FREIE UNIVERSITÄT BERLIN
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Algebraic Groups

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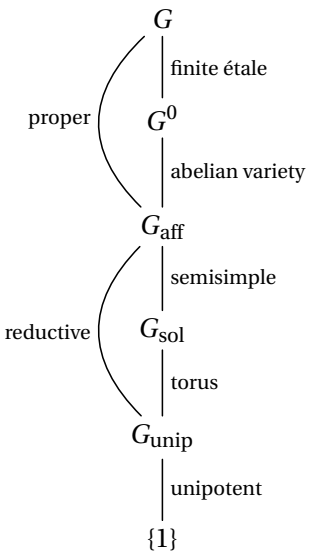
DESCRIPTION

Group schemes are "groups" in the category of schemes, just as the usual groups are "groups" in the category of sets and Lie groups are "groups" in the category of smooth manifolds. An algebraic group is a group scheme whose underlying space is of finite type. In this course we are going to study algebraic groups in certain generality. Given a smooth algebraic group G, we first split it into the connected part G^0 and the étale part $G_{\text{ét}} := G/G^0$ using the connected-étale sequence. For the connected algebraic group G^0 we can split it further into an affine algerbaic group G_{aff} and an abelian variety A using the theorem of Chevalley, i.e. we have the following exact sequence.

$$1 \to G_{\text{aff}} \to G^0 \to A \to 1$$

Then we can find a maximal connected normal solvable subgroup G_{sol} of the affine algebraic group G_{aff} so that the quotient is semisimple. This normal subgroup G_{sol} is called the radical. There is also a maxiaml connected unipotent normal subgroup G_{unip} of G_{aff} which is called the unipotent-radical. The unipotent radical of G_{aff} and G_{sol} are actually the same and the quotient $G_{\text{sol}}/G_{\text{unip}}$ is a torus. The aim of the course is to understand how can one decompose a smooth algebraic group in this way. We will introduce all the notions involved and prove

most of theorems used. To summarize we have the following diagram.



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1 GROUP SCHEMES (20/04/2016)

Aim of the talk: INTRODUCTION AND BACKGROUND MATERIAL

DETAILS:

- Group object of a category;
- Examples: group, Lie Group, topological group, group scheme;
- What is a Hopf-algebra? What is the relation between algebraic groups and Hopf-algebras?
- Functorial view point of a group scheme;
- Examples of group schemes: $\mathbb{G}_a$, $\mathbb{G}_m$, GL_n, elliptic curves, constant group schemes;
- Group schemes as a category. Kernel und cokenel of a morphism of group schemes. In the category of affine group schemes, what is a subgroup and what is a quotient group?

2 ALGEBRAIC GROUPS (27/04/2016)

Aim of the talk: ALGEBRAIC GROUPS ARE QUASI-PROJECTIVE, AFFINE ALGEBRAIC GROUPS ARE MATRIX GROUPS, COMMUTATIVE ALGEBRAIC GROUP

DETAILS:

- Theorem: Every algerbaic group is quasi-projective;
- Theorem: Every affine algebraic group is a closed subgroup of the general linear algebraic group;
- Theorem: The category of commutative affine algebraic groups is an abelian category.

3 THE CONNECTED COMPONENT AND FINITE ÉTALE GROUP SCHEMES (04/05/2016)

Aim of the talk: THE CONNECTED-ÉTALE SEQUENCE

DETAILS:

- The connected components of an algebraic group;
- The category of finite étale group schemes is equivalent to the category of finite groups equipped with a Galois action;
- The connected-étale sequence.

4 GROUPS OF MULTIPLICATIVE TYPE (I) (11/05/2016)

Aim of the talk: DIAGONALISABLE ALGEBRAIC GROUP

DETAILS:

- Group characters;
- There is an equivalence between the category of diagonalisable algebraic groups and the category of commutative finitely generated abstract groups;
- Representations of diagonalisable groups;
- Tori.

5 Groups of Multiplicative Type (II) (18/05/2016)

Aim of the talk: Groups of multiplicative type

Details:

- The character function induces an equivalence between the category of groups of multiplicative type and the category of commutative finitely generated abstract groups with a Galois action;
- Representations of diagonalisable group schemes;
- A criterion for an algebraic group to be a group of multiplicative type.

6 Unipotent Group (I) (25/05/2016)

Aim of the talk: Classification of the unipotent groups

Details: Let G be an affine algebraic group over k. The following statements are equivalent.

- The only irreducible representation of G is the trivial 1-dimensional representation;
- There is an imbedding from G to the subgroup $\mathbb{U}_n \subseteq \mathrm{GL}_n$, where U_n is the subgroup of upper triangular matrix with 1s on the diagonals;
- There is a composition series $G \times_k \bar{k}$, in which all the successive quotients are isomorphic to a subgroup $\mathbb{G}_{a,\bar{k}}$.

7 Unipotent Group (II) (01/06/2016)

Aim of the talk: Unipotent group in characteristic 0 or characteristic $p > 0$

Details:

- Lie-algebra of an algebraic group;
- Classification of the unipotent groups in characteristic 0 via their Lie-algebras;
- Classification of the unipotent groups in characteristic $p > 0$.

8 Trigonalisable Algebraic Group (08/06/2016)

Aim of the talk: Trigonalisable algebraic group

Details:

- The definition;
- The unipotent radical of a trigonalisable algebraic group;
- The structure of a trigonalisable algebraic group.

9 Semisimple and Reductive Algebraic Group (I) (15/06/2016)

Aim of the talk: Solvable algebraic group

Details:
- The commutator subgroup;
- Solvable algebraic group;
- Lie-Kolchin's Theorem.

10 Borel Subgroup (22/06/2016)

Aim of the talk: Borel group und parabolic subgroup

Details:
- Borel fixed point theorem;
- Borel subgroup;
- Parabolic subgroup;
- Radical of an algebraic group.

11 Semisimple and Reductive Algebraic Group (II) (29/06/2016)

Aim of the talk: Semisimple group, reductive group, pseudo-reductive group

Details:
- Semisimple group, reductive group, pseudo-reductive group and examples;
- Simply connected algebraic groups and examples;
- Maximal Tori.

12 Abelian Varieties (06/07/2016)

Aim of the talk: Basics about Abelian varieties

Details:
- Abelian Variety, Jacobian Variety, Albanese Variety;
- Picard-Functor und Picard-Scheme;
- Néron Model.

13 Finite Group Schemes (13/07/2016)

Aim of the talk: To be decided

14 Representations and Tannakian Duality (20/07/2016)

Aim of the talk: To be decided