

## Zahlentheorie II

### Exercise sheet 4

**Exercise 1** (2 Points). Let  $A$  be a discrete valuation ring,  $K$  be its fraction field. Show that if  $B \subseteq K$  is a subring containing  $A$  then either  $B = A$  or  $B = K$ .

**Exercise 2.** (6 Points) Let  $A$  be a commutative ring with a unit. An  $A$ -module  $M$  is called flat if one of the following two equivalent conditions is satisfied: (1) for any injective map of  $A$ -modules  $N' \rightarrow N$  the induced map  $N' \otimes_A M \rightarrow N \otimes_A M$  is still injective; (2) for any ideal  $I \subseteq A$  the natural map  $I \otimes_A M \rightarrow M$  is injective.

- (1) If  $A$  is a PID, show that an  $A$ -module is flat if and only if for any two non-zero elements  $a \in A$ ,  $m \in M$ ,  $am \neq 0$ . (2 points)
- (2) (Nakayama lemma) Show that if  $A$  is a local ring with maximal ideal  $\mathfrak{p}$ ,  $M$  a finitely generated  $A$ -module, then  $\mathfrak{p}M = M$  implies  $M = 0$ . (2 points) (*Hint:* Let  $\{x_1, x_2, \dots, x_n\}$  be a set of generators of  $M$ . So we have equations:

$$x_i = \sum_{j=1}^n a_{ij}x_j \quad (1 \leq i \leq n, a_{ij} \in \mathfrak{p}). \quad (*)$$

Let  $T$  be the matrix with entries  $(\delta_{ij} - a_{ij})$  where  $\delta_{ij}$  is the Kronecker notation:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j ; \\ 0 & \text{if } i \neq j . \end{cases}$$

Let  $X$  be the vector

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}.$$

Then  $(*)$  implies  $TX = 0$ . Let  $T^*$  be the adjoint matrix of  $T$ . Then  $T^*TX = 0$ . So  $\det(T)x_i = 0$  for all  $1 \leq i \leq n$ . Thus  $\det(T)M = 0$ . But  $\det(T)$  is invertible in  $A$ .

- (3) Show that if  $A$  is a DVR with maximal ideal  $\mathfrak{p}$ , then  $M$  is a finitely generated flat  $A$ -module if and only if it is a free  $A$ -module of finite rank. (2 points) (*Hint:*  $M \otimes_A A/\mathfrak{p} = M/\mathfrak{p}M$  is a finite dimensional  $A/\mathfrak{p}$ -vector space. After choosing a basis we get a map  $f : A^{\oplus n} \rightarrow M$  so that the induced map  $(A/\mathfrak{p})^{\oplus n} \rightarrow$

$M/\mathfrak{p}M$  is an isomorphism. Let  $N$  be the kernel of  $f$  and  $N'$  be the cokernel of  $f$ . Prove that  $\mathfrak{p}N' = N'$  and deduce from (2) that  $N' = 0$ . So we have an exact sequence

$$0 \rightarrow N \rightarrow A^{\oplus n} \xrightarrow{f} M \rightarrow 0.$$

Let  $\pi \in A$  be the uniformizer. Now multiplying the exact sequence by  $\pi$  we get a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & A^{\oplus n} & \xrightarrow{f} & M \longrightarrow 0 \\ & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ 0 & \longrightarrow & N & \longrightarrow & A^{\oplus n} & \xrightarrow{f} & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & N/\mathfrak{p}N & \longrightarrow & (A/\mathfrak{p})^{\oplus n} & \xrightarrow{\cong} & M/\mathfrak{p}M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Prove that the maps  $N \xrightarrow{\pi} N$  and  $M \xrightarrow{\pi} M$  in the above diagram are injective and deduce from this that the map  $N/\mathfrak{p}N \rightarrow (A/\mathfrak{p})^{\oplus n}$  is injective too. Now applying (2) to prove that  $N = 0$  and deduce that  $f$  is an isomorphism.)

**Exercise 3.** (7 Points) Let  $A$  be a commutative ring with a unit,  $I, J$  be two ideals of  $A$ . We say  $I, J$  are coprime ideals if  $I + J = A$ .

- (1) Show that for two coprime ideals  $I, J \subseteq A$  we have  $IJ = I \cap J$ . (1 point)
- (2) Let  $n \in \mathbb{N}^+$ ,  $I_i$  be ideals of  $A$  for  $1 \leq i \leq n$ . Suppose  $I_i$  and  $I_j$  are mutually coprime when  $i \neq j$ . Use induction to show that  $\prod I_i = \cap I_i$ . (2 points)
- (3) Let

$$\phi : A \rightarrow \prod_{i=1}^n (A/I_i)$$

be the natural map of rings induced by the natural quotient map.

- (a) Show that  $\phi$  is an injection if and only if  $\cap I_i = 0$ . (1 point)
- (b) Show also that  $\phi$  is surjective if and only if  $I_i, I_j$  are coprime ideals when  $i \neq j$ . (2 points) (*Hint:* For the surjectivity

- one only needs to show that the element  $(1,0,0,\dots) \in \prod_{i=1}^n (A/I_i)$  is mapped onto by some element of  $A$ .
- (4) Now let  $A$  be a Dedekind domain,  $k \in \mathbb{N}^+$ . For every  $i$ ,  $1 \leq i \leq k$ , let  $\mathfrak{p}_i$  be distinct prime ideals of  $A$ ,  $x_i$  elements of  $A$ , and  $n_i \in \mathbb{Z}$ . Use the above conclusions to show that there exists an  $x \in A$  such that  $v_{\mathfrak{p}_i}(x - x_i) \geq n_i$  for all  $i$ . (See Serre's book "Local Fields" page 12, Approximation Lemma.) (1 point)

If you have any questions concerning these exercises you can contact Lei Zhang via [l.zhang@fu-berlin.de](mailto:l.zhang@fu-berlin.de) or come to Arnimalle 3 112A.