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Algebraic Number Theory

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Exercise sheet 5[1]

Exercise 1. Let k be a field, V a finite dimensional k-vector space, and $f : V \to V$ a k-linear endomorphism.

(a) Show that there is a unique structure of a $k[X]$-module on V, such that $X \cdot v := f(v)$.

(b) Show that the $k[X]$-module V defined in the previous part is a torsion module.

(c) Assume that k contains all eigenvalues of f. Use the structure theorem for finitely generated modules over a Dedekind domain to prove that there exists a basis of V, in which f has Jordan canonical form, i.e. in which f is given by a matrix

$$\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix},$$

where the J_i are square matrices of the shape

$$\begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}.$$

(d) Conclude that there is a unique way to write $f = f_s + f_n$, where f_s is diagonizable and f_n nilpotent. *Optional*: If you are very bored and/or want to practice applying Galois theory, you can try to prove: If k is any perfect field (not necessarily algebraically closed), then there is a unique way to write $f = f_s + f_n$, where f_n is nilpotent and f_s is diagonizable over some finite extension k' of k.

Exercise 2. This exercise is intended for those who have never seen the definition of a projective module before this week. Let A be a commutative ring and P an A-module. Show that the following properties are equivalent:

[1] If you want your solutions to be corrected, please hand them in just before the lecture on May 21st. If you have any questions concerning these exercises you can contact Lars Kindler via kindler@math.fu-berlin.de or come to Arnimallee 3 112A.

(a) For any short exact sequence
$$0 \to L \to M \to N \to 0$$
the associated sequence
$$0 \to \mathrm{Hom}_A(P, L) \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N) \to 0$$
is also exact.

(b) For any surjective morphism $\phi : M \to N$ of A-modules, and any A-linear morphism $f : P \to N$, there exists a morphism $F : P \to N$ lifting f, i.e. such that the diagram
$$\begin{array}{ccc} & & P \\ & \overset{F}{\swarrow} & \downarrow f \\ M & \overset{\phi}{\twoheadrightarrow} & N \end{array}$$
commutes.

(c) Every short exact sequence of A-modules
$$0 \to L \to M \xrightarrow{\tau} P \to 0$$
splits, i.e. there exists an A-module homomorphism $\sigma : P \to M$, such that $\tau \circ \sigma = id_P$.

(d) P is a direct summand of a free A-module.

If P satisfies the above equivalent properties, then P is called *projective*.

Exercise 3. Now let A be a Dedekind domain.

(a) If $J \subset \mathrm{Frac}(A)$ is a fractional ideal, show that J is a projective A-module.

(b) Show that a finitely generated A-module P is projective if and only if it is torsion-free, i.e. if and only if for any $a \in A$ and $x \in P$, $ax = 0$ implies $a = 0$ or $x = 0$.

(c) If A is a principal ideal domain, show that a finitely generated A-module P is projective if and only if it is free.

(d) Find an example of a Dedekind domain A, and a finitely generated projective A-module P, such that P is not free (*Hint:* Think about $\mathbb{Z}[\sqrt{-5}]$.)

Exercise 4. Let A be a ring, and define $P(A)$ to be the free abelian group on the set of isomorphism classes of finitely generated projective A-modules. In other words, if we write $[M]$ for the isomorphism class of a finitely generated projective A-module M, then $P(A)$ is the group of finite formal sums $\sum_{i=1}^{n} a_i[M_i]$, $a_i \in \mathbb{Z}$, $M_1, \ldots, M_n$ pairwise nonisomorphic finitely generated projective A-modules.

On $P(A)$ define the following relations: Whenever there exists a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

then $[M] = [M'] + [M'']$. Define $K_0(A)$ to be the quotient of $P(A)$ by the subgroup generated by these relations. This group is called the *Grothendieck group of finitely generated projective A-modules*. (*Note*: It is not important that we are dealing with projective modules. The same construction works in any abelian category, or even more generally.)

(a) Let A be a Dedekind domain. Show that the map $P(A) \to \mathbb{Z}$, $[M] \mapsto \mathrm{rank}(M)$ induces a well-defined surjection of abelian groups $r : K_0(A) \twoheadrightarrow \mathbb{Z}$.

(b) If $[I] \in \mathrm{Cl}(A)$ is a class of ideals, show that the map $\phi : \mathrm{Cl}(A) \to K_0(A)$, $[I] \mapsto [I] - [A]$ is well-defined, and fits in a short exact sequence

$$0 \to \mathrm{Cl}(A) \xrightarrow{\phi} K_0(A) \xrightarrow{r} \mathbb{Z} \to 0.$$

(c) Conclude that $K_0(A) \cong \mathrm{Cl}(A) \oplus \mathbb{Z}$.