Addendum to the June 4th 2014 class

## 1. Finiteness

Please forgive the schlamasel, you saw a living example of something one should never ever do, and which one does every now and then...

**Theorem 1.1.** *Let $A$ be a Dedekind domain, $K = \mathrm{Frac}(A)$ be its field of fractions, $K \subset L$ be a finite separable field extension, $B \subset L$ be the integral closure of $A$ in $L$. We assume that all residue fields $A/\mathfrak{p}$, where $0 \neq \mathfrak{p} \subset A$ is a prime ideal, are perfect (e.g. finite). Then there are finitely many prime ideals in $A$ which ramifiy.*

*Proof.* If $B$ is free over $A$ then the theorem has been proven in the course.

Let $b_i, i = 1, \ldots, m$ be a basis of the $L$ as a $K$-vector space, such that the $b_i$ lie in $B$. Let

$$\psi : \oplus_{i=1}^m A \to B,\ (a_i) \mapsto \sum_{i=1}^m a_i b_i$$

be the $A$-linear map defined by the $b_i$. Then the $b_i$ being linearly independent over $K$ implies that $\psi$ is injective.

The $A$-module $B/\mathrm{Im}(\psi)$ is finitely generated by the residue classes of finitely many elements $\beta_i, i = 1, \ldots, c$. Thus there are coefficients $\lambda_{ij} \in K, i = 1, \ldots, c, j = 1, \ldots, m$ such that $\beta_i = \sum_{j=1}^m \lambda_{ij} b_j$. Write $\lambda_{ij} = \frac{\mu_{ij}}{\delta}$ for $\mu_{ij} \in A, \delta \in A \setminus \{0\}$. Thus

$$\beta_i = \sum_{j=1}^m \frac{\mu_{ij}}{\delta} b_j,\ \mu_{ij} \in A,\ \delta \in A \setminus \{0\}.$$

Define the multiplicative subset $S = \{\delta^n, n \in \mathbb{N}\} \subset A$. Let $A_S \subset K$ be the localized ring $A$ in $S$. As $A \subset B$ is a ring homomorphism, thus in particular the multiplication on $B$, restricted to $A$, is the multiplicaton in $A$, $S \subset B$ is a multiplicative subset. Then the ring homomorphism

$$\iota : B \otimes_A A_S \to B_S,\ \iota(\sum_i b_i \otimes \frac{a_i}{s_i}) = \sum_i \frac{b_i a_i}{s_i}$$

is an isomorphism. Here $B_S$ is the localization of $B$ in $S \subset B$. Indeed, it is clearly a homomorphism, and is also surjective as $\frac{b}{s} = \iota(b \otimes \frac{1}{s})$. It is injective as well. Indeed,

$$\sum_i b_i \otimes \frac{a_i}{s_i} = \sum_i b_i a_i t_i \otimes \frac{1}{s} \in B \otimes_A A_S$$

where $s = \prod_i s_i,\ t_i = \frac{s}{s_i} \in S$. Thus $\iota(\sum_i b_i \otimes \frac{a_i}{s_i}) = \sum_i b_i a_i t_i \otimes \frac{1}{s} = 0$ is equivalent to $\sum_i b_i a_i t_i = 0$ which in turn implies $\sum_i b_i \otimes \frac{a_i}{s_i} = 0$.

Now the $A_S$-linear map

$$\psi \otimes_A A_S : (\oplus_{i=1}^m A) \otimes_A A_S = \oplus_{i=1}^m A_S \to B \otimes_A A_S = B_S$$

is an isomorphism. Indeed, as an $A_S$-module, $B_S$ is spanned by $\frac{b_i}{1}, i = 1, \ldots, m$ and $\frac{\beta_i}{1}, i = 1, \ldots, c$. But by definition, $\frac{\beta_i}{1}$ lies in the $A_S$-submodule of $B_S$ spanned by the $\frac{b_i}{1}, i = 1, \ldots, m$. So $\psi \otimes_A A_S$ is surjective. It is injective as well. Indeed, if $(\frac{a_1}{s_1}, \ldots, \frac{a_m}{s_m}) \in \oplus_{i=1}^m A_S$ lies in $\mathrm{Ker}(\psi \otimes_A S)$, then, as $\psi \otimes_A S$ is a ring homomorphism, for any $s \in S$, $(\frac{a_1 s}{s_1}, \ldots, \frac{a_m s}{s_m})$ lies in $\mathrm{Ker}(\psi \otimes_A S)$ as well. In particular for $s = \prod_i s_i$, in which case $(\frac{a_1 s}{s_1}, \ldots, \frac{a_1 s}{s_1}) = (\frac{a_1 t_1}{1}, \ldots, \frac{a_m t_m}{1})$ with the previous notations. Thus $(\psi \otimes_A A_S)(\frac{a_1 s}{s_1}, \ldots, \frac{a_m s}{s_m}) = \frac{\psi(x)}{1} = 0$ with $x = (a_1 t_1, \ldots, a_m t_m)$. By the injectivity of $\psi$, we conclude $x = 0$, thus $x \otimes \frac{1}{s} = (\frac{a_1}{s_1}, \ldots, \frac{a_m}{s_m}) = 0$.

Yet again by theorem of the course, there are finitely many prime ideals in $A_S$ which ramify in $B_S$. We denote them by $P_1, \ldots, P_\alpha$. On the other hand, one has the prime ideal decomposition $\langle \delta \rangle = \prod_{i=1}^s \mathfrak{q}_j^{m_j}$ of the principal ideal spanned by $\delta$ in $A$. Via the localization homomorphism $A \to A_S$, the prime ideals of $A_S$ are all of the shape $\mathfrak{p} A_S$, where $\mathfrak{p} \subset A$ is a prime ideal and $\mathfrak{p} \notin \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$. Write $P_i = \mathfrak{p}_i A_S$ for $\mathfrak{p}_i = P_i \cap A$. Then $B/A$ ramifies in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_\alpha\}$ and possibly in some of the $\mathfrak{q}_j, j = 1, \ldots, s$. This finishes the proof.

## 2. Remarks

If we wish to understand the theorem for $K$ being a number field and $A$ being its ring of integers, that is the integral closure of $\mathbb{Z}$ in $K$, or $K$ being a function field and $A$ being the integral closure of $\mathbb{F}_p[X]$ in $K$, then we can give a slicker argument. Write $F = \mathbb{Q}$ in the number field case, $F = \mathbb{F}_p(X)$ in the function field case, write $R = \mathbb{Z}$ or $\mathbb{F}_p[X]$. Now if $\mathfrak{p} \subset A$ ramifies in $L$, then a fortiori $\mathfrak{p} \cap R$ ramifies in $L$. But $B$, the integral closure of $A$ in $L$, is also the integral closure of $R$ in $L$, thus is free as a $R$-module as $R$ is a PID, thus a UFD. So by the theorem of the course applied to $L \supset F$, there are finitely many such primes in $\mathfrak{p} \cap R \subset R$ which ramifiy in $L$. The finiteness of the number of prime ideals in $K$ dividing a given prime in $R$ finishes the proof.

□

## 3. Apologies, with Auden: The Hard Question

To ask the hard question is simple:
Asked at a meeting
With the simple glance of acquaintance
To what these go
And how these do;
To ask the hard question is simple,
The simple act of the confused will.

But the answer
Is hard and hard to remember:
On steps or on shore
The ears listening
To words at meeting,
The eyes looking
At the hands helping,
Are never sure
Of what they learn
From how these things are done,
And forgetting to listen or see
Makes forgetting easy,
Only remembering the method of remembering,
Remembering only in another way,
Only the strangely exciting lie,
Afraid
To remember what the fish ignored,
How the bird escaped, or if the sheep obeyed.

Till, losing memory,
Bird, fish, and sheep are ghostly,
And ghosts must do again
What gives them pain.
Cowardice cries
For windy skies,
Coldness for water,
Obedience for a master.

Shall memory restore
The steps and the shore,
The face and the meeting place;
Shall the bird live,
Shall the fish dive,
And sheep obey
In a sheep's way;
Can love remember
The question and the answer,
For love recover
What has been dark and rich and warm all over?