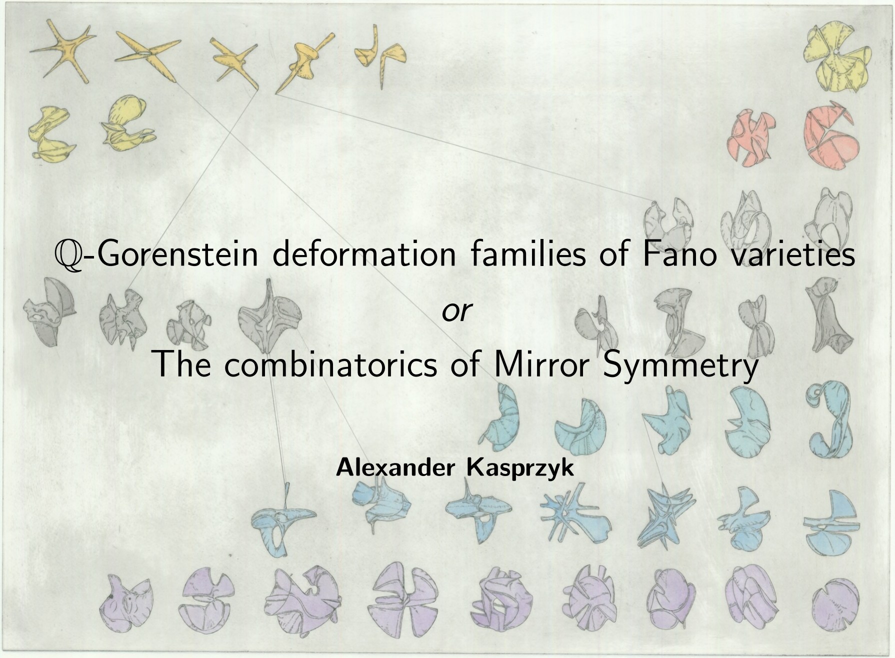


$\mathbb{Q}$-Gorenstein deformation families of Fano varieties

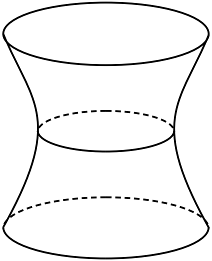
or

The combinatorics of Mirror Symmetry

Alexander Kasprzyk

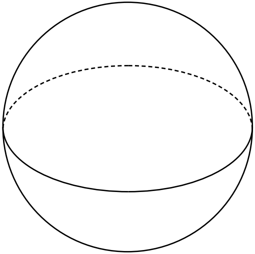
Fano manifolds

Smooth varieties, called *manifolds*, come with a natural notion of curvature, and fall into one of three classes.



Negative curvature
General type

Flat
Calabi–Yau



Positive curvature
Fano

There are finitely many Fano manifolds in each dimension.

Fano manifolds: Basic building blocks of geometry

Fano manifolds are the building blocks from which other varieties are formed.

- Both from the *Minimal Model Program*
- And in terms of explicit constructions

Fano art by Gemma Anderson

Fano manifolds: Classification

The classification of Fano manifolds is known up to dimension 3.

- **Dimension 1**:
  - $\mathbb{P}^1$ (i.e. the Reimann sphere)
- **Dimension 2** (del Pezzo, 1880s):
  - $\mathbb{P}^2$
  - $\mathbb{P}^1 \times \mathbb{P}^1$
  - The blow-up of $\mathbb{P}^2$ in at most 8 points.

  These are called *del Pezzo* surfaces.
- **Dimension 3** (Mori–Mukai, 1980s):
  - 105 cases

Very little is known in dimension ≥ 4.

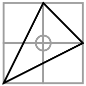
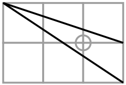
Fano polytopes and toric geometry

Fix a lattice $N \cong \mathbb{Z}^n$. A convex lattice polytope $P \subset N \otimes \mathbb{Q} = N_{\mathbb{Q}}$ is *Fano* if:

- $\dim(P) = n$;
- $0 \in \operatorname{int}(P)$;
- each $v \in \operatorname{vert}(P)$ is a primitive lattice point of N.

Two Fano polytopes P and Q are considered to be isomorphic if there exists a change of basis of N sending P to Q. That is,

$$P \cong Q \qquad \Longleftrightarrow \qquad \varphi(P) = Q, \text{ for some } \varphi \in \mathrm{GL}_n(\mathbb{Z})$$

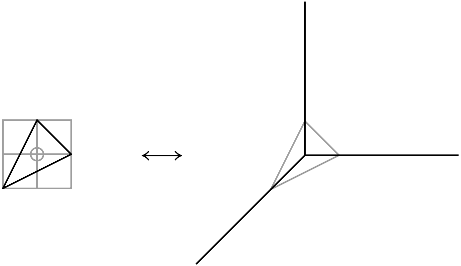
 $\cong$  via

$$\varphi : \begin{array}{l} e_1 \mapsto e_1 \\ e_2 \mapsto e_1 - e_2 \end{array}$$

We consider Fano polytopes only up to isomorphism.

Fano polytopes and toric geometry

To a Fano polytope $P \subset N_{\mathbb{Q}}$ we associate the *spanning fan*.
The spanning fan describes a *toric* Fano variety X_P.



$\longleftrightarrow$ $X_P = \mathbb{P}^2$

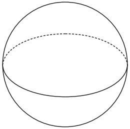
The geometry of X_P is encoded in the combinatorics of P. For example, the singularities of X_P can be read off P.

Toric Fano manifolds: Classification

A Fano polytope P is *smooth* if:

- For each facet F of P, $\mathrm{vert}(F)$ are a $\mathbb{Z}$-basis of N.

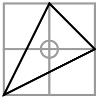
n-dimensional toric Fano manifold X



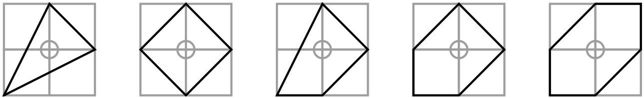
toric geometry

$\longleftrightarrow$

smooth Fano polytope P with $\dim(P) = n$



- **Dimension 2**:
  - $\mathbb{P}^2$; $\mathbb{P}^1 \times \mathbb{P}^1$; the blow-up of $\mathbb{P}^2$ in at most 3 points.



Toric Fano manifolds: Classification

Being toric is *unusual*:

- **Dimension 2**:
  - 5 of the 10 del Pezzo surfaces are toric.
- **Dimension 3**:
  - 18 of the 105 Fano manifolds are toric.

But being toric is *good*: we can use the combinatorics of lattice polytopes to study them.

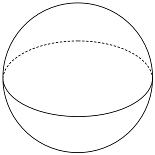
For example, Øbro (2007) gave an efficient algorithm for classifying smooth Fano polytopes in any dimension.

Dimension	1	2	3	4	5	6	7	8
Number	1	5	18	124	866	7 622	72 256	749 892

They grow slowly – approximately by a power of 10 per dimension.

Mirror Symmetry

n-dimensional Fano manifold X



Mirror Symmetry $\longleftrightarrow$

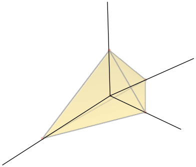
Laurent polynomial f in n variables

$$f = x + y + z + \frac{1}{xyz}$$

deformation $\downarrow$

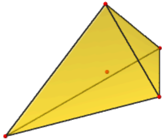
$\downarrow$ Newt(f)

n-dimensional toric Fano variety X_P



toric geometry $\longleftrightarrow$

Fano polytope P with $\dim(P) = n$



Example: $\mathbb{P}^2$

Illustrate this equivalence in the case of $X = \mathbb{P}^2$. We start with the Laurent polynomial

$$f = x + y + \frac{1}{xy} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$$

Associated with f is its *period*

$$\pi_f(t) = \left(\frac{1}{2\pi i}\right)^2 \int_{|x|=|y|=1} \frac{1}{1 - tf} \frac{dx}{x} \frac{dy}{y}, \quad t \in \mathbb{C}, |t| \ll \infty.$$

The Taylor expansion of the period has coefficients given by the constant term of successive powers of f

$$\begin{aligned} \pi_f(t) &= \sum_{k \geq 0} \operatorname{coeff}_1(f^k) t^k \\ &= 1 + 6t^3 + 90t^6 + 34650t^9 + 756756t^{12} + 17153136t^{15} + \dots \\ &= \sum_{k \geq 0} \frac{(3k)!}{(k!)^3} t^{3k} \end{aligned}$$

Example: $\mathbb{P}^2$

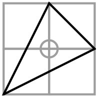
$$\pi_f(t) = 1 + 6t^3 + 90t^6 + 34650t^9 + 756756t^{12} + 17153136t^{15} + \dots$$

The coefficients of π_f agree with certain Gromov–Witten invariants of X. Roughly speaking, they count curves in X with given degree and a certain constraint on the $\mathbb{C}$-structure. This is called the *regularised quantum period* $\widehat{G}_X$.

f is *mirror dual* to X if $\pi_f = \widehat{G}_X$

The Newton polytope $P \subset N_{\mathbb{Q}}$ of f gives a toric Fano variety X_P $\mathbb{Q}$-Gorenstein deformation equivalent to X. In this case we recover $\mathbb{P}^2$.

$$f = x + y + \frac{1}{xy}$$

$P = \mathrm{Newt}(f) =$  $\subset N_{\mathbb{Q}}$

Example: $\mathbb{P}^2$

The mirror f for X is typically not unique. One way of transforming f to a mirror-equivalent Laurent polynomial g is via a *mutation*.

- This is a change of variables $\varphi : (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$ such that $g = \varphi^* f$ is a Laurent polynomial with the same period:

$$\pi_f(t) = \pi_g(t)$$

In the case $f = x + y + \frac{1}{xy}$ we can apply the mutation

$$\varphi : \begin{array}{l} x \mapsto \dfrac{x}{1 + \frac{x}{y}} \\ y \mapsto \dfrac{y}{1 + \frac{x}{y}} \end{array}$$

Then:

$$g = \varphi^* f = \varphi^* \left(x + y + \frac{1}{xy} \right) = \frac{x}{1 + \frac{x}{y}} + \frac{y}{1 + \frac{x}{y}} + \frac{\left(1 + \frac{x}{y}\right)^2}{xy}$$

Example: $\mathbb{P}^2$

$$
\begin{aligned}
g = \varphi^* f &= \frac{x}{1+\frac{x}{y}} + \frac{y}{1+\frac{x}{y}} + \frac{\left(1+\frac{x}{y}\right)^2}{xy} \\
&= \frac{y(y+x)}{y+x} + \frac{y^2+2xy+x^2}{xy^3} \\
&= y + \frac{1}{xy} + \frac{2}{y^2} + \frac{x}{y^3} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]
\end{aligned}
$$

One can compute the period of g:

$$\pi_g(t) = 1 + 6t^3 + 90t^6 + 34650t^9 + 756756t^{12} + \cdots = \pi_f(t)$$

g is also a mirror for $\mathbb{P}^2$

Mutation of a Laurent polynomial

A mutation of $f \in \mathbb{C}[\underline{x}^{\pm 1}]$ requires two pieces of data:

- a *grading* on monomials;
- a *factor* $F \in \mathbb{C}[\underline{x}^{\pm 1}]$.

The grading is a map $w : \underline{x}^a \mapsto w(a)$ from monomials to $\mathbb{Z}$.
The factor is a Laurent polynomial with $w(F) = \{0\}$ such that

$$f_h = F^{-h} r_h,$$

for all $h < 0$, where $r_h \in \mathbb{C}[\underline{x}^{\pm 1}]$. Here

$$f_h = \text{“the terms of } f \text{ in graded piece } h\text{”}, \qquad \text{i.e. } w(f_h) = \{h\}.$$

Then $\varphi : \underline{x}^a \mapsto \underline{x}^a F^{w(a)}$ is a *mutation* of f with

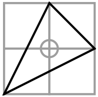
$$g = \varphi^* f = \sum_{h<0} r_h + \sum_{h \geq 0} f_h F^h$$

Example: $\mathbb{P}^2$

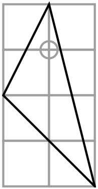
Mutation is a combinatorial operation on the Newton polytopes

At the level of Newton polytopes we have transformed the Fano polygon for $\mathbb{P}^2$ into the Fano polygon for $\mathbb{P}(1,1,4)$:

$$\mathrm{Newt}\Big(x + y + \frac{1}{xy}\Big) =$$



$$\longmapsto$$



$$= \mathrm{Newt}\Big(y + \frac{1}{xy} + \frac{2}{y^2} + \frac{x}{y^3}\Big)$$

Notice that $\mathbb{P}(1,1,4)$ is a singular toric Fano variety. It has two smooth cones, and one singular cone corresponding to a $\frac{1}{4}(1,1)$ singularity.

Mutation of $P \subset N_{\mathbb{Q}}$

A mutation of $P \subset N_{\mathbb{Q}}$ requires two pieces of data:

- a *grading* on N;
- a *factor* of P.

The grading is given by a primitive lattice vector $w \in M = \mathrm{Hom}(N, \mathbb{Z})$. The factor is a convex lattice polytope $F \subset w^{\perp} \subset N_{\mathbb{Q}}$ such that

$$\{v \in \mathrm{vert}(P) \mid w(v) = h\} \subset (-h)F + R_h \subset P_h,$$

for all $h < 0$, where $R_h \subset N_{\mathbb{Q}}$ is a convex lattice polytope. Here

$$P_h = \mathrm{conv}(v \in P \cap N \mid w(v) = h).$$

The the *mutation* of P is

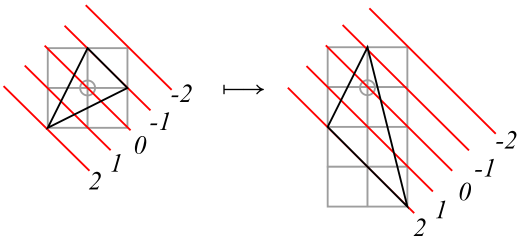
$$Q = \mathrm{conv}\left(\bigcup_{h<0} R_h \cup \bigcup_{h\geq 0} (P_h + hF)\right)$$

Mutation of $P \subset N_{\mathbb{Q}}$

In the example of $\mathbb{P}^2$ we pick

$$w = (-1,-1) \in M, \qquad F = \mathrm{conv}\{(0,0),(1,-1)\} \subset w^{\perp} \subset N_{\mathbb{Q}}.$$

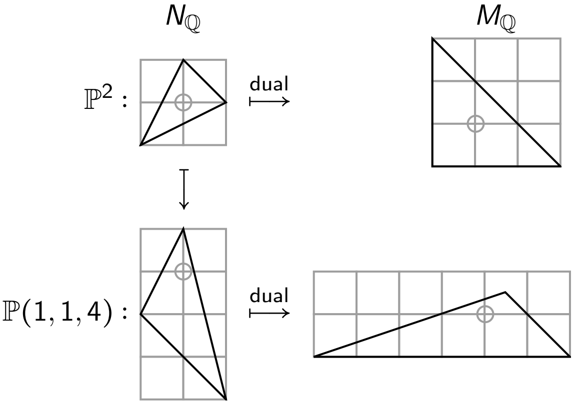
Then mutation adds or subtracts dilates of F depending on height:



Example: $\mathbb{P}^2$

Now consider the dual polytope to $P \subset N_{\mathbb{Q}}$:

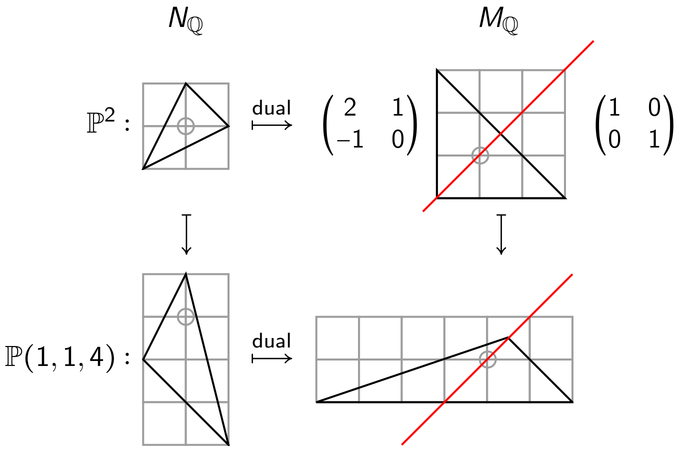
$$P^* = \{u \in M_{\mathbb{Q}} \mid u(v) \geq -1 \text{ for all } v \in P\}$$



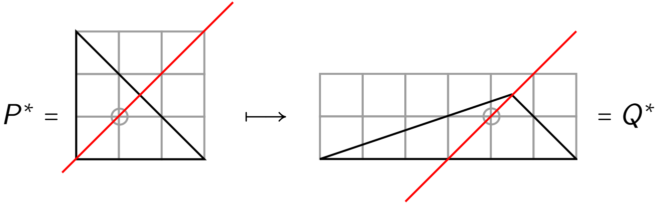
Mutation of $P^* \subset M_{\mathbb{Q}}$

Mutation acts via a piecewise $\mathrm{GL}_n(\mathbb{Z})$ map on M:

$$u \longmapsto u - w \min\{w(v) \mid v \in \mathrm{vert}(F)\}$$



Mutation of $P^* \subset M_{\mathbb{Q}}$



Mutation has straightened out the bottom-left corner of Q^*. Since this is a piecewise $GL_n(\mathbb{Z})$ map on M, we have that:

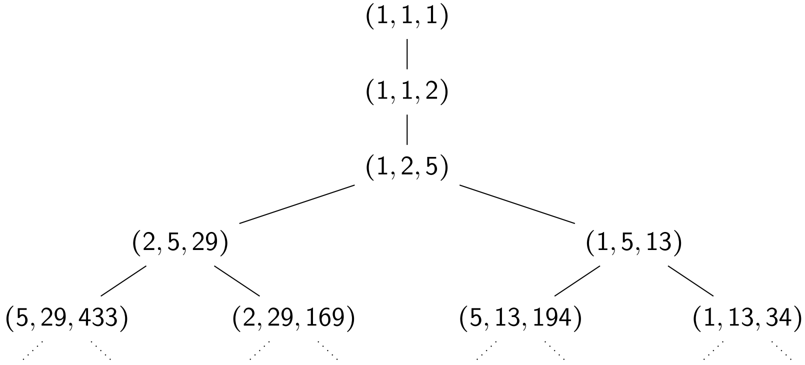
$$\mathrm{Vol}(P^*) = \mathrm{Vol}(Q^*), \qquad \mathrm{Ehr}(P^*) = \mathrm{Ehr}(Q^*)$$

Equivalently:

$$(-K_{X_P})^n = (-K_{X_Q})^n, \qquad \mathrm{Hilb}(X_P, -K_{X_P}) = \mathrm{Hilb}(X_Q, -K_{X_Q})$$

Mutation of Markov triples

We can continue mutating $\mathbb{P}^2$, moving from Fano triangle to Fano triangle:



The vertices (a, b, c) correspond to the Fano triangles for $\mathbb{P}(a^2, b^2, c^2)$.
The vertices (a, b, c) correspond to solutions to the *Markov equation*:

$$a^2 + b^2 + c^2 = 3abc$$

Mutation of Markov triples

A solution $(a, b, c) \in \mathbb{Z}^3_{>0}$ of the Markov equation

$$a^2 + b^2 + c^2 = 3abc$$

is called a *Markov triple*. All Markov triples can be obtained from $(1, 1, 1)$ via *mutation*:

$$(a, b, c) \longmapsto (3bc - a, b, c)$$

Mutations of the Markov triples correspond to mutations of the Fano triangles arising from $\mathbb{P}^2$.

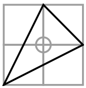
Quiver mutation

We can associate a quiver $\mathcal{Q}_P$ to a Fano polygon $P \subset N_{\mathbb{Q}}$.

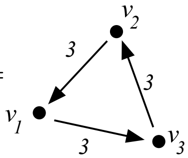
- We have a vertex v_i for each edge E_i of P.
- Let $w_i \in M$ be the primitive (inner) normal vector to E_i. Then the number of arrows between vertices v_i and v_j is given by

$$w_i \wedge w_j = \det \begin{pmatrix} w_i \\ w_j \end{pmatrix},$$

where the sign determines the orientation. For $\mathbb{P}^2$ we get:

$P =$ 

$$w_1 = (-1,-1)$$
$$w_2 = (-1,2)$$
$$w_3 = (2,-1)$$

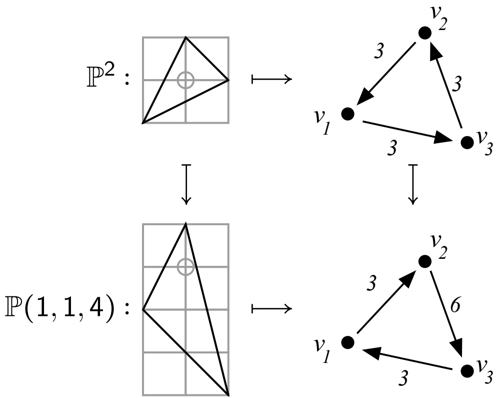
$\mathcal{Q}_P =$ 

Quiver mutation

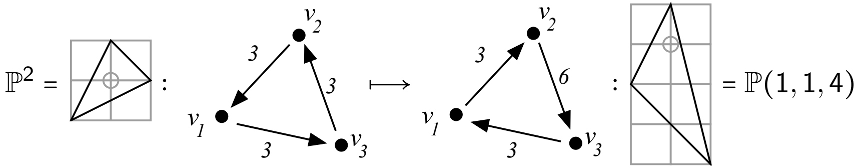
We can *mutate* $\mathcal{Q}_P$ about a vertex v_i.

- For every path $v_j \longrightarrow v_i \longrightarrow v_k$ add in a new edge $v_j \longrightarrow v_k$;
- Reverse the direction of every arrow that starts or ends at v_i;
- Cancel opposing edges.

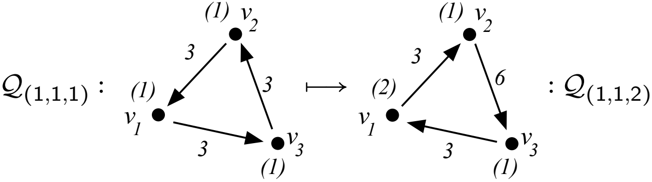
We recover the quiver for $\mathbb{P}(1,1,4)$:



Mirrors for $\mathbb{P}^2$



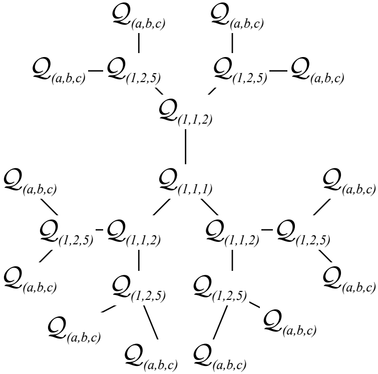
Notice that the quiver for $\mathbb{P}(1,1,4)$ isn't *balanced*.
We re-balance by adding multiplicities for to vertices v_i given by the edge lengths E_i of the Fano polygon.



This re-balancing condition *is* the Markov equation.

Mirrors for $\mathbb{P}^2$

We obtain a tree of quiver mutations



where the quiver $\mathcal{Q}_{(a,b,c)}$ corresponding to $\mathbb{P}(a^2, b^2, c^2)$ is balanced via assigning weights a, b, c to the vertices v_1, v_2, v_3.
Here (a, b, c) is a solution to the Markov equation $a^2 + b^2 + c^2 = 3abc$.
This corresponds to the space of mirrors for $\mathbb{P}^2$ via Mirror Symmetry.