Addendum to the May 15, 2013 course

1. Topological versus algebraic completion

Let $A, \mathfrak{m}, k, K$ be a DVR, $v : K \to \mathbb{Z} \cup \infty$ be the discrete valuation defining A, $a \in)0\ 1($ defining the norm

$$||x|| = a^{v(x)} \tag{1.1}$$

on K, inducing on K the normed topology, $\hat{K}$ be the completed field wrt this normed topology, $\hat{v} : \hat{K} \to \mathbb{Z}$ be the extension of v on $\hat{K}$ defined by $\hat{v}((x_n)_{n\geq 1}) = \limsup\{v(x_n), n \geq 1\}$, which, as we have seen, makes $\hat{K}$ into a discrete valued field. We define $\hat{A}$ to be the ring of valuation of $\hat{v}$. So $\hat{A}$ consists of the classes of Cauchy sequences $(x_n)_{n\geq 1}$ for which $v(x_n) \geq 0$ for n large enough.

We define $\aleph = \varprojlim_n A/\mathfrak{m}^n = \{(x_n)_{n\geq 1} \in \prod_{n\geq 1} A/\mathfrak{m}^n,\ x_{n+1} \mapsto x_n\}$. Then $\aleph$ is discretely valued via $v_\aleph = \max\{v(\widetilde{x}_n), n \geq 1\}$, where $\widetilde{x}_n \in A$ is any lifting of x_n in A. This is straightforward to see that $v_\aleph$ is well defined and is a discrete valuation. One has $v_\aleph(x_n)_{n\geq 1} = \infty \iff (x_n)_{n\geq 1} = 0$. One has the injective homomorphism of rings $\iota_\aleph : A \to \aleph,\ x \mapsto (x_n = \text{residue class of } x \text{ in } A/\mathfrak{m}^n)$.

Proposition 1.1. *The completion homomorphism $\iota : A \to \hat{A},\ x \mapsto (x_n = x)_{n\geq 1}$, which is injective, factors as $\alpha \circ \iota_\aleph$, where $\alpha : \aleph \to \hat{A},\ (x_n)_{n\geq 1} \mapsto (\widetilde{x}_n)_{n\geq 1}$, and α is an isomorphism of discretely valued rings.*

Proof. The map α is well defined as for two choices of lifts $\widetilde{x}_n, \widetilde{x}'_n$ the Cauchy sequences $(\widetilde{x}_n)_{\geq 1}$ and $(\widetilde{x}'_n)_{\geq 1}$ are equivalent. Clearly if $\alpha((x_n)_{n\geq 1})$ is a sequence converging to 0, then $(x_n)_{\geq 1} \in \aleph$ is 0. Further one can give explicitely the inverse map to α. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence. Let $n_i \geq 1$ be a strictly increasing sequence such that $v(x_n - x_{n_i}) \geq (i+1)\ \forall n \geq n_i$. Set y_i = residue class of x_{n_i} in $A/\mathfrak{m}^i$. Then $(y_n)_{n\geq 1}$ is an element in $\aleph$ and then $(\alpha)^{-1}((x_n)_{n\geq 1}) = (y_n)_{n\geq 1}$. One easily sees that α and $(\alpha)^{-1}$ are inverse homomorphisms. □

2. For A a complete DVR, A locally compact implies A compact and thus $k = A/\mathfrak{m}$ finite

As $\mathfrak{m}^n$ for a fundamental system of closed neighbourhoods of 0, for n large, $\mathfrak{m}^n$ lies in a compact neighborhood, thus is compact as well. Thus A, which is the image of $\mathfrak{m}^n$ via the continuous map $K \to K,\ x \mapsto \frac{x}{\pi^n}$, where π is a uniformizer, is compact as well. Thus $A/\mathfrak{m}$ is compact and discrete, thus finite.