Zahlentheorie II

Probeklausur

Exercise 1. In the following table, indicate which properties the rings in the first column have by making an **X** in the corresponding boxes.

	domain	local	complete local	1-dimensional[1]	Dedekind	discrete valuation ring
$\mathbb{Z}$						
$\mathbb{F}_3[X,Y]/(XY-1)$						
$\mathbb{Z}_2[X]$						
$\mathbb{Z}/6\mathbb{Z}$						
$\mathbb{Q}(\sqrt{7})[\![X]\!]$						
$\mathbb{Z}_{23}$						
$\mathbb{Q}(\sqrt{2},\sqrt{5})\otimes_{\mathbb{Q}}\mathbb{Q}(\sqrt{2})$						
$\mathbb{Z}[\sqrt{23}]$						
$\mathbb{Z}[\sqrt{7}]$						
$\mathbb{F}_2[X]_{(x^2+x+1)}$						
$\varprojlim_n \mathbb{Q}(\sqrt{-1})[\![X]\!]/(X^n)$						

Exercise 2. Which of the following rational numbers has square-roots in the field $\mathbb{Q}_3$? $2, -2, \frac{1}{2}, 3, 9, -1$. Prove your claims!

Exercise 3. Let A be a Dedekind domain with fraction field K, L/K a finite separable field extension and B the integral closure of A in L. Let $\mathfrak{p}$ be a maximal ideal in A and assume that $\mathfrak{p}$ ramifies in B. Does there exist a finite separable field extension L'/L such that $\mathfrak{p}$ splits completely in L'?

(Recall that by definition a prime $\mathfrak{p}$ of A splits completely in L' if and only if $\mathfrak{p}$ is the product of $[L':K]$-many different prime ideals in the integral closure of A in L').

Exercise 4. For $n \in \mathbb{N}$ define

$$B_n := \{x \in \mathbb{Q}_p \mid |x|_p \leq 1/n\},$$

where $|x|_p := (1/p)^{v_p(x)}$. Show that B_n is an ideal in $\mathbb{Z}_p$, and give a generator.

Exercise 5. The integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}[X]/(X^3-2)$ is isomorphic to $A := \mathbb{Z}[X]/(X^3-2)$. (You don't have to prove this!) Describe the ramification behavior of the prime ideal $(5) \subset \mathbb{Z}$ in A i.e.

(1) Is $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ a Galois extension?
(2) How many prime ideals of A are lying over (5)?
(3) What are the ramification indices and the degrees the residue extensions of the primes lying over (5)?
(4) Describe the primes over (5) explicitly by giving 2 generators of each prime.
(5) For each prime $\mathfrak{q}$ of A lying over (5) give a local parameter of $A_{\mathfrak{q}}$.

[1] Recall that this means that for any inclusion of prime ideals $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$, the ideal $\mathfrak{p}_2$ is maximal.

Exercise 6. Set $A = \mathbb{Z}_3[\zeta_3]$, where $\zeta_3 \in \bar{\mathbb{Q}}_3$ is a 3rd-primitive root of unity. Recall from Exercise sheet 9, that $\mathbb{Z}_3[\zeta_3]$ is a complete discrete valuation ring with local prameter $\zeta_3 - 1$. Thus $f := X^3 - (\zeta_3 - 1)$ is an Eisenstein polynomial and you know from the lecture that $B = A[X]/(f)$ is a complete discrete valuation ring with local parameter $x =$ the image of X in B. Set $K = \mathrm{Frac}(A)$ and $L = \mathrm{Frac}(B)$.

Show that L/K is Galois and compute the ramification subgroups G_i, $i \geq -1$ of $\mathrm{Gal}(L/K)$.