Optimizing volume with prescribed diameter or minimum width

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(joint with T. Jahn, A. Polyanskii, M. Schymura, and G. Wachsmuth)

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*Author partially funded by Fundación Séneca, proyect 19901/GERM/15, and by MINECO, project MTM2015-63699-P, Spain.

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Einstein Workshop Discrete Geometry and Topology, FU Berlin, Germany, π-Day 2018.

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Which region D minimizes area for which a needle of length 1 can be rotated inside D by 2π radians?

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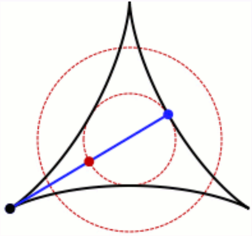

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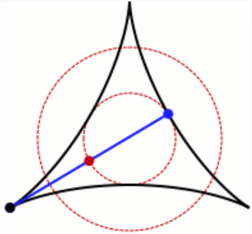

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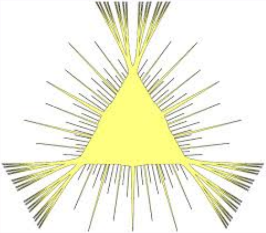

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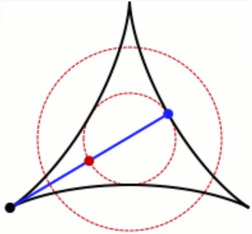

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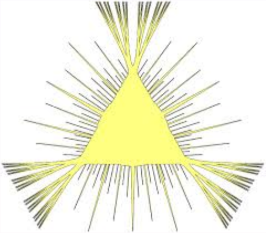

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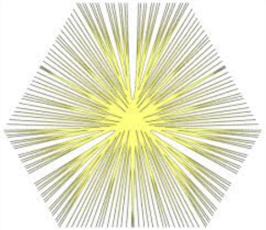

Arbitrary small area

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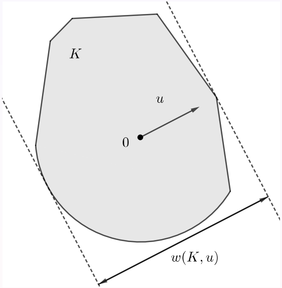

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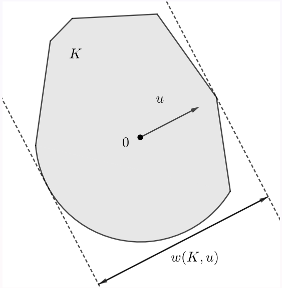
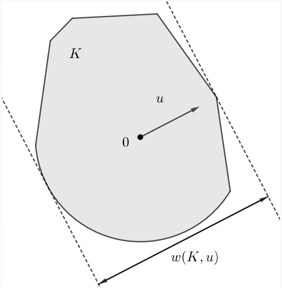


$$w(K) = \min_u w(K, u)$$

Convex Kakeya problem:

Which convex region K minimizes area for which $w(K) \geq 1$?



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Theorem (Pál, 1921)

Let K be a planar convex set. Then

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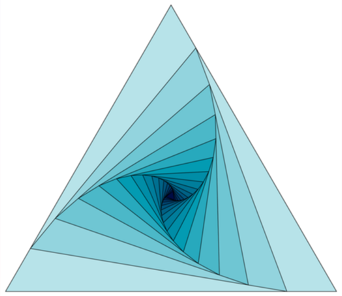

Pál's problem

Let $K \in \mathcal{K}^n$. Find $K_0 \in \mathcal{K}^n$ and $C_n > 0$ s.t.

$$\frac{vol(K)}{w(K)^n} \geq \frac{vol(K_0)}{w(K_0)^n} = C_n.$$

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Observation:

K_0 is w-minimal under inclusion.

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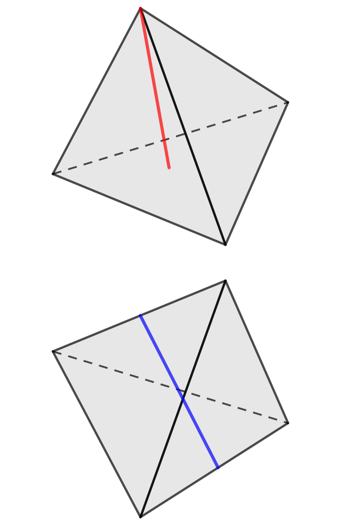
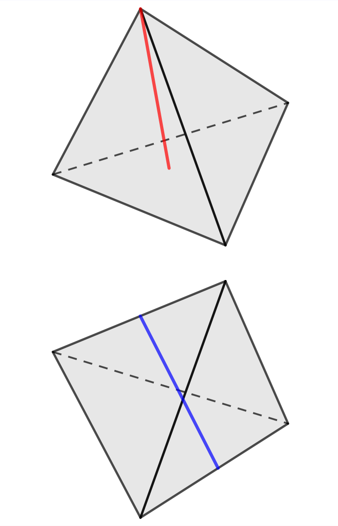
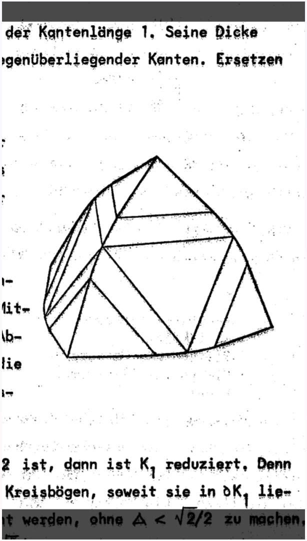
Definition (Reduced set):

K is w-minimal under inclusion.







der Kantenlänge 1. Seine Dicke
egenüberliegender Kanten. Ersetzen

-
Mit-
Ab-
die
-

2 ist, dann ist K_1 reduziert. Denn
Kreisbögen, soweit sie in ∂K_1 lie-
t werden, ohne $\Delta < \sqrt{2}/2$ zu machen.

Lassak's question (1990):

Does it exist reduced polytopes in dimension $n \geq 3$?

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2. *it is a pyramid, then ...*
3. *it has $n+2$ facets or $n+2$ vertices, then ...*

... P is not reduced.

Theorem 1 (G.M., Jahn, Polyanskii, Wachsmuth '17)

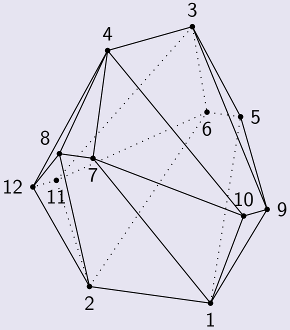

Figure: A reduced polytope with 12 vertices and 16 facets

Question (f-vector)

If $P \in \mathcal{K}^n$ is a reduced polytope, find best $c(n), C(n) > 0$ s.t.

$$f_0(P) + c(n) \leq f_{n-1}(P) \leq f_0(P) + C(n)$$

$(c(n), C(3) - 4 \geq 0)$.

Isodiametric inequality (Bieberbach 1915)

Let $K \in \mathcal{K}^n$. Then

$$\frac{vol(K)}{D(K)^n} \leq \frac{vol(\mathbb{B}_2^n)}{D(\mathbb{B}_2^n)^n} = 2^{-n}\kappa_n.$$

Equality holds iff $K = \mathbb{B}_2^n$.

Reverse isodiametric?

If $K = [-a, a] \times [-a^{-1}, a^{-1}]$, for $a > 0$ arbitrarily large, then

$$\frac{A(K)}{D(K)^2} = \frac{4}{4\left(a^2 + \frac{1}{a^2}\right)} \to 0.$$

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Idea: Affine Geometry

Definition (Isodiametric position, Behrend '37, G.M., Schymura '18+)

$K \in \mathcal{K}^n$ is in *isodiametric position* if

$$\frac{vol(K)}{D(K)^n} = \sup_{A \in \mathrm{GL}(n,\mathbb{R})} \frac{vol(A(K))}{D(A(K))^n}.$$

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Definition

$V \subset \mathbb{S}^{n-1}$ is the *set of diameters of* K, and fulfills

$$v \in V \quad \text{iff} \quad \exists \quad x \in K \quad s.t. \quad x + D(K)[0, v] \subset K.$$

Theorem 2 (G.M., Schymura '18+)

Let $K \in \mathcal{K}^n$ be in IDP and V be its set of diameters. Then for every L i-dimensional subspace, $i \in [n-1]$,

$$\exists \quad v \in V \quad \text{s.t.} \quad \sphericalangle(L, v) \geq \arccos\left(\sqrt{\frac{i}{n}}\right).$$

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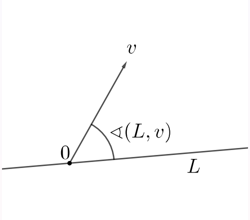

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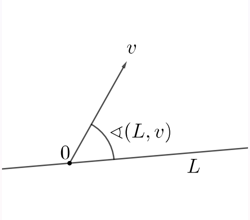

Sharpness?

Identifying WHEN K is in isodiametric position!

Löwner position

$K \subseteq \mathbb{B}_2^n$ is in *Löwner position* if $\mathbb{B}_2^n$ is the ellipsoid of minimum volume containing K.

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Theorem (John 1948, Ball 1992)

Let $K \in \mathcal{K}^n$ be 0-symmetric s.t. $K \subseteq \mathbb{B}_2^n$. The following are equivalent:

- K is in Löwner position.
- There exist $u_i \in K \cap \mathbb{S}^{n-1}$, $\lambda_i \geq 0$, $i \in [m]$, $n \leq m \leq \binom{n+1}{2}$, s.t.
$$\sum_{i=1}^{m} \lambda_i u_i u_i^T = \mathrm{I}_n.$$

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- $\frac{1}{D(K)}(K - K)$ is in Löwner position.

Remark 1 (Dvoretzky-Rogers Factorization 1950)

Let $u_i \in \mathbb{S}^{n-1}$, $\lambda_i \geq 0$, s.t. $\mathrm{I}_n = \sum_{i=1}^m \lambda_i u_i u_i^T$. Then there exist $1 \leq i_1 < \cdots < i_n \leq m$ s.t.

$$\sphericalangle(L_j, u_{i_{j+1}}) \geq \arccos\left(\sqrt{\frac{j}{n}}\right) \quad \text{where} \quad L_j = \mathrm{span}(u_{i_1}, \ldots, u_{i_j}).$$

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Remark 2

The vectors $u_i = \frac{1}{\sqrt{n}}(\pm 1, \ldots, \pm 1)$, $i \in [2^n]$ and $L_j = \mathrm{span}(e_1, \ldots, e_j)$ shows that Theorem 2 is best possible.

Theorem 4 (G.M., Schymura '18+)

Let $u_i \in \mathbb{S}^{n-1}$, $\lambda_i \geq 0$, $i \in [m]$, with $n \leq m \leq \binom{n+1}{2}$, be such that

$$\mathrm{I}_n = \sum_{i=1}^{m} \lambda_i u_i u_i^T.$$

Then

$$\min_{1 \leq i < j \leq m} |u_i^T u_j| \leq \sqrt{1 - \frac{\binom{n}{2}}{\binom{m}{2}} \left(\frac{m}{n}\right)^2}.$$

The inequality is sharp for $m = n, n+1$, and *sometimes* for $\binom{n+1}{2}$.

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$$\sphericalangle(u_{i_1}, u_{i_2}) \geq \begin{cases} \arccos\left(\frac{1}{\sqrt{n}}\right) & \text{DR-F} \\ \arccos\left(\frac{1}{\sqrt{n+2}}\right) & \text{GMS} \end{cases}$$

Lemma 1 (Gen. Cauchy-Binet formula)

Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times n}$, and $I, J \in \binom{[n]}{i}$. Then

$$\det((AB)_{I,J}) = \sum_{P \in \binom{[m]}{i}} \det(A_{I,P}) \det(B_{P,J}).$$

Lemma 2

Let $u_i \in \mathbb{S}^{n-1}$, $\lambda_i \geq 0$, $i \in [m]$, $n \leq m \leq \binom{n+1}{2}$, be s.t. $\mathrm{I}_n = \sum_{i=1}^{m} \lambda_i u_i u_i^T$. Then for every $i \in [n]$

$$\binom{n}{i} = \sum_{J \in \binom{[m]}{i}} \lambda_J \det((U_J)^T U_J),$$

where $\lambda_J = \prod_{j \in J} \lambda_j$ and $U_J = (u_j : j \in J)$.

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For instance, $n = \sum_{i=1}^m \lambda_i$.

Lemma 3

Let $m \in \mathbb{N}$, $m \geq 2$, $\lambda_i \geq 0$, $i \in [m]$, and $c \geq 0$ be such that $c = \sum_{i=1}^{m} \lambda_i$. If

$$f(\lambda_1, \ldots, \lambda_m) = \sum_{1 \leq i < j \leq m} \lambda_i \lambda_j,$$

then

$$\max_{\lambda_i \geq 0} f(\lambda_1, \ldots, \lambda_m) = f(c/m, \ldots, c/m) = \frac{c^2(m-1)}{2m}.$$

Theorem 4 (G.M., Schymura '18+)

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The inequality is sharp for $m = n, n+1$, and *sometimes* for $\binom{n+1}{2}$.

Proof.

$$\binom{n}{2} = \sum_{1 \leq i < j \leq m} \lambda_i \lambda_j \det \begin{pmatrix} 1 & u_i^T u_j \\ u_i^T u_j & 1 \end{pmatrix}$$ [Lem. 2]

□

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$\square$

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$$= \left(\frac{n}{m}\right)^2 \binom{m}{2} \left(1 - \min_{1\le i<j\le m} (u_i^T u_j)^2\right) \text{ [Lem. 3]}$$

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Proof of equality.

$$\lambda_1 = \cdots = \lambda_m = \frac{n}{m} \quad \text{and} \quad |u_i^T u_j| = \sqrt{\frac{m-n}{n(m-1)}},$$

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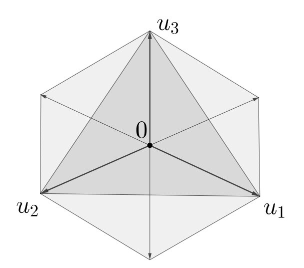
i.e., $\{u_1, \ldots, u_m\}$ is *set of equiangular lines* with

$$\sphericalangle(u_i, u_j) = \arccos\left(\sqrt{\frac{m-n}{n(m-1)}}\right).$$

. □

$m = n : |u_i^T u_j| = 0$



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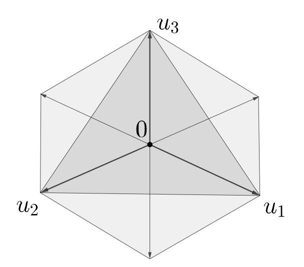
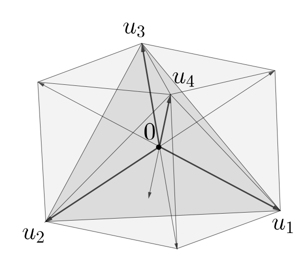


$m = n + 1 : |u_i^T u_j| = \frac{1}{n}$



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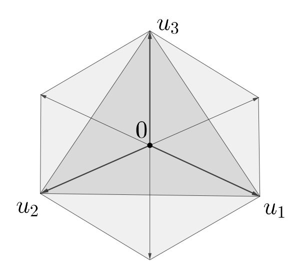

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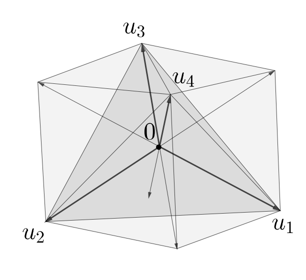

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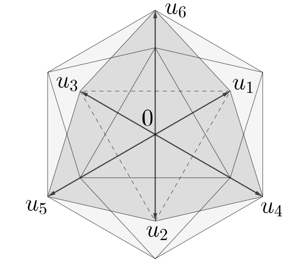

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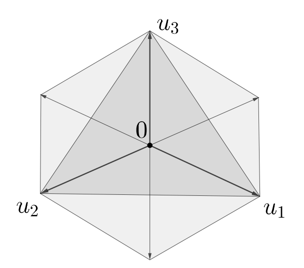

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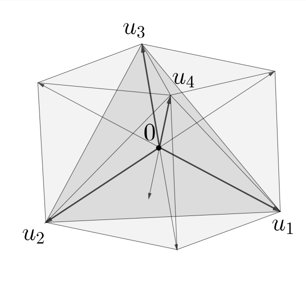
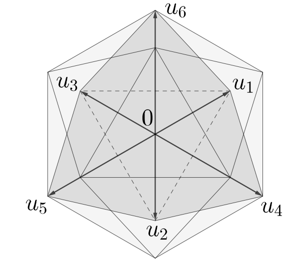


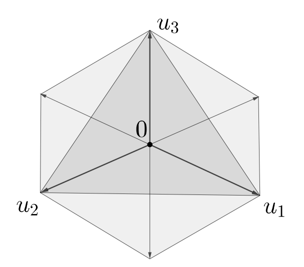
$m = \binom{n+1}{2} : |u_i^T u_j| = \frac{1}{\sqrt{n+2}}$



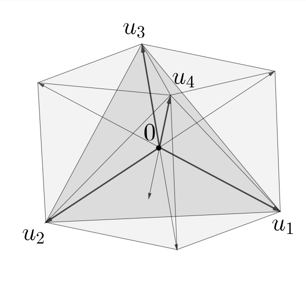
Lemmens-Seidel '73:

- $n = 4$ Not sharp :(

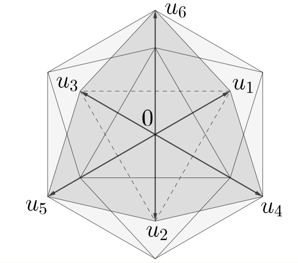
$m = n : |u_i^T u_j| = 0$



$m = n + 1 : |u_i^T u_j| = \frac{1}{n}$



$m = \binom{n+1}{2} : |u_i^T u_j| = \frac{1}{\sqrt{n+2}}$



Lemmens-Seidel '73:

▼ $n = 4$ Not sharp :(

▲ *but* for $n = 7$ and $n = 23$ *is sharp too*.

Corollary (Behrend 1937)

Let $K \in \mathcal{K}^2$ be in isodiametric position. Then

$$\frac{vol(K)}{D(K)^2} \geq \frac{\sqrt{3}}{4}.$$

Equality holds iff K is an equilateral triangle.

Proof.

$D(K) = 1$, $V = \{u_i\}_{i \in [m]}$, $C := \mathrm{conv}([x_i, x_i + u_i] : i \in [m]) \subseteq K$.

$\square$

Proof.

$D(K) = 1$, $V = \{u_i\}_{i \in [m]}$, $C := \mathrm{conv}([x_i, x_i + u_i] : i \in [m]) \subseteq K$. Then

$$vol(K) \geq vol(C)$$

□

Proof.

$D(K) = 1$, $V = \{u_i\}_{i \in [m]}$, $C := \mathrm{conv}([x_i, x_i + u_i] : i \in [m]) \subseteq K$. Then

$$\begin{aligned} vol(K) &\geq vol(C) \\ &\geq vol(\mathrm{conv}([x_1, x_1 + u_1] \cup [x_2, x_2 + u_2])) \quad \text{[Thm. 4]} \end{aligned}$$

□

Proof.

$D(K) = 1$, $V = \{u_i\}_{i\in[m]}$, $C := \mathrm{conv}([x_i, x_i + u_i] : i \in [m]) \subseteq K$.
Then

$$\begin{aligned} vol(K) &\geq vol(C) \\ &\geq vol(\mathrm{conv}([x_1, x_1 + u_1] \cup [x_2, x_2 + u_2])) \quad \text{[Thm. 4]} \\ &\geq vol(\frac{1}{2}\mathrm{conv}(\{\pm u_1, \pm u_2\})) \quad \text{[Betke-Henk'93]} \end{aligned}$$

□

Proof.

$D(K) = 1$, $V = \{u_i\}_{i \in [m]}$, $C := \mathrm{conv}([x_i, x_i + u_i] : i \in [m]) \subseteq K$. Then

$$\begin{aligned} vol(K) &\geq vol(C) \\ &\geq vol(\mathrm{conv}([x_1, x_1 + u_1] \cup [x_2, x_2 + u_2])) \quad \text{[Thm. 4]} \\ &\geq vol(\frac{1}{2}\mathrm{conv}(\{\pm u_1, \pm u_2\})) \quad \text{[Betke-Henk'93]} \\ &= \frac{1}{2}\sqrt{1 - (u_1^T u_2)^2} \end{aligned}$$

□

Proof.

$D(K) = 1$, $V = \{u_i\}_{i \in [m]}$, $C := \mathrm{conv}([x_i, x_i + u_i] : i \in [m]) \subseteq K$.
Then

$$\begin{aligned}
vol(K) &\geq vol(C) \\
&\geq vol(\mathrm{conv}([x_1, x_1 + u_1] \cup [x_2, x_2 + u_2])) \quad \text{[Thm. 4]} \\
&\geq vol(\frac{1}{2}\mathrm{conv}(\{\pm u_1, \pm u_2\})) \quad \text{[Betke-Henk'93]} \\
&= \frac{1}{2}\sqrt{1 - (u_1^T u_2)^2} \\
&\geq \frac{1}{2}\sqrt{1 - 1/4} = \frac{\sqrt{3}}{4} \quad \text{[Thm. 4]}
\end{aligned}$$

□

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Thank you for your attention!!