Addendum to the April 17, 2013 course

**Lemma 0.1.** *Let $A$ be a domain (commutative with $1 \in A$) with field of fraction $K$. Let $\mathfrak{A} \subset K$ be a $A$-submodule of $K$.*

i) *If there is a $A$-submodule $\mathfrak{A}' \subset K$ such that $\mathfrak{A} \cdot \mathfrak{A}' = A$, i.e. if $\mathfrak{A}$ is invertible, then both $\mathfrak{A}$ and $\mathfrak{A}'$ are finitely generated $A$-submodules, that is they are fractional ideals.*

ii) *If two $A$-submodules $\mathfrak{A}_i \subset K,\ i = 1, 2$ fulfill $\mathfrak{A} \cdot \mathfrak{A}_i = A$, then $\mathfrak{A}_1 = \mathfrak{A}_2$. We denote $\mathfrak{A}_i$ by $\mathfrak{A}^{-1}$.*

*Proof.* i) There are finitely many elements $x_i \in \mathfrak{A}, y_i \in \mathfrak{A}',\ i = 1, \ldots, N$ such that $1 = \sum_1^N x_i y_i$. Let $x \in \mathfrak{A}$. Then $x = \sum_1^N x_i(xy_i)$. Since $xy_i \in A$, $x \in \langle x_i \rangle_A$, thus $\mathfrak{A} \subset \langle x_i \rangle_A$, thus $\mathfrak{A} = \langle x_i \rangle_A$. Similarly $\mathfrak{A}' = \langle y_i \rangle_A$.

ii) Write $1 = \sum_1^N x_i y_i$ with $x_i \in \mathfrak{A}, y_i \in \mathfrak{A}_1$. Then for any $z \in \mathfrak{A}_2$, $z = \sum_1^N (x_i z) y_i$. Since $x_i z \in A$, we conclude $z \in \langle y_i \rangle_A = \mathfrak{A}_1$, thus $\mathfrak{A}_2 \subset \mathfrak{A}_1$ and similarly $\mathfrak{A}_2 \subset \mathfrak{A}_1$. □

**Lemma 0.2.** *Let $A$ be a Dedekind domain, let $\mathfrak{A} \subset K$ be a fractional ideal. Then*

$$\mathfrak{A} = \cap_{0 \neq p \in \operatorname{Spec} A} \mathfrak{A} A_p.$$

*Proof.* Let $bc^{-1} = a \in \cap_{0 \neq p \in \operatorname{Spec} A} \mathfrak{A} A_p,\ b \in \mathfrak{A}, c \in A \setminus 0$. Set $\mathfrak{B} = \{x \in A, xb \in c\mathfrak{A}\}$. Then $\mathfrak{B}$ is an ideal of $A$ which is not $0$ as it contains $c$. On the other hand, $\mathfrak{B} A_p = A_p$, thus $\mathfrak{B}$ is contained in no maximal ideal of $A$, thus $\mathfrak{B} = A$, thus $\mathfrak{B} \ni 1$ and $a \in \mathfrak{A}$. Thus $\cap_{0 \neq p \in \operatorname{Spec} A} \mathfrak{A} A_p \subset \mathfrak{A}$, while the other inclusion $A \subset \cap_{0 \neq p \in \operatorname{Spec} A} \mathfrak{A} A_p$ is trivial.

□

**Proposition 0.3.** *Let $A$ be a Dedekind domain. Then any fractional ideal is invertible.*

*Proof.* Let $\mathfrak{A}$ be a fractional ideal. For any fractional ideal, define $\iota(\mathfrak{A}) = \{x \in K, x\mathfrak{A} \subset A\}$. If $\mathfrak{A}$ is spanned by $x_1, \ldots, x_N \in A$, then $\iota(\mathfrak{A}) \subset \iota(x_1 A) = x_1^{-1} A$. As $A$ is noetherian, $\iota(A)$ is a fractional ideal. If $x \in \iota(\mathfrak{A})$, then $x\mathfrak{A} A_p \subset A_p$ thus by Lemma **??**

$$\iota(\mathfrak{A}) = \cap_{0 \neq p \in \operatorname{Spec} A} \iota(\mathfrak{A} A_p),$$

where $\iota(\mathfrak{A} A_p) = \{x \in K, x\mathfrak{A} A_p \subset A_p\} = (\mathfrak{A} A_p)^{-1}$. By definition, one has $\mathfrak{A}\iota(\mathfrak{A}) \subset A$ is an ideal of $A$. One has $(\mathfrak{A}\iota(\mathfrak{A}))_p = \mathfrak{A}_p \iota(\mathfrak{A})_p = A_p$. Thus $\mathfrak{A}\iota(\mathfrak{A}) = A$. □

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