x_2) + 2P_e(x_1, x_2)) = -7 + (4x_1^2 + 4x_1x_2 - 3)^2$$

showing that

$$(\widehat{\mu}_{\text{vertices}} + 2\widehat{\mu}_{\text{edges}})/3$$

has minimum $-7/18$, leading to the bound $7/25 = 0.28$.

Point measures, polynomial optimization, and SOS

- For fixed set of rational points $\mathcal{Q}$, optimizing over the weights w_j amounts to solve:

$$\max\{\ m\ :\ \sum_{j=1}^{m} w_j P_j(x_1,\dots,x_d) \geq m,\ \sum_{j=1}^{m} w_j = 1\ \}$$

where $P_j(x_1,\dots,x_d) = \widehat{\delta}_{Q_j}(\xi)$ and $x_j \in [-1,1]$.

- sums of squares relaxations allow to approximate the above by semidefinite programing:

$$P_j(\underline{x}) = m + S_0(\underline{x}) + (1-x_1^2)S_1(\underline{x}) + \cdots + (1-x_d^2)S_d(\underline{x})$$

where $S_0,\dots,S_d$ are SOS.

Numerical results

- Optimizing the weights for the middle of all faces:
  - For the hexagon, the weights $1/3$, $2/3$ are optimal leading to bound 0.28
  - For the crosspolytopes it seems the bound is $1/2d$ (verified for $d = 2, 3, 4$).
- Optimizing the weights for more points:
  - For the hexagon it does not improve.
  - For the crosspolytopes, it does and we obtain:

d	$\mathcal{Q}$	m	bound
3	$\mathbb{Z}/8$	-0.186	0.155
4	$\mathbb{Z}/6$	-0.104	0.095