CLASSIFICATION OF EMPTY LATTICE 4-SIMPLICES

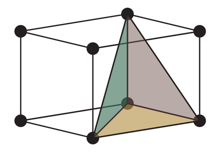
ÓSCAR IGLESIAS-VALIÑO AND FRANCISCO SANTOS (OSCAR.IGLESIAS@UNICAN.ES, FRANCISCO.SANTOS@UNICAN.ES).
Department of Mathematics, Statistics and Computer Science, University of Cantabria, SPAIN

Supported by grants MTM2014-54207-P (both authors) and BES-2015-073128 (O.Iglesias) of the Spanish Ministry of Economy and Competitiveness. F. Santos is also supported by an Einstein Visiting Professorship of the Einstein Foundation Berlin

Einstein Stiftung Berlin
Einstein Foundation Berlin

BASIC DEFINITIONS

- **Lattice** d**-polytope** P: convex hull of a finite set of points in $\mathbb{Z}^d$ with $\text{aff}(P) = \mathbb{R}^d$.
- **Simplex:** P is a n-simplex if its vertices are affinely independent.
- **Empty:** P is empty if its only lattice points are its vertices.
- **Facet:** a $d-1$-dimensional face of a d-polytope.
- **Volume of** P : Volume normalized to the lattice (e.g., volume of a simplex equals its determinant).
- **Width of** P : minimum lattice distance between 2 parallel lattice hyperplanes enclosing P. Equiv., minimum length of $f(P)$ among all affine **integer** maps $f : \mathbb{R}^d \to \mathbb{R}$.



Lattice Simplex in $\mathbb{R}^3$.

CLASSIFICATION OF EMPTY 3-SIMPLICES

In dimension 3, all empty simplices have width 1, i.e., they lie between 2 consecutive lattice planes.
Even more, there is a complete classification of empty lattice 3-simplices:

THEOREM: [White, 1963]
Every empty lattice tetrahedron of volume q is $\mathbb{Z}$-equivalent to one of the form $T(p,q)$, for some $p \in \{1, \cdots, q\}$ with $\gcd(p,q) = 1$:

$$T(p,q) := \text{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\}$$

Moreover, $T(p,q)$ is $\mathbb{Z}$-equivalent to $T(p',q)$ if and only if $p' = \pm p^{\pm 1} \pmod q$

PREVIOUS WORK IN 4-SIMPLICES

1. There are infinitely many lattice emtpy 4-simplices of width 1 (e.g., cones over empty tetrahedra).
2. There are infinitely many lattice emtpy 4-simplices of width 2. [4]
3. The amount of empty lattice 4-simplices of width greater than 2 is finite. [3] (**Note:** A previous proof from [2] is wrong).

Haase and Ziegler did an exhaustive computer enumeration of empty 4-simplices up to determinant 1000, obtaining:

THEOREM [Haase-Ziegler, 2000] Among the four-dimensional empty lattice simplices of determinant $D \leq 1000$,

1. There are no simplices of width ≥ 5,
2. There is a unique equivalence class of simplices of width 4, whose determinant is D = 101,
3. All simplices of width 3 have determinant between 41 and 179 (both extremal cases are unique, but for intermediate determinants non-equivalent simplices exist).

Based on this, Haase and Ziegler stated the following Conjecture:

CONJECTURE [Haase-Ziegler, 2000] There is no empty 4-simplex of width greater than 2 of determinant greater than 179.

OUR GOAL: Prove this conjecture by (a) proving a good upper bound $D_{\max}$ for the determinant of empty 4-simplices of width > 2, and (b) continuing the Haase-Ziegler computations up to $D_{\max}$.

A FIRST UPPER BOUND FOR THE VOLUME OF EMPTY 4-SIMPLICES

The fist approach tryng to bound to volume of empty lattice 4-simplices with general results of convex theory allow to give the following bounds for the volume of a simplex P :

$$\text{vol}(P) \leq \text{vol}(P-P)/2^d \leq 1/\lambda_1(P-P)^d,$$

where $\lambda_1(P-P)$ is the first succesive minima. That is the lowest number such that λK contains i linear independent vectors of $\mathbb{Z}^4$. Considering π the projection along the direction where $\lambda_1(P-P)$ is attained for P, and considering $z \in \pi(P) = Q$ the point projected by the line where $\lambda_1(P-P)$ is attained, it can be obtained a bound that depends on the distances between this point z and the intersection between the facets of Q and the line from z to this facets.

$$\lambda = \lambda_1(P-P)^{-1} \leq \pi^{-1}(x)/\pi^{-1}(y) \leq |xz|/|yz|.$$

And the ratio $|xz|/|yz|$ can be expressed in terms of the *coefficient of assymetry* of a polytope. Using bounds for the coefficient of assymetry in dimension 3 it is possible to obtain a general bound for empty simplices that project into not hollow 3-polytopes, which we can classify:

$$vol(P) = vol(P-P)/\binom{8}{4} \leq 24 \cdot 2^4 \lambda^{-4}/\binom{8}{4} \sim 3\lambda^{-4} \simeq 16\,000\,000.$$

So, with this general method we do not get a useful upper bound. Enumerating empty 4-simplices uo to volume 1 million is computationally **unreachable.**

VOLUME VS. WIDTH IN HOLLOW 3-POLYTOPES

For dimension 3, some results are known for the maximum volume of hollow polytopes, polytopes without lattice points in the interior:

- **PROPOSITION:** [Averkov et al., 2015] Let K be a lattice-free (without lattice point in its interior) convex body in dimension three and with width at least 3.
  (a) $\lambda_1(K-K) > 1/4$,
  (b) $\text{vol}(K-K) \leq 2^3 \cdot 27$,

We generalize this result for lattice-free convex bodies of width bounded from below.

- **THEOREM:** [Iglesias-Valiño,Santos, 2016]:
Let $w > 2.155$ and let $\mu = w^{-1}$. Then, the following statements hold for any lattice-free convex body K in dimension three of width at least w:
  (a) $\lambda_1(K-K) \geq 1 - (1 + 2/\sqrt{3})\mu$.
  (b)

$$\text{vol}(K) \leq \begin{cases} 3/4\mu^2(1-\mu(1+2/\sqrt{3})) & \text{if} \quad w \leq 2.427, \\ 8/(1-\mu)^3 = \frac{8w^3}{(w-1)^3} & \text{if} \quad w \geq 2.427. \end{cases}$$

MAIN THEORETICAL RESULTS

The generalized results obtained in dimension 3 help us to get a bound in the volume of empyt lattice 4-simplices if we can rewrite the relation between successive minima with the property of being lattice free in dimension 3 :

LEMMA Let P be a lattice-free convex body and let $\lambda_1 := \lambda_1(P-P)$. Consider the projection $\pi : P \to Q$ of P along the direction where λ_1 is achieved and assume Q is not lattice-free. Then, $\lambda_1 \geq 1 - r$ where $r > 0$ is the maximum value such that rQ is lattice-free.

By using, the projection approach together with the generalization of the Averkov et al. result in dimension 3, we are able to give a computionally reachable bound for the volume of hollow simplices of width greater than 3

THEOREM [Iglesias-Valiño,Santos, 2016]
There is no hollow 4-simplex of width greater than 3 with volume greater than 7588.

REMARK: 4-simplices of width one are easy to classify. Thus, if the list of simplices up to volume 7588 is known, then only simplices of width two remain to be classified.

ENUMERATION ALGORITHMS

Let Δ be an empty lattice 4-simplex. Since the quotient of $\mathbb{Z}^4$ by the lattice generated by vertices of Δ is always a cyclic group [2], Δ can be represented by its volume $D \in \mathbb{N}$ plus a quintuple $(a_0, a_1, a_2, a_3, a_4) \in \mathbb{Z}_D{}^5$ with $\sum a_i = 0$ (the barycentric coordinates, dilated by D, of a generator of this quotient).
Enumerating all such tuples up to $D = 7588$ is too much (10^{15} possibilities). The following ideas reduce this considerably:

- Simplices of prime-power volume have (at least) one unimodular facet. Thus, they are equivalent to a simplex of the form $\sigma(v) := \text{conv}\{e_1, e_2, e_3, e_4, v\}$, the convex hull of the standard unit vectors and another $v \in \mathbb{Z}^4$.

  Remark $\sigma(v)$ has determinant $D = ||v||_1 - 1$ and $v \setminus v' = 0 \pmod D$ implies $v \equiv v'$.

ENUMERATION ALGORITHM (PRIME-POWER):
For each prime-power $D \in \{1, \ldots, 7588\}$, and each $v \in \mathbb{Z}_{\mathbb{D}}{}^4 \cap \{\sum x_i = D+1\}$, consider the simplex $\sigma(v)$.

- Simplices of non-prime-power volume D could, a priori, have all facets non-unimodular. But they can be constructed as a "common refinement" of two simplices of smaller volumes:

LEMMA: Let $D = p \cdot q$, with $\gcd(p,q) = 1.$. Let Δ_p and Δ_q be simplices generated by quintuples $u = (u_0, u_1, u_2, u_3, u_4) \mod p$ and $v = (v_0, v_1, v_2, v_3, v_4) \mod q$. Then, the simplex generated by the quintuple $p \cdot \Delta_q + q \cdot \Delta_p$ has volume D. Moreover, every lattice 4-simplex of volume D arises in this way.

ENUMERATION ALG. (NON-PRIME-POWER):
For each non-prime-power $D \in \{1, \ldots, 7588\}$

1. Factor $D = p \cdot q$, with $p, q \geq 2$ and $\gcd(p,q) = 1$. All empty lattice 4-simplices of volumes p and q are precalculated.
2. For each possible pair of simplices of volume p and q, Δ_p and Δ_q, calculate the glued simplex $p \cdot \Delta_q + q \cdot \Delta_p \pmod D$.

REMARKS:

- In the prime-power case some "tricks" are implemented to quickly discard choices of v that lead to non-empty simplices (e.g., $\gcd(D, v_i, v_j) = 1$ for all i, j).
- In the non-prime-power case the choice of p and q in point 1 affects the computation time. Experimentally, it seems best to have both $p \simeq q$.

COMPUTATIONS

We have enumerated empty lattice 4-simplices up to volume 7588 with the algorithms above. The calculations have been done with the Altamira node of the Spanish Computing Network. The code for the algorithms has been written in python.
The total CPU time has been about 10 000 hours. The following table gives computation times for some specific values of the volume D.

Table 1. Computation time (hours) for all empty lattice 4-simplices of a given volume

Volume	Primes Algorithm	Gluing Algorithm(p similar to q)	Gluing Algorithm ($p << q$)
$\simeq 2000$	0.53	0.29	1.06
$\simeq 3000$	1.14	0.65	7.32
$\simeq 4000$	5.31	1.17	17.76
$\simeq 5000$	11.45	2.32	10.31
$\simeq 6000$	21.88	4.42	21.38
$\simeq 7000$	38.59	6.69	26.95

COMPUTATION REMARKS:

- The time we give to the gluing algorithm does not take into account the precalculation of the simplices of volumes p and q that are needed to glue.
- As seen in the table, gluing part is much faster if p is about the size of q, which is not always possible.

CLASSIFICATION OF EMPTY LATTICE 4- SIMPLICES

By combining the theoretical result from Main Theorem and the Computations up to volume 7588 we can solve the conjecture of Haase and Ziegler [4] and state following:

THEOREM [Iglesias-Valiño, Santos, 2016]

- There is no empty 4-simplex of width greater than 4.
- There is only one equivalence class of empty lattice 4-simplex of width 4. It has volume 101 and is given by $\sigma[(6,14,17,65)]$. (Its quintuple is $(-1,6,14,17,65) \mod 101$).
- All empty lattice 4-simplices that have width 3 have volume between 41 and 179. The whole list is as stated in Haase and Ziegler [4] and the complete list can be found in Master Thesis of Perriello [9].
- All simplices other than that, have width 2 or 1. Those of prime volume are as classified in [2], but infinite families of empty 4-simplices of width 2 and not present in the classification of [2] do exist.

RESULT: This implies the conjecture of Haase and Ziegler and gives a classification of empty lattice 4-simplices, except perhaps some of width two.

REFERENCES

[1] G. Averkov, J. Krumplemann and S. Weltge. Notions of maximality for integral lattice-free polyhedra: the case of dimension three. arXiv:1509.05200
[2] M. Barile, D. Bernardi, A. Borisov and J.-M. Kantor, On empty lattice simplices in dimension 4, prepint 2009. arXiv:0912.5310 [math.AG].
[3] M. Blanco, C. Haase, J. Hoffman, F. Santos, The finiteness threshold width of lattice polytopes, arXiv:1607.00798v2
[4] C. Haase, G.M. Ziegler, On the maximal width of empty lattice simplices. Europ. J. Combinatorics 21 (2000), 111-119.15
[5] O. Iglesias-Valiño, F.Santos, Classification of empty lattice 4-simplices, In preparation.
[6] S. Mori; D.R. Morrison; I.Morrison: On four-dimensional terminal quotient singularities. Math. Comput. 51 (1988), no. 184, 769-786.
[7] B. Nill and G. M. Ziegler. Projecting lattice polytopes without interior lattice points. *Math. Oper. Res.*, 36:3 (2011), 462–467.
[8] O. Pikhurko, Lattice points in lattice polytopes, Mathematika 48 (2001), no. 1-2, 15-24 (2003).
[9] A. Perriello, Lattice-Free simplexes in dimension 4. Master Thesis
[10] G. K. White. Lattice tetrahedra. *Canadian J. Math.* 16 (1964), 389–396.