Lattice Simplices of Bounded Degree

Einstein Workshop on Lattice Polytopes 2016

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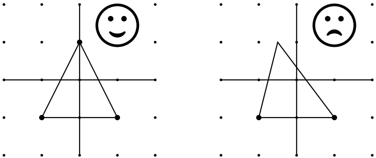
Basic Definitions

- *Lattice polytope*: $P = \mathrm{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n) \subseteq \mathbb{R}^d$ where $\mathbf{v}_i \in \mathbb{Z}^d$.



- *Ehrhart polynomial*: $L_P(k) := \left|kP \cap \mathbb{Z}^d\right| \; (k \in \mathbb{Z}_{\geq 0})$.
- h^**-polynomial*: $\sum_{k \geq 0} L_P(k) t^k = \frac{h_P^*(t)}{(1-t)^{d+1}}$ for $h_P^* \in \mathbb{Z}_{\geq 0}[t]$.
- *degree of P*: $\deg(P) = \deg h_P^*(t)$.

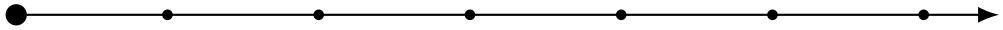
Motivation

Question A

What are the polynomials which can be interpreted as the h^*-polynomial of a lattice polytope?

Linear Polynomials

$P = [0, a] \subseteq \mathbb{R}$ $(a \in \mathbb{Z})$



$L_P(0) = 1$

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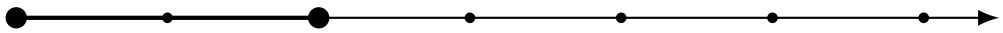

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$L_P(0) = 1,\ L_P(1) = a + 1,\ L_P(2) = 2a + 1,\ L_P(3) = 3a + 1,\ \ldots$
$\Rightarrow L_P(k) = ka + 1$

$$\sum_{k \geq 0} (ka + 1)t^k = a \sum_{k \geq 0} kt^k + \sum_{k \geq 0} t^k = a \tfrac{t}{(1-t)^2} + \tfrac{1}{1-t}$$
$$= \tfrac{(a-1)t+1}{(1-t)^2} \qquad \Rightarrow h_P^*(t) = (a-1)t + 1.$$

Degree 1

All lin. polynomials $1 + at \in \mathbb{Z}^2_{\geq 0}[t]$ can be interpreted as h^*-vectors.

Quadratic Polynomials

Degree 2[Henk & Tagami, Treutlein]

All polynomials $1 + a_1 t + a_2 t^2 \in \mathbb{Z}_{\geq 0}[t]$ with

$$a_1 \leq \begin{cases} 7 & \text{if } a_2 = 1 \\ 3a_2 + 3 & \text{if } a_2 \geq 2 \end{cases}$$

can be interpreted as h^*-polynomials.

(Need polytopes up to dimension 3.)

Simpler Question

Question B

What are the h^*-polynomials coming from lattice **simplices**?

Question B: Degree 1

All lin. polynomials $1 + at \in \mathbb{Z}_{\geq 0}[t]$ can be interpreted as h^*-polynomials of lattice simplices.

Question B: Degree 2

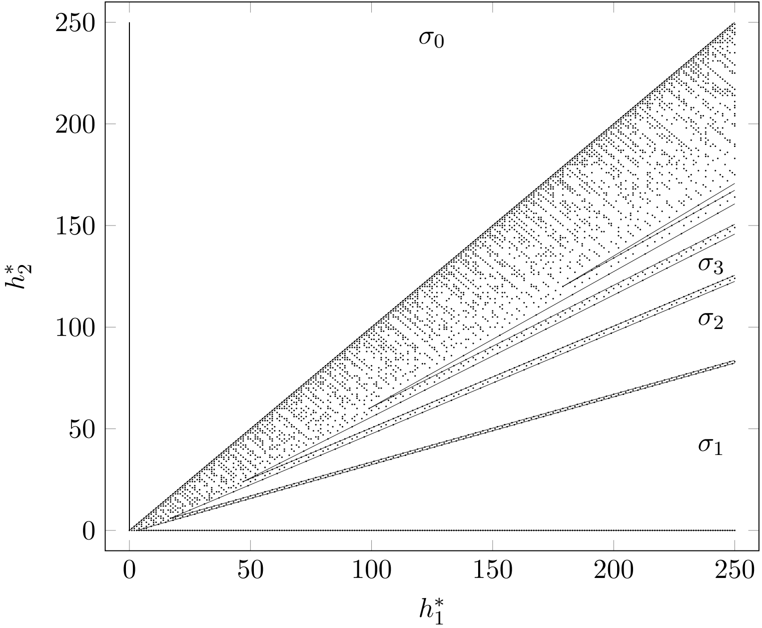
Interpret $h^* = 1 + a_1 t + a_2 t^2 \in \mathbb{Z}_{\geq 0}[t]$ as point in the positive orthant $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2_{\geq 0}$.

$\mathcal{M} := \{\mathbf{a} \in \mathbb{Z}^2_{\geq 0} \colon h^*_P(t){=}1{+}a_1 t{+}a_2 t^2$ for lattice **triangle** $P \subseteq \mathbb{R}^2\}$

Proposition[H.,Nill,Oeberg]

There is a family $(\sigma_i)_{i \in \mathbb{Z}_{\geq 0}}$ of affine cones $\sigma_i \subseteq \mathbb{R}^2_{\geq 0}$ such that $\mathcal{M} \cap \sigma_i^\circ = \emptyset$ for all i.

Question B: Degree 2



Simplices of given degree

Question C

What are the simplices of a given degree (any dimension)?

Idea: Question C $\Rightarrow$ Question B.

Idea from Alg. Geom.

Alg. Geometer are interested in

$$M_{g,n} = \{\text{smooth proj. curves } C \text{ of genus } g \text{ with } n \text{ distinct marked points}\}/\sim$$

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Completeness is a desirable property. For a (possible) compactification, relax the smoothness condition

$$\overline{M}_{g,n} = \{\text{proj. connected nodal curves } C \text{ of genus } g \text{ with } n \text{ distinct, nonsing. marked points with a stability condition}\}/\sim$$

"Moduli" of Lattice Simplices

We are interested in

$$M_{0,s} = \{\text{lattice simplices } \Delta \text{ of deg. } s \text{ with marked vertices}\}/\sim$$

Here "$\sim$" means: "up to affine unimodular transformations".

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Question

What could be $\overline{M}_{0,s}$?

Recall Definition

Usually: $\Delta = \operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}) \subseteq \mathbb{R}^d$ d-dimensional lattice simplex if $\mathbf{v}_i \in \mathbb{Z}^d$ (aff. indep.)

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Idea

Let's do it vice versa.

- Lattice changes: Λ.
- Vertices stay the same: What is a good choice?

Lattice of a Lattice Simplex

From now on: $\mathbf{e}_1, \ldots, \mathbf{e}_d \in \mathbb{R}^d$ standard basis vectors.

All vertices should be "equivalent" $\rightsquigarrow$ bad choice: $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d$.

Better choice: $\mathbf{e}_1, \ldots, \mathbf{e}_{d+1} \in \mathbb{R}^{d+1}$ (Dimension increases by 1).

Lattice of a Lattice Simplex

$\Delta = \operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}) \subseteq \mathbb{R}^d$ d-dimensional lattice simplex. Cone over Δ

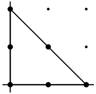
$$C = \operatorname{cone}(\{1\} \times \Delta) \subseteq \mathbb{R}^{d+1}.$$

Exists unique linear iso. $\varphi\colon \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ with $(1, \mathbf{v}_i) \mapsto \mathbf{e}_i$.

- $\varphi(\Delta) = \operatorname{conv}(\mathbf{e}_1, \ldots, \mathbf{e}_{d+1})$
- $\Lambda_\Delta := \varphi(\mathbb{Z}^{d+1}) \subseteq \mathbb{R}^d$ lattice

Example

$$\Delta = \operatorname{conv}(\mathbf{0}, 2\mathbf{e}_1, 2\mathbf{e}_2) \subseteq \mathbb{R}^2$$



$$\underbrace{\begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}}_{\varphi} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_\Delta = \mathbb{Z}^3 + \mathbb{Z}\begin{pmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \frac{-1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

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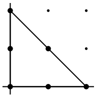

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Properties of Λ_Δ

Proposition

$\Delta = \mathrm{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}) \subseteq \mathbb{R}^d$ d-dim. lattice simplex.
$\varphi\colon \mathbb{R}^{d+1} \mapsto \mathbb{R}^{d+1}; \varphi(1, \mathbf{v}_i) = \mathbf{e}_i$. $\Lambda_\Delta = \varphi\big(\mathbb{Z}^{d+1}\big)$.

1. $\mathbb{Z}^{d+1} \subseteq \Lambda_\Delta$
2. $\Lambda_\Delta \subseteq \left\{\mathbf{x} \in \mathbb{R}^{d+1} \colon \sum_{i=1}^{d+1} x_i \in \mathbb{Z}\right\}$.

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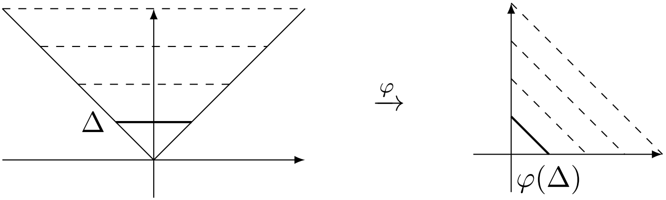
Idea of Proof:

1. $\mathbf{v}_i \in \mathbb{Z}^d \Rightarrow \mathbb{Z}^{d+1} \subseteq \Lambda_\Delta$
2.



Correspondence

Definition

A lattice $\Lambda_\Delta \subseteq \mathbb{R}^{d+1}$ we call *simplicial* if

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Theorem

The assignment $\Delta \mapsto \Lambda_\Delta$ induces a bijection

$$\left\{\begin{matrix}d\text{-dim. lattice sim-}\\ \text{plices } \Delta \subseteq \mathbb{R}^d\end{matrix}\right\} / \sim_1 \leftrightarrow \left\{\begin{matrix}\text{simplicial lattices}\\ \Lambda \subseteq \mathbb{R}^{d+1}\end{matrix}\right\} / \sim_2$$

1. $\sim_1 =$ up to affine unimodular equivalence
2. $\sim_2 =$ up to permutation of the coordinates

Chabauty Topology

$\mathscr{C}\ell :=\{C \subseteq X$ closed subgroup$\}$

$X := \left\{\mathbf{x} \in \mathbb{R}^{d+1} \colon \sum_{i=1}^{d+1} x_i \in \mathbb{Z}\right\} \subseteq \mathbb{R}^{d+1}$ closed subgp.

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Basis of neighborhoods of $C \in \mathscr{C}\ell$

$$\mathcal{N}_{K,U}(C)$$

where $K \subseteq X$ compact and $U \subseteq X$ open with $0 \in U$.

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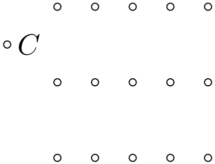

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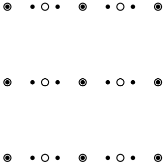

$\circ\, C$

$\bullet\, D$

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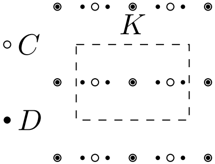

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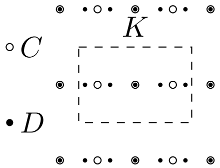

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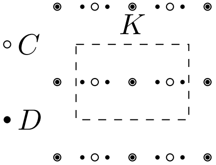

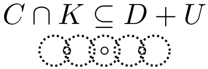

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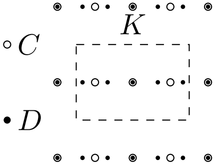

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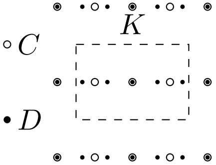

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Simplicial Lattices

Proposition

- $\mathscr{C} := \{C \in \mathscr{C}\ell \colon \mathbb{Z}^{d+1} \subseteq C\} \subseteq \mathscr{C}\ell$ closed
- $\mathcal{D} := \{C \in \mathscr{C}\ell$ discrete$\} \subseteq \mathscr{C}\ell$ open and dense.

In particular $\{\Lambda \subseteq \mathbb{R}^{d+1}$ simplicial lattice$\} = \mathcal{D} \cap \mathscr{C} \subseteq \mathscr{C}\ell$ is locally closed.

Degree

Definition

Degree $\deg(C)$ of $C \in \mathscr{C}$:

$$\deg(C) := \max\left\{\sum_{i=1}^{d+1} x_i \colon (x_1, \ldots, x_{d+1}) \in C \cap [0,1[^{d+1}\right\}.$$

Proposition

For $\Delta \subseteq \mathbb{R}^d$ a d-dim. lattice simplex, we have $\deg(\Lambda_\Delta) = \deg(\Delta)$.

The Moduli Space

Clearly: $M_{0,s} = M_{0,\leq s} \setminus M_{0,\leq s-1}$.
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Proposition

$\overline{M}_{0,\leq s}(d) := \{C \in \mathscr{C} \colon \deg(C) \leq s\} \subseteq \mathscr{C}$ closed.

Proposition

The assignment $\Delta \mapsto \Lambda_\Delta$ induces a bijection between the d-dim. lattice simplices with $\deg(\Delta) \leq s$ and $\overline{M}_{0,\leq s}(d) \cap \mathcal{D}$.

Lattice Pyramids

Definition

Let $\Delta \subseteq \mathbb{R}^d$ be a lattice polytope. *Lattice pyramid* over Δ:

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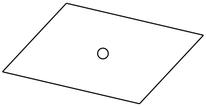

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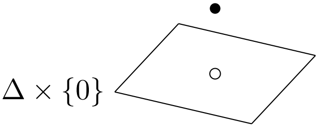

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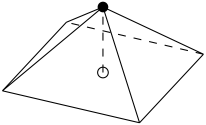

Simplices of given Degree

Proposition

The d-dim. lattice simplex $\Delta \subseteq \mathbb{R}^d$ is a lattice pyramid iff $\pi_i(\Lambda_\Delta) = \mathbb{Z}$ where $\pi_i \colon \mathbb{R}^{d+1} \to \mathbb{R}; \mathbf{x} \mapsto x_i$ for $i = 1, \ldots, d+1$.

Theorem[Nill]

Let $\Delta \subseteq \mathbb{R}^d$ be a d-dim. lattice simplex. If $d \geq 4\deg(\Delta) - 1$, then Δ is a lattice pyramid.

Corollary

$\overline{M}_{0,\leq s} \subseteq \mathcal{P}\left(\mathbb{R}^{4s-1}\right)$ (power set).

Finiteness

Proposition [Chabauty]

X locally compact $\Rightarrow \mathscr{C}\ell$ is compact.
In particular $\overline{M}_{0,\leq s} \subseteq \mathscr{C}\ell$ compact.

Partial ordering on $\mathscr{C}\ell$: $C \leq D :\Leftrightarrow C \subseteq D \qquad (C, D \in \mathscr{C}\ell)$.

Theorem

The set of maximal elements in $\overline{M}_{0,\leq s}$ is finite.

Idea of Proof

Set $\mathcal{M} := \left\{ C \in \overline{M}_{0,\leq s} \text{ maximal} \right\}$.

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Set $\mathcal{M} := \left\{C \in \overline{M}_{0,\leq s} \text{ maximal}\right\}$.
Assume $|\mathcal{M}| = \infty$.

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Chabauty-Pontryagin Duality[Cornulier]

The duality map

$$*\colon \left\{\text{cl. subgrp. } \subseteq \mathbb{R}^{d+1}\right\} \to \left\{\text{cl. subgrp. } \subseteq \mathbb{R}^{d+1}\right\};$$

$$C \mapsto C^* := \left\{\mathbf{x} \in \mathbb{R}^{d+1} \colon \sum_{i=1}^{d+1} x_i y_i \in \mathbb{Z}\ \forall \mathbf{y} \in C\right\}$$

is an involutory homeomorphism.

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Remark

Could be also proved using a result due to Lawrence.

Degree 1

Theorem [Batyrev, Nill]

A lattice simplex of degree ≤ 1 is either

- a lattice pyramid over an interval or
- a lattice pyramid over twice a unimodular simplex.

Corollary

The maximal elements in $\overline{M}_{0,\leq 1} \subseteq \mathcal{P}(\mathbb{R}^3)$ are the following:

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \qquad \text{and} \qquad \mathbb{Z}^3 + \mathbb{R}(\mathbf{e}_1 - \mathbf{e}_2) =: \left(\overline{1 \ -1 \ 0} \right)$$

Degree 2 case

Recall: $\mathbf{a}_1, \dots, \mathbf{a}_p, \mathbf{b}_1, \dots, \mathbf{b}_q \in \mathbb{R}^{d+1}$ linearly indep.

$$\begin{pmatrix} - \ \mathbf{a}_1 \ - \\ \vdots \\ - \ \mathbf{a}_p \ - \\ \hline - \ \mathbf{b}_1 \ - \\ \vdots \\ - \ \mathbf{b}_q \ - \end{pmatrix} := \bigoplus_{i=1}^{p} \mathbb{Z}\mathbf{a}_i \oplus \bigoplus_{j=1}^{q} \mathbb{R}\mathbf{b}_j$$

Degree 2 case (continued)

Theorem[Higashitani, H.]

The maximal elements in $\overline{M}_{0,\leq 2} \subseteq \mathcal{P}(\mathbb{R}^7)$ are the following:

1. $\begin{pmatrix} \overline{\begin{matrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \end{matrix}} \end{pmatrix}$
2. $\begin{pmatrix} \overline{\begin{matrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{matrix}} \end{pmatrix}$ $\left(\begin{array}{cccccc} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right)$ $\left(\begin{array}{cccccc} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \hline 1 & 1 & -2 & 0 & 0 & 0 \end{array}\right)$ $+9$ more discr. subgrp.
3. $\left(\begin{array}{cccccc} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right)$ $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$
4. All other max. subgrp. are discrete and contained in $\frac{1}{2}\mathbb{Z}^7$.

Cayley Conjecture

Definition

$\Delta \subseteq \mathbb{R}^d$ is a *Cayley Polytope* of lattice polytopes $\Delta_1, \dots, \Delta_k \subseteq \mathbb{R}^m$ if $k \geq 2$ and Δ is unimodularly equivalent to $\mathrm{conv}(\Delta_1 \times \mathbf{e}_1, \dots, \Delta_k \times \mathbf{e}_k) \subseteq \mathbb{R}^m \times \mathbb{R}^k$.

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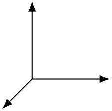
$\Delta_1, \Delta_2 \subseteq \mathbb{R}$ two (lattice) intervals.



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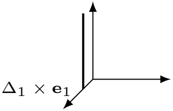

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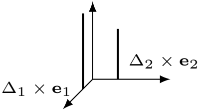

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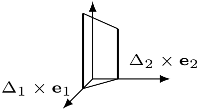


Cayley Conjecture

"Weak" Cayley Conjecture[Dickenstein, Nill]

A d-dim. lattice polytpe with degree s is a Cayley polytope, if $d > 2s$.

Proposition[Higashitani,H.]

The "Weak" Cayley Conjecture holds for degree 2 simplices.

Idea of Proof

Proposition

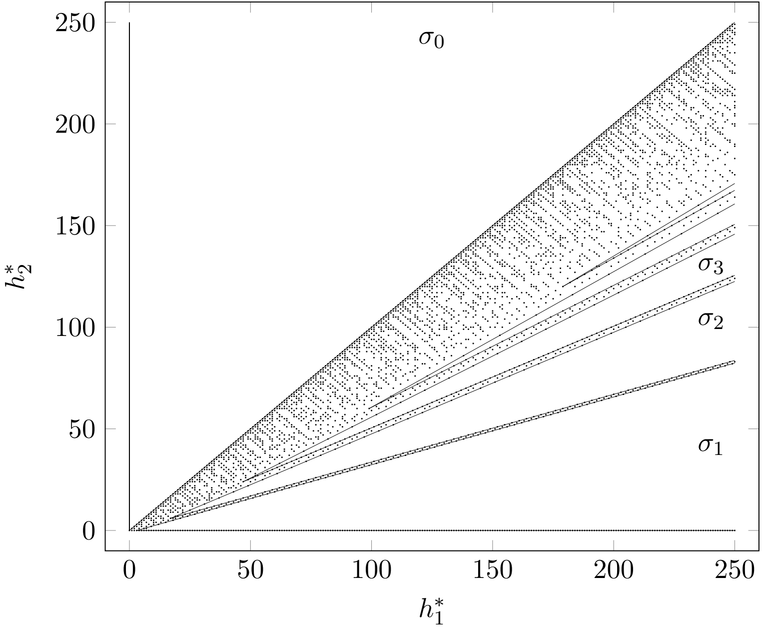
Let $\Lambda \in \mathscr{C} \cap \mathcal{D} \subseteq \mathbb{R}^{d+1}$. Λ corresponds to a Cayley polytope iff there is a proper subset $I \subsetneq \{1, \ldots, d+1\}$ such that $f_I(\Lambda) \subseteq \mathbb{Z}$ where $f_I : \Lambda \to \mathbb{R}; \mathbf{x} \mapsto \sum_{i \in I} x_i$.

Need to consider dim. at least 5, i. e., the max. subgrp. satisfy $\Lambda \subseteq \frac{1}{2}\mathbb{Z}^7$.

Example (6-dim.):

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \qquad \text{for instance } I = \{1, 2, 3, 4\}. \qquad \square$$

Not realizable h^*-vectors



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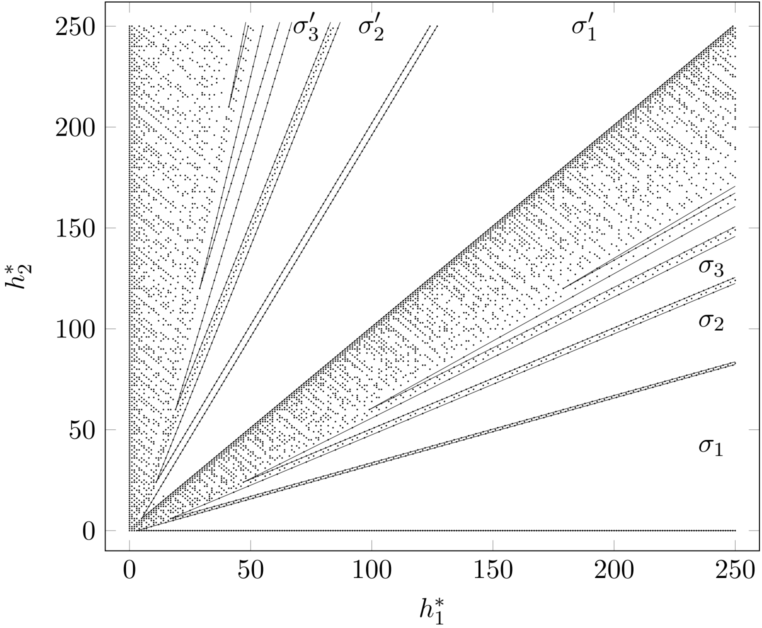

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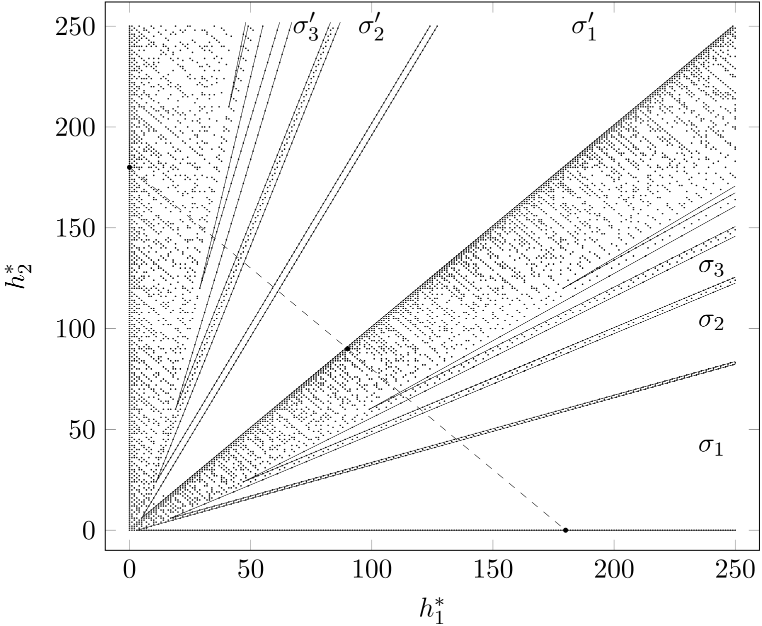

Thank you!