April 16th, 2014

Algebraic Number Theory

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Exercise Sheet 1[1]

Exercise 1.1 (2 Points)**.** Let A be an integral domain. An element $a \neq 0$ is called *irreducible* if it is not a unit, and if whenever one can write $a = b\,c$ with $b, c \in A$ then b or c is a unit.

Now consider the ring $A = \mathbb{Z}[\sqrt{-6}]$, i.e. $A = \mathbb{Z} \oplus \sqrt{-6}\,\mathbb{Z}$ as a $\mathbb{Z}$-module, endowed with the ring structure induced by the ring structure of $\mathbb{C}$ via the obvious injection $\mathbb{Z} \oplus \sqrt{-6}\,\mathbb{Z} \hookrightarrow \mathbb{C}$. ($A$ is the integral closure of $\mathbb{Z}$ in the number field $K = \mathbb{Q}(\sqrt{-6})$, but this is not part of the exercise.) Show that in A the number -6 has two different factorizations as products of irreducible elements:

$$-6 = -2 \cdot 3 = (\sqrt{-6})^2.$$

(*Hint:* Let $\mathrm{N} : \mathbb{Z}[\sqrt{-6}] \longrightarrow \mathbb{Z}_{\geq 0}$ be the map induced by the usual complex norm $|\cdot|^2_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$, $x + iy \longmapsto x^2 + y^2$. If $a = b\,c$ in A, then $\mathrm{N}(a) = \mathrm{N}(b)\,\mathrm{N}(c)$ in $\mathbb{Z}$.)

Exercise 1.2 (4 Points)**.** Let A be a discrete valuation ring, $\mathfrak{m}$ its maximal ideal, K its field of fractions, $K^\times$ the multiplicative group of non-zero elements of K. You know from the lecture that given the choice of a uniformizer π (that is a generator of $\mathfrak{m}$), any element $x \neq 0$ of A can be uniquely written as $x = u\,\pi^n$, with u invertible and $n \in \mathbb{N}$.

(a) Show that if $x = a/b$ is any element of $K^\times$, one can again write x in the form $u\,\pi^n$, with $n \in \mathbb{Z}$ this time. Set $v(x) = n$. Show v does not depend on the choice of the uniformizer π. Show:
   (i) The map $v : K^\times \longrightarrow \mathbb{Z}$ is a surjective homomorphism.
   (ii) One has $v(x + y) \geq \min\big(v(x), v(y)\big)$.
   The map v is called the *valuation associated to A*.

(b) Conversely, the knowledge of the function v determines the local ring $(A, \mathfrak{m})$: Let K be a field, and let $v : K^\times \longrightarrow \mathbb{Z}$ be a map satisfying the properties (i) and (ii) above. By definition, we set $v(0) := +\infty$. Show that the set $A := \{x \in K \mid v(x) \geq 0\}$ is a discrete valuation ring with maximal ideal $\mathfrak{m} = \{x \in K \mid v(x) > 0\}$, having v as its associated valuation.

(c) In the situation of (b), reprove that A is integrally closed using the valuation.

[1]If you want your solutions of this exercise sheet to be corrected, please hand them in just before the lecture on April 23rd. Questions or comments to `henrik.russell@math.fu-berlin.de` or come to office A3, 112.

Exercise 1.3 (4 Points)**.** Let k be a field.

(a) Show that the set
$$k((T)) = \left\{ f = \sum_{n\in\mathbb{Z}} a_n T^n \;\middle|\; \begin{array}{l} a_n \in k,\ a_n = 0 \quad \forall n < n_0 \\ \text{for some } n_0 \in \mathbb{Z} \end{array} \right\}$$
becomes a field in a natural way (in particular determine the ring structure), called the field of *formal Laurent series* over k.

(b) Show that the subset
$$k[[T]] = \left\{ f = \sum_{n\geq 0} a_n T^n \;\middle|\; a_n \in k \right\},$$
of $k((T))$ is a discrete valuation ring, called the ring of *formal power series* (with non-negative exponents) over k. Describe explicitly its maximal ideal, its residue field and its associated valuation.
(*Hint:* It is probably more elegant to determine the associated valuation first, prove that it satisfies the conditions (i) and (ii) from Exc. 1.2 (a) and then use Exc. 1.2 (b).)

Exercise 1.4 (5 Points)**.** Let A be a Dedekind domain with field of fractions $K = \mathrm{Frac}(A)$, let $L \mid K$ be a finite field extension and B the integral closure of A in L.

(a) Let $\mathfrak{p}$ be a prime ideal of A. Show that $\mathfrak{p}B \neq B$.
(*Hint:* You may use results from Commutative Algebra about integral ring extensions.)

(b) Let $\mathfrak{a}$ be an ideal of A. Show that $\mathfrak{a} = \mathfrak{a}B \cap A$.
(*Hint:* Via localization, reduce to the case that A is a discrete valuation ring, hence a principal ideal domain. Use that $B \cap K = A$.)

(c) Let $\mathfrak{a}$ and $\mathfrak{a}'$ be ideals of A. Show $\mathfrak{a} \supseteq \mathfrak{a}' \iff \mathfrak{a}B \supseteq \mathfrak{a}'B$.