POLYTOPES IN THE 0/1-CUBE WITH BOUNDED CHVÁTAL-GOMORY RANK

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CUTTING-PLANE PROOFS AND CHVÁTAL-GOMORY CLOSURES
Given linear inequalities

\[ a_i^T x \geq b_i \quad (i = 1, \ldots, m) \quad (1) \]

an inequality \( a^T x \geq b \) with \( a \in \mathbb{Z}^n \) is derived from (1) if

- \( a = \sum_{i=1}^{m} \lambda_i a_i \) for some \( \lambda_1, \ldots, \lambda_m \geq 0 \)
- \( \lfloor \sum_{i=1}^{m} \lambda_i b_i \rfloor \geq b \)

Clear: every \( x \in \mathbb{Z}^n \) that satisfies (1) also satisfies \( a^T x \geq b \)
Example

\[ \begin{align*}
  x_1 + x_2 &\leq 1, 
  x_2 + x_3 &\leq 1, 
  x_3 + x_4 &\leq 1, 
  x_4 + x_5 &\leq 1, 
  x_1 + x_5 &\leq 1
\end{align*} \]

\[ \Rightarrow 2x_1 + \cdots + 2x_5 \leq 5 \]

\[ \Rightarrow x_1 + \cdots + x_5 \leq 2.5 \]

\[ \Rightarrow x_1 + \cdots + x_5 \leq \lfloor 2.5 \rfloor = 2 \]
**Definition**

Given linear inequalities

\[ a_i^T x \geq b_i \quad (i = 1, \ldots, m) \]

a sequence of linear inequalities

\[ a_{m+k}^T x \geq b_{m+k} \quad (k = 1, \ldots, M) \]

is a **cutting-plane proof** for \( a^T x \geq b \) if for every \( k = 1, \ldots, M \)

- \( a_{m+k} \in \mathbb{Z}^n \),
- \( a_{m+k}^T x \geq b_{m+k} \) is derived from the previous inequalities,

and \( a^T x \geq b \) is a nonnegative multiple of \( a_{m+M}^T x \geq b_{m+M} \).

Its length is \( M \).
Theorem (Gomory)

If $a^i x \geq b_i$ ($i = 1, \ldots, m$) define a polytope $P$, then every linear inequality with integer coefficients that is valid for $P \cap \mathbb{Z}^n$ has a cutting-plane proof of finite length.

How long do cutting-plane proofs need to be?
Chvátal-Gomory

<table>
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<th>Definition</th>
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| Given a polytope $P \subseteq \mathbb{R}^n$, the first Chvátal-Gomory (CG) closure of $P$ is  

$$P' := \{x \in \mathbb{R}^n : c^T x \geq \left\lfloor \min_{y \in P} c^T y \right\rfloor \ \forall \ c \in \mathbb{Z}^n\}$$

$P^{(0)} := P$, $P^{(t)} := (P^{(t-1)})'$ is the $t$-th CG closure of $P$. |

<table>
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<tr>
<th>Definition</th>
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<td>The smallest $t$ such that $P^{(t)} = \text{conv}(P \cap \mathbb{Z}^n)$ is the CG-rank of $P$.</td>
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<th>Theorem (Chvátal)</th>
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<td>The CG-rank of every polytope is finite.</td>
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Fact
Let $a_i^T x \geq b_i$ ($i = 1, \ldots, m$) define a polytope $P$ with CG-rank $k$. Then every linear inequality with integer coefficients that is valid for $P \cap \mathbb{Z}^n$ has a cutting-plane proof of length at most
\[
(n^{k+1} - 1)/(n - 1).
\]

Fact
Even in dimension 2, the CG-rank of a polytope can be arbitrarily large.

Eisenbrand, Schulz 2003; Rothvoß, Sanità 2013
The CG-rank of any polytope contained in $[0, 1]^n$ is at most $O(n^2 \log n)$; and this bound is tight up to the log-factor.
Definition

Let $S \subseteq \{0, 1\}^n$. A polytope $R \subseteq [0, 1]^n$ is a relaxation of $S$ iff $R \cap \mathbb{Z}^n = S$.

Question

Let $S \subseteq \{0, 1\}^n$. What properties of $S$ ensure that every relaxation of $S$ has bounded CG rank (by a constant independent of $n$)?
Fix $k$ to be a constant.

**Remark**
Polytopes in $\mathbb{R}^n$ with CG-rank $k$ have cutting-plane proofs of length polynomial in $n$.

**Remark**
Maximizing/minimizing a linear functional over the integer points of a polytope with CG-rank $k$ is in $\text{NP} \cap \text{coNP}$ (but not known to be in $\text{P}$).
Previous work

- $\tilde{S} := \{0, 1\}^n \setminus S$
- $H[\tilde{S}] :=$ undirected graph with vertices $\tilde{S}$, two vertices are adjacent iff they differ in one coordinate

**Easy**

If $H[\tilde{S}]$ is a stable set, then the CG-rank of any relaxation of $S$ is at most 1.

**Cornuéjols, Lee (2016)**

If $H[\tilde{S}]$ is a forest, then the CG-rank of any relaxation of $S$ is at most 3.

**Cornuéjols, Lee (2016)**

If the treewidth of $H[\tilde{S}]$ is at most 2, then the CG-rank of any relaxation of $S$ is at most 4.
WHAT MAKES THE CG-RANK LARGE?
A large pitch!

**Definition**

The pitch of $S \subseteq \{0, 1\}^n$ is the smallest number $p \in \mathbb{Z}_{\geq 0}$ such that every $p$-dimensional face of $[0, 1]^n$ intersects $S$.

(If the pitch is $p$, there is a $p-1$-dimensional face of $[0, 1]^n$ disjoint from $S$)

**Fact**

Let $S \subseteq \{0, 1\}^n$ with pitch $p$. Then there is a relaxation of $S$ with CG-rank at least $p - 1$. 
Large coefficients!

**Definition**

The gap of $S \subseteq \{0, 1\}^n$ is the smallest number $\Delta \in \mathbb{Z}_{\geq 0}$ such that $\text{conv}(S)$ can be described by inequalities of the form

$$
\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta
$$

with $I, J \subseteq [n]$ disjoint, $\delta, c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}$ with $\delta \leq \Delta$.

**Fact**

Let $S \subseteq \{0, 1\}^n$ with gap $\Delta$. Then there is a relaxation of $S$ with CG-rank at least $\frac{\log \Delta}{\log n} - 1$. 
Theorem

Let \( S \subseteq \{0, 1\}^n \) with pitch \( p \) and gap \( \Delta \). Then the CG-rank of any relaxation of \( S \) is at most \( p + \Delta - 1 \).

Corollary

Let \( S \subseteq \{0, 1\}^n \) and let \( t \) be the treewidth of \( H[\bar{S}] \). Then the CG-rank of any relaxation of \( S \) is at most \( t + 2t^{t/2} \).
Comparing to treewidth

Bounded treewidth implies bounded pitch and gap:

**Proposition**

Let $S \subseteq \{0, 1\}^n$ with pitch $p$ and gap $\Delta$. If $t$ is the treewidth of $H[\bar{S}]$, then we have $p \leq t + 1$ and $\Delta \leq 2t^{t/2}$. 

\[\text{clique of size 6}\]
Proof idea

• induction on the rhs of the inequality to obtain

• every inequality of the form \( \sum_{i \in I} x_i \geq 1 \) can be obtained after \( n + 1 - |I| \) rounds of CG.

• note that \( n + 1 - |I| \leq p \)

• \( \Rightarrow \) all inequalities with rhs 1 can be obtained after \( p \) rounds.

• for inequalities with larger rhs, proof by example
Proof idea (2)

- suppose that $7x_1 + 3x_2 + 2x_3 \geq 5$ is valid for $S$, then also
  
  $$(7 - 1)x_1 + 3x_2 + 2x_3 \geq 4$$
  
  $$7x_1 + (3 - 1)x_2 + 2x_3 \geq 4$$
  
  $$7x_1 + 3x_2 + (2 - 1)x_3 \geq 4$$

  are valid for $S$

- thus, $(7 - \varepsilon)x_1 + (3 - \varepsilon)x_2 + (2 - \varepsilon)x_3 \geq 4$ is valid for $S$

- thus, $7x_1 + 3x_2 + 2x_3 \geq 4 + \varepsilon''$ is valid for $S$

- induction ...

- rounding up the rhs, we obtain the desired inequality
FURTHER PROPERTIES OF SETS WITH BOUNDED PITCH
Proposition

For every $S \subseteq \{0, 1\}^n$ with pitch $p$ and every $c \in \mathbb{R}^n$, the problem $\min\{c^T s : s \in S\}$ can be solved using $\mathcal{O}(n^p)$ oracle calls to $S$.

Why?

- may assume that $0 \leq c_1 \leq \cdots \leq c_n$
- note: optimal solution over $\{0, 1\}^n$ would be $\emptyset$
- claim: only need to check all vectors with support at most $p$
Bounded pitch allows for fast approximation:

**Corollary**

Let \( S \subseteq \{0, 1\}^n \) with pitch \( p \) and let \( R \) be any relaxation of \( S \). Let \( \varepsilon \in (0, 1) \) with \( p\varepsilon^{-1} \in \mathbb{Z} \). If

\[
\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta
\]

with \( \delta \geq c_1, \ldots, c_n \geq 0 \) is valid for \( S \), then the inequality

\[
\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq (1 - \varepsilon)\delta
\]

is valid for \( R^{(p\varepsilon^{-1} - 1)} \).
Theorem

Let $S \subseteq \{0, 1\}^n$ with pitch $p$ such that there exists a depth-$D$ Boolean circuit (with AND and OR gates of fan-in 2, and NOT gates of fan-in 1) that decides $S$.

Then $\text{conv}(S)$ is a linear projection of a polytope with $O(n \cdot 2^{pD})$ many facets.