April 30, 2014

Algebraic Number Theory

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Exercise sheet 3[1]

Exercise 1. Let A be an integral domain.

(a) Prove that $A = \bigcap_{\mathfrak{m} \text{ maximal ideal}} A_{\mathfrak{m}}$.
(b) Conclude that if A is also noetherian (and not a field), then A is a Dedekind domain if and only if for every maximal ideal $\mathfrak{m} \subset A$, the localization $A_{\mathfrak{m}}$ is a discrete valuation ring.

Exercise 2. Let A be a Dedekind domain and $I \subset A$ a nonzero ideal.

(a) Show that every ideal of A/I is principal. (Hint: First prove this for I =power of a prime ideal, then use the Chinese Remainder Theorem.)
(b) Conclude that I can be generated by two elements. More generally, conclude that I satisifies the property

($\star$) for every $a \in I \setminus \{0\}$ there exists some $b \in I$, such that $I = (a, b)$.

Exercise 3. This is a converse to the previous exercise: Let A be an integral domain such that every nonzero ideal $I \subset A$ has the property ($\star$) from the previous exercise. Show that if A is not a field, then it is a Dedekind domain. (Hint: Let $\mathfrak{p}$ be a nonzero prime ideal of A. To show that $A_{\mathfrak{p}}$ is a discrete valuation ring, show that for every ideal I of $A_{\mathfrak{p}}$ there exists $b \in I$ with $I = I\mathfrak{p}A_{\mathfrak{p}} + bA_{\mathfrak{p}}$. Apply Nakayama's Lemma (why can you?) to conclude $I = (b)$.)

Exercise 4. From the lecture you know that if A is a Dedekind domain, then multiplication of fractional ideals makes the set $\mathrm{Id}(A)$ of fractional ideals of A into an abelian group. In this exercise, you prove the converse statement which goes back to Emmy Noether: If A is an integral domain such that multiplication of fractional ideals makes $\mathrm{Id}(A)$ into an abelian group, then A is a Dedekind domain. Write K for the field of fractions of A and proceed as follows:

(a) First show that A is noetherian. (Hint: If $I \subset A$ is an ideal, then there exist $x_1, \dots, x_n \in I$ and $y_1, \dots, y_n \in I^{-1} \subset K$, such that $1 = \sum_{i=1}^n x_i y_i$.)

[1] If you want your solutions to be corrected, please hand them in just before the lecture on May 7th. If you have any questions concerning these exercises you can contact Lars Kindler via kindler@math.fu-berlin.de or come to Arnimallee 3 112A.

(b) Next, show that A is integrally closed. (Hint: If $x \in K$ is the root of a monic polynomial $T^m + a_{m-1}T^{m-1} + \ldots + a_0 = 0$ with $a_0, \ldots, a_{m-1} \in A$, then x^m belongs to the fractional ideal I generated by $1, x, \ldots, x^{m-1}$. Use the fact that I is invertible to show that $I \subset A$, so $x \in A$.)

(c) Finally, show that every nonzero prime ideal $\mathfrak{p} \subset A$ is maximal. (Hint: Let $a \in A \setminus \mathfrak{p}$, and define $I := \mathfrak{p} + (a)$. Show that $\mathfrak{p} = I^{-1}\mathfrak{p}$. Conclude that $I = A$ and that $\mathfrak{p}$ is maximal.)