June 19th, 2013

Zahlentheorie II

Exercise sheet 9[1]

Exercise 1. Let A be a DVR with maximal ideal $\mathfrak{p}$ and fraction field K and L/K a finite Galois extension with Galois group G. Denote by B the integral closure of A in L. Then there are finitely many prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ in B lying over $\mathfrak{p}$ and you saw in the lecture that G acts transitively on the set of $\mathfrak{q}_i$'s. Let $\mathfrak{q}$ be a prime in B over $\mathfrak{p}$ and denote by $D_{\mathfrak{q}} \subset G$ the decomposition group of $\mathfrak{q}$. Denote by $\hat{A}_{\mathfrak{p}}$ and $\hat{B}_{\mathfrak{q}}$ the completions of A and B along $\mathfrak{p}$ and $\mathfrak{q}$, respectively, by $\hat{K}_{\mathfrak{p}}$ and $\hat{L}_{\mathfrak{q}}$ their fraction fields. You know from the lecture that $\hat{L}_{\mathfrak{q}}/\hat{K}_{\mathfrak{p}}$ is Galois.

(i) Show that an element $\sigma \in D_{\mathfrak{q}}$ extends uniquely to an element $\hat{\sigma}$ in $\mathrm{Gal}(\hat{L}_{\mathfrak{q}}/\hat{K}_{\mathfrak{p}})$.

(ii) Show that $\sigma \mapsto \hat{\sigma}$ induces an isomorphism of groups

$$D_{\mathfrak{q}} \xrightarrow{\cong} \mathrm{Gal}(\hat{L}_{\mathfrak{q}}/\hat{K}_{\mathfrak{p}}).$$

(*Hint*: From (i) you get an inclusion and then you can use $\hat{K}_{\mathfrak{p}} \otimes_K L \cong \prod_i \hat{L}_{\mathfrak{q}_i}$ to show that both sides have the same cardinality.)

Exercise 2. Let p be a prime number, and $\bar{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. For $n \in \mathbb{N}$ let $\zeta_{p^n} \in \bar{\mathbb{Q}}_p$ be a p^n-th primitive root of unity (i.e. a generator of the cyclic subgroup of order p^n of the p^n-th root of unity in $\bar{\mathbb{Q}}_p^{\times}$). Set $K_0 = \mathbb{Q}_p$ and $K_n = \mathbb{Q}_p(\zeta_{p^n})$, $n \in \mathbb{N}$. Show the following for all $n \in \mathbb{N}$:

(i) $[K_n : K_0] = (p-1)p^{n-1}$. (*Hint:* Set $f(x) := x^{(p-1)p^{n-1}} + x^{(p-2)p^{n-1}} + \ldots + x^{p^{n-1}} + 1 \in \mathbb{Z}_p[x]$. Show that $f(x+1)$ is an Eisenstein polynomial and that $f(\zeta_{p^n}) = 0$.)

(ii) The extension K_n/K_0 is Galois with Galois group $\mathrm{Gal}(K_n/K) \cong (\mathbb{Z}/p^n\mathbb{Z})^{\times}$.

(iii) K_n/K is totally ramified, $\mathcal{O}_{K_n} = \mathbb{Z}_p[\zeta_{p^n}]$ and a local parameter of $\mathcal{O}_{K_n}$ is given by $\zeta_{p^n} - 1$. (*Hint:* (i).)

Exercise 3. With the notation from Exercise 2 we denote by G_i, $i \geq -1$, the higher ramification groups of the Galois extension K_n/K_0.

(i) Show that the isomorphism $\mathrm{Gal}(K_n/K_0) \cong (\mathbb{Z}/p^n\mathbb{Z})^{\times}$ from part (ii) of Exercise 2 induces group isomorphisms

$$G_{-1} = G_0 \cong (\mathbb{Z}/p^n\mathbb{Z})^{\times} \text{ and } G_i = \{1\}, \text{ for } i \geq p^{n-1},$$

[1] Questions or comments to kay.ruelling@fu-berlin.de or come to A3, Room 108.

and for $1 \leq i \leq p^{n-1} - 1$
$$G_i \cong \mathrm{Ker}((\mathbb{Z}/p^n\mathbb{Z})^\times \to (\mathbb{Z}/p^r\mathbb{Z})^\times),$$
where $r \in \{1, \ldots, n-1\}$ is the unique natural number with $p^{r-1} \leq i \leq p^r - 1$.

(ii) Show that the quotients G_i/G_{i+1}, $i \geq 1$, are cyclic groups of order 0 or p.

Exercise 4. Let k be an algebraically closed field of characteristic $p > 0$ and $K = k((t))$. Let $L = K[u]/(u^p - u - \frac{1}{t})$ be the extension of K described in Exercise 6 on Exercise sheet 3. Denote by $\mathcal{O}_L$ the algebraic closure of $k[[t]]$ in L. In particular we saw that $\mathcal{O}_L$ is a complete DVR with local parameter u^{-1}. Show:

(i) L/K is Galois and for $a \in \mathbb{F}_p$ the homomorphism of K-algebras $K[u] \to K[u]$, $u \mapsto u+a$, naturally induces a K-linear automorphism $p_a : L \to L$ and the map $\mathbb{F}_p \xrightarrow{\simeq} \mathrm{Gal}(L/K)$, $a \mapsto p_a$ is a group isomorphism.

(ii) Compute the ramification groups G_i of L/K for $i \geq -1$.