POLYTOPES IN THE 0/1-CUBE WITH BOUNDED CHVÁTAL-GOMORY RANK

Yohann Benchetrit, Samuel Fiorini, Tony Huynh
Université Libre de Bruxelles

Stefan Weltge
ETH Zürich

CUTTING-PLANE PROOFS AND CHVÁTAL-GOMORY CLOSURES

Cutting-plane proofs

Definition

Given linear inequalities

$$a_i^\intercal x \geq b_i \quad (i = 1, \ldots, m) \tag{1}$$

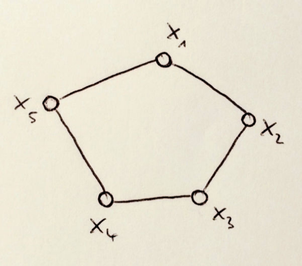
an inequality $a^\intercal x \geq b$ with $a \in \mathbb{Z}^n$ is derived from (1) if

- $a = \sum_{i=1}^m \lambda_i a_i$ for some $\lambda_1, \ldots, \lambda_m \geq 0$
- $\lceil \sum_{i=1}^m \lambda_i b_i \rceil \geq b$

Clear: every $x \in \mathbb{Z}^n$ that satsifies (1) also satisfies $a^\intercal x \geq b$

Cutting-plane proofs (2)

Example



$x_1 + x_2 \leq 1$, $x_2 + x3 \leq 1$, $x_3 + x_4 \leq 1$, $x_4 + x_5 \leq 1$, $x_1 + x_5 \leq 1$

$\Rightarrow 2x_1 + \cdots + 2x_5 \leq 5$

$\Rightarrow x_1 + \cdots + x_5 \leq 2.5$

$\Rightarrow x_1 + \cdots + x_5 \leq \lfloor 2.5 \rfloor = 2$

Cutting-plane proofs (3)

Definition

Given linear inequalities

$$a_i^\mathsf{T} x \geq b_i \quad (i = 1, \ldots, m)$$

a sequence of linear inequalities

$$a_{m+k}^\mathsf{T} x \geq b_{m+k} \quad (k = 1, \ldots, M)$$

is a <u>cutting-plane proof</u> for $a^\mathsf{T} x \geq b$ if for every $k = 1, \ldots, M$

- $a_{m+k} \in \mathbb{Z}^n$,
- $a_{m+k}^\mathsf{T} x \geq b_{m+k}$ is derived from the previous inequalities,

and $a^\mathsf{T} x \geq b$ is a nonnegative multiple of $a_{m+M}^\mathsf{T} x \geq b_{m+M}$.

Its length is M.

Cutting-plane proofs (4)

Theorem (Gomory)

If $a_i^\intercal x \geq b_i$ $(i = 1, \ldots, m)$ define a polytope P, then every linear inequality with integer coefficients that is valid for $P \cap \mathbb{Z}^n$ has a cutting-plane proof of finite length.

How long do cutting-plane proofs need to be?

Chvátal-Gomory

Definition

Given a polytope $P \subseteq \mathbb{R}^n$, the first Chvátal-Gomory (CG) closure of P is

$$P' := \{x \in \mathbb{R}^n : c^\intercal x \geq \lceil \min_{y \in P} c^\intercal y \rceil \; \forall\, c \in \mathbb{Z}^n\}$$

$P^{(0)} := P$, $P^{(t)} := (P^{(t-1)})'$ is the t-th CG closure of P.

Definition

The smallest t such that $P^{(t)} = \mathrm{conv}(P \cap \mathbb{Z}^n)$ is the CG-rank of P.

Theorem (Chvátal)

The CG-rank of every polytope is finite.

Chvátal-Gomory (2)

Fact

Let $a_i^\intercal x \geq b_i$ $(i = 1, \ldots, m)$ define a polytope P with CG-rank k. Then every linear inequality with integer coefficients that is valid for $P \cap \mathbb{Z}^n$ has a cutting-plane proof of length at most

$$(n^{k+1} - 1)/(n - 1).$$

Fact

Even in dimension 2, the CG-rank of a polytope can be arbitarily large.

Eisenbrand, Schulz 2003; Rothvoß, Sanità 2013

The CG-rank of any polytope contained in $[0, 1]^n$ is at most $\mathcal{O}(n^2 \log n)$; and this bound is tight up to the log-factor.

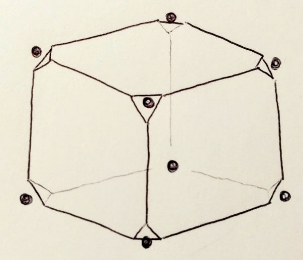
Today

Definition

Let $S \subseteq \{0,1\}^n$. A polytope $R \subseteq [0,1]^n$ is a relaxation of S iff $R \cap \mathbb{Z}^n = S$.

Question

Let $S \subseteq \{0,1\}^n$. What properties of S ensure that every relaxation of S has bounded CG rank (by a constant independent of n)?



Constant CG-rank

Fix k to be a constant.

Remark

Polytopes in $\mathbb{R}^n$ with CG-rank k have cutting-plane proofs of length polynomial in n.

Remark

Maximizing/minimizing a linear functional over the integer points of a polytope with CG-rank k is in $\mathrm{NP} \cap \mathrm{coNP}$ (but not known to be in P).

Previous work

- $\bar{S} := \{0,1\}^n \setminus S$
- $H[\bar{S}]$:= undirected graph with vertices $\bar{S}$, two vertices are adjacent iff they differ in one coordinate

Easy

If $H[\bar{S}]$ is a stable set, then the CG-rank of any relaxation of S is at most 1.

Cornuéjols, Lee (2016)

If $H[\bar{S}]$ is a forest, then the CG-rank of any relaxation of S is at most 3.

Cornuéjols, Lee (2016)

If the treewidth of $H[\bar{S}]$ is at most 2, then the CG-rank of any relaxation of S is at most 4.

WHAT MAKES THE CG-RANK LARGE?

A large pitch!

Definition

The pitch of $S \subseteq \{0,1\}^n$ is the smallest number $p \in \mathbb{Z}_{\geq 0}$ such that every p-dimensional face of $[0,1]^n$ intersects S.

(If the pitch is p, there is a $p-1$-dimensional face of $[0,1]^n$ disjoint from S)

Fact

Let $S \subseteq \{0,1\}^n$ with pitch p. Then there is a relaxation of S with CG-rank at least $p-1$.

Large coefficients!

Definition

The gap of $S \subseteq \{0,1\}^n$ is the smallest number $\Delta \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{conv}(S)$ can be described by inequalities of the form

$$\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta$$

with $I, J \subseteq [n]$ disjoint, $\delta, c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}$ with $\delta \leq \Delta$.

Fact

Let $S \subseteq \{0,1\}^n$ with gap Δ. Then there is a relaxation of S with CG-rank at least $\frac{\log \Delta}{\log n} - 1$.

Main result

Theorem

Let $S \subseteq \{0,1\}^n$ with pitch p and gap Δ. Then the CG-rank of any relaxation of S is at most $p + \Delta - 1$.

Corollary

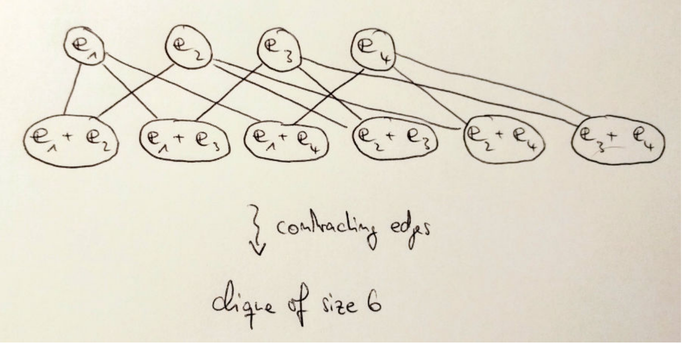
Let $S \subseteq \{0,1\}^n$ and let t be the treewidth of $H[\bar{S}]$. Then the CG-rank of any relaxation of S is at most $t + 2t^{t/2}$.

Comparing to treewidth

Bounded treewidth implies bounded pitch and gap:

Proposition

Let $S \subseteq \{0,1\}^n$ with pitch p and gap Δ. If t is the treewidth of $H[\bar{S}]$, then we have $p \leq t+1$ and $\Delta \leq 2t^{t/2}$.



Proof idea

- induction on the rhs of the inequality to obtain
- every inequality of the form $\sum_{i \in I} x_i \geq 1$ can be obtained after $n + 1 - |I|$ rounds of CG.
- note that $n + 1 - |I| \leq p$
- $\rightsquigarrow$ all inequalities with rhs 1 can be obtained after p rounds.
- for inequalities with larger rhs, proof by example

Proof idea (2)

- suppose that $7x_1 + 3x_2 + 2x_3 \geq 5$ is valid for S, then also

$$\begin{aligned} (7-1)x_1 + 3x_2 + 2x_3 &\geq 4 \\ 7x_1 + (3-1)x_2 + 2x_3 &\geq 4 \\ 7x_1 + 3x_2 + (2-1)x_3 &\geq 4 \end{aligned}$$

  are valid for S

- thus, $(7-\varepsilon)x_1 + (3-\varepsilon)x_2 + (2-\varepsilon)x_3 \geq 4$ is valid for S
- thus, $7x_1 + 3x_2 + 2x_3 \geq 4 + \varepsilon''$ is valid for S
- induction ...
- rounding up the rhs, we obtain the desired inequality

FURTHER PROPERTIES OF SETS WITH BOUNDED PITCH

Optimizing

Proposition

For every $S \subseteq \{0,1\}^n$ with pitch p and every $c \in \mathbb{R}^n$, the problem $\min\{c^\intercal s : s \in S\}$ can be solved using $\mathcal{O}(n^p)$ oracle calls to S.

Why?

- may assume that $0 \leq c_1 \leq \cdots \leq c_n$
- note: optimal solution over $\{0,1\}^n$ would be $\mathbb{O}$
- claim: only need to check all vectors with support at most p

Approximating

Bounded pitch allows for fast approximation:

Corollary

Let $S \subseteq \{0,1\}^n$ with pitch p and let R be any relaxation of S. Let $\varepsilon \in (0,1)$ with $p\varepsilon^{-1} \in \mathbb{Z}$. If

$$\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta$$

with $\delta \geq c_1, \dots, c_n \geq 0$ is valid for S, then the inequality

$$\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq (1 - \varepsilon)\delta$$

is valid for $R^{(p\varepsilon^{-1}-1)}$.

Extended formulations

Theorem

Let $S \subseteq \{0,1\}^n$ with pitch p such that there exists a depth-D Boolean circuit (with AND and OR gates of fan-in 2, and NOT gates of fan-in 1) that decides S.

Then $\text{conv}(S)$ is a linear projection of a polytope with $\mathcal{O}(n \cdot 2^{pD})$ many facets.