June 18, 2014

Algebraic Number Theory

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Exercise sheet 11[1]

Exercise 1. Let $n, m \in \mathbb{N}$ be coprime. Show that $\mathbb{Q} = \mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\zeta_m)$ if ζ_n (resp. ζ_m) is a primitive n-th (resp. m-th) root of unity. Conclude that if p is a prime, then every root of unity in $\mathbb{Q}(\zeta_p)$ is of the form $\pm\zeta_p^a$, $a = 0, \ldots, p-1$.

Exercise 2. Let p be a prime number and ζ_p a primitive p-th root of unity. Show that $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ has at most one quadratic subextension. (*Hint:* The Galois group of $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ is cyclic.)

Exercise 3. Let $p > 2$ be a prime number and ζ_p a primitive p-th root of unity. Define $p^* := (-1)^{\frac{p-1}{2}} p$, and show that $\mathbb{Q}(\sqrt{p^*}) \subset \mathbb{Q}(\zeta_p)$. (*Hint:* If Φ_p is the p-th cyclotomic polynomial, rewrite the right hand side of $p = \Phi_p(1)$ to show that p is a square in $\mathbb{Q}(\zeta_p)$ up to a power of -1.)

Exercise 4. Let $n \in \mathbb{N}$ and ζ_n a primitive n-th root of unity. Write $\mathbb{Q}(\zeta_n)^+ := \mathbb{Q}(\zeta_n + \zeta_n^{-1})$. Show that

(i) $\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n)^+$ is an extension of degree 2,

(ii) For any embedding $\iota : \mathbb{Q}(\zeta_n) \hookrightarrow \mathbb{C}$, we have $\iota(\mathbb{Q}(\zeta_n)^+) \subset \mathbb{R}$,

(iii) The integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\zeta_n)^+$ is $\mathbb{Z}[\zeta_n + \zeta_n^{-1}]$.

[1] If you want your solutions to be corrected, please hand them in just before the lecture on July 2nd. If you have any questions concerning these exercises you can contact Lars Kindler via kindler@math.fu-berlin.de or come to Arnimallee 3 112A.