Balanced shellings on combinatorial manifolds

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(joint work with Lorenzo Venturello)

Einstein Workshop on Discrete Geometry and Topology 2018, Berlin

March 14, 2018

UNIVERSITÄT OSNABRÜCK

1. Balanced combinatorial manifolds

2. Moves on simplicial complexes

3. A balanced analog of Pachner's theorem for manifolds with boundary

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- Δ is a combinatorial d-manifold with boundary if all its vertex links are combinatorial $(d-1)$-spheres or $(d-1)$-balls and its boundary is

$$\partial\Delta := \{F \in \Delta \ : \ \mathrm{lk}_\Delta(F) \text{ is a combinatorial } (d-|F|)\text{-ball}\} \cup \{\emptyset\}.$$

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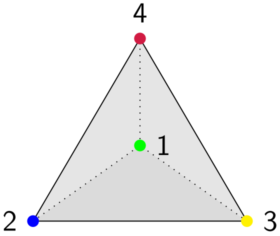
Δ is balanced if it is properly $(\dim \Delta + 1)$-colorable.

The (boundary) of the d-simplex

Let σ^d be the d-simplex.



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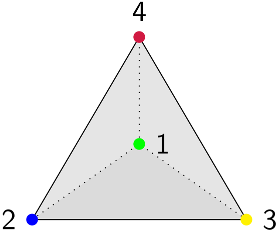

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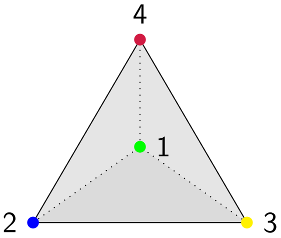
As the 1-skeleton of σ^d is a complete graph on $d+1$ vertices, a proper coloring uses at least $d+1$ colors.

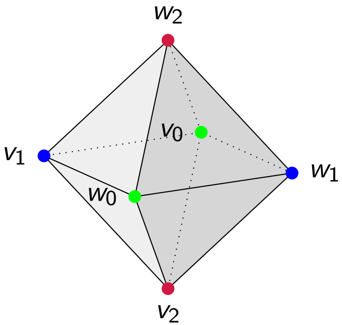
$\Rightarrow$ σ^d is balanced, whereas its boundary $\partial\sigma^d$ is not balanced.



The boundary of the $(d+1)$-dimensional cross-polytope

Let $\mathcal{C}_d$ be the boundary of the $(d+1)$-dimensional cross-polytope:

$$\mathcal{C}_d = \{v_0, w_0\} * \cdots * \{v_d, w_d\}.$$



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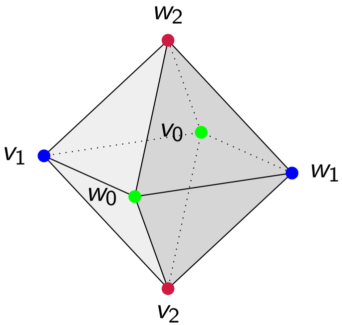
A $(d+1)$-coloring ϕ is given by setting $\phi(v_i) = \phi(w_i) = i$ for $0 \leq i \leq d$.



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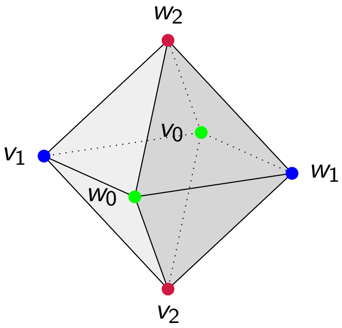

$\Rightarrow \mathcal{C}_d$ is balanced.

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Stellar moves and bistellar moves

Δ d-dimensional simplicial complex.

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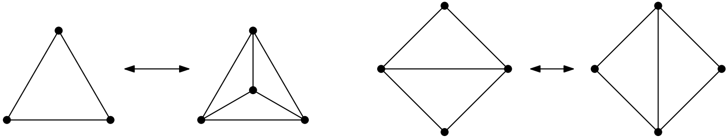
- A bistellar move replaces an induced subcomplex $A \subseteq \Delta$ that is isomorphic to a d-dimensional subcomplex of $\partial\sigma^{d+1}$ with its complement:

$$\Delta \to (\Delta \setminus A) \cup (\partial\sigma^{d+1} \setminus A).$$



What about combinatorial manifolds with boundary?

Shellings and their inverses

Δ pure d-dimensional simplicial complex.

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- A shelling on Δ corresponds to a bistellar flip on $\partial\Delta$.

What about balanced combinatorial manifolds?

Cross-flips

Δ balanced d-dimensional simplicial complex.

- A cross-flip replaces an induced subcomplex $D \subseteq \Delta$ that is isomorphic to a shellable and coshellable subcomplex of $\mathcal{C}_d$ with its complement:

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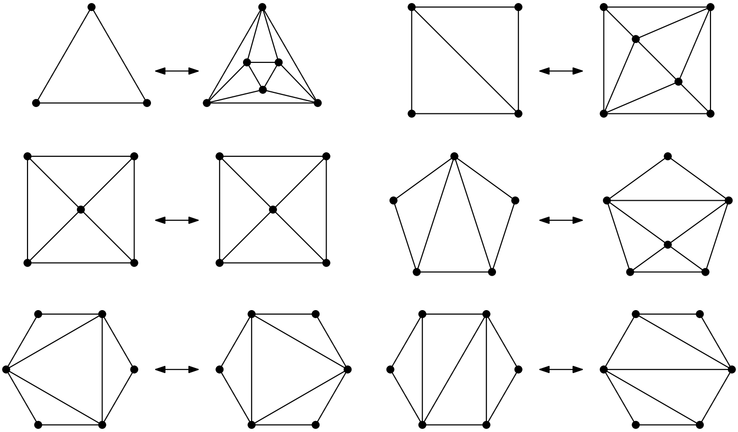
- Cross-flips preserve balancedness.
- Cross-flips preserve the PL homeomorphism type.

Cross-flips in dimension 2



What about balanced combinatorial manifolds with boundary?

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The main result

Theorem (J.-K., Venturello; 2018+)

Balanced combinatorial manifolds with boundary are PL homeomorphic if and only if they are connected by a sequence of shellings and inverse shellings preserving balancedness in each step.

Sketch of the proof

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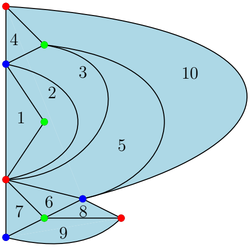
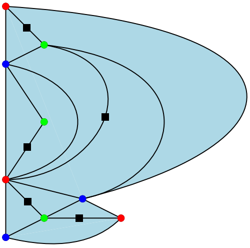
Step 3: The shellable pseudo-cobordism encodes a sequence of cross-flips Δ' and Γ.

Step 4: Convert each cross-flip into a sequence of shellings and balanced inverse shellings.

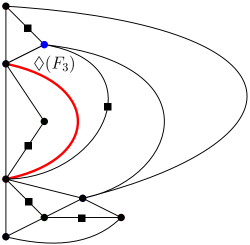
Step 3: From pseudo-cobordisms to cross-flips



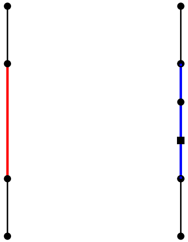
Ω



$\lozenge(\Omega)$

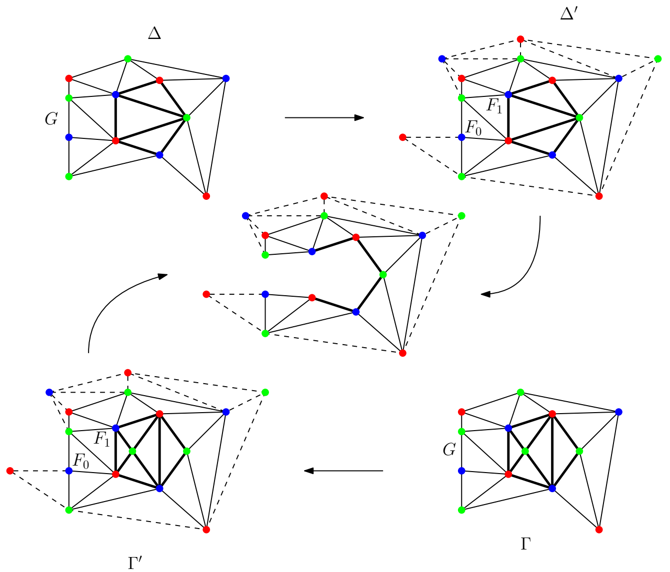


$F_3 = A_3 * R_3$



Δ_1 $\qquad$ $\Delta_1 \setminus \lozenge(A_3) \cup \mathcal{C}_2 \setminus (\lozenge(A_3))$

Step 4: From cross-flips to shellings



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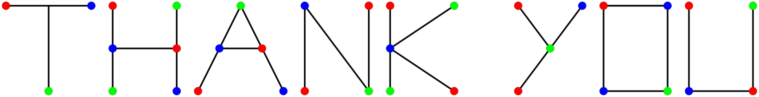
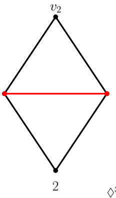
- Lorenzo wrote a program in Sage to apply cross-flips.
- He found balanced vertex-minimal triangulations of several surfaces and 3-manifolds.

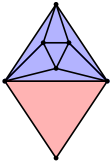

THANK YOU

Reductions

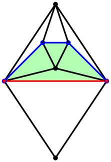
$\Diamond^3(\Gamma_1) = \Diamond^3(\Gamma_2)\#_{\Diamond^2(\Gamma_1)}\Diamond^3(\Gamma_2)$



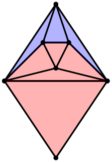
$\Diamond^3(\Gamma_2)\#_{\Diamond^2(\Gamma_1)}\Diamond^3(\Gamma_{\{0,1,2\}})$



$\Diamond^3(\Gamma_2)\#_{\Diamond^2(\Gamma_1)}\Diamond^3(\Gamma_0)\#_{\Diamond^2(\Gamma_{\{0,1\}})}\Diamond^3(\Gamma_{\{1,2\}})$



$\Diamond^3(\Gamma_{\{0,2\}})\#_{\Diamond^2(\Gamma_{\{0,1\}})}\Diamond^3(\Gamma_{\{1,2\}})$



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