April 10th, 2013

Zahlentheorie II

Exercise sheet 1

Exercise 1 (3 Points)**.** Let k be a field and $k[t]$ the ring of polynomials in one variable t with coefficients in k. We denote by $k(t) = \mathrm{Frac}(k[t])$ the field of fractions. For $f \in k[t] \setminus \{0\}$ we denote by $\deg(f) \in \mathbb{N}_0$ the degree of the polynomial f.

(1) For $x \in k(t)^\times$ with $x = \frac{a}{b}$, $a, b \in k[t] \setminus \{0\}$, set
$$v_\infty(x) := -\deg(a) + \deg(b) \in \mathbb{Z}.$$
Show that $v_\infty(x)$ is well-defined, i.e. independent of the presentation $x = a/b$.

(2) Show that
$$k(t)^\times \to \mathbb{Z}, \quad x \mapsto v_\infty(x)$$
is a normalized discrete valuation.

(3) Describe the DVR $\mathcal{O}_{v_\infty}$, its maximal ideal $\mathfrak{m}$, its units $\mathcal{O}_{v_\infty}^\times$, its residue field $\mathcal{O}_{v_\infty}/\mathfrak{m}$ and the local parameters of $\mathcal{O}_{v_\infty}$.

Exercise 2. (6 Points) Let k be a field. A formal power series with coefficients in k is a formal (infitie) sum
$$a_0 + a_1 t + a_2 t^2 + \dots a_n t^n + \dots, \quad a_i \in k.$$
(In particular any polynomial with coefficients in k is a formal power series.) We denote by $k[[t]]$ the set of formal power series. (As a set it is bijective to $k^{\mathbb{N}_0}$.) For two formal power series $\alpha = \sum_{n=0}^\infty a_n t^n$ and $\beta = \sum_{n=0}^\infty b_n t^n$ in $k[[t]]$ we define:
$$\alpha + \beta := \sum_{n=0}^\infty (a_n + b_n) t^n, \quad -\alpha := \sum_{n=0}^\infty (-a_n) t^n$$
and
$$\alpha \cdot \beta := \sum_{n=0}^\infty c_n t^n, \quad \text{with } c_n = \sum_{i+j=n} a_i b_j.$$

(1) Show that with the above operations $k[[t]]$ becomes a commutative ring with 1 such that the natural inclusion $k[t] \hookrightarrow k[[t]]$ is an injective homomorphism.

(2) Show that $\alpha = \sum_n a_n t^n$ is a unit in $k[[t]]$ if and only if $a_0 \in k^\times$. (*Hint:* If $a_0 \in k^\times$ we can multiply α with a_0^{-1}. Thus we may assume $\alpha = 1 - t \cdot \beta$ for some $\beta \in k[[t]]$. Then use the standard geometric series trick.)

(3) Show that any element $\alpha \in k[[t]]$ can uniquely be written in the form $\alpha = t^n u$ with $n \geq 0$ and $u \in k[[t]]^\times$. Conclude that $k[[t]]$ is an integral domain with field of fractions

$$k((t)) := k[[t]][\frac{1}{t}].$$

It is called the field of *formal Laurent series* with coefficients in k.

(4) Show that

$$v : k((t))^\times \to \mathbb{Z}, \quad t^n u \text{ (with } u \in k[[t]]^\times) \ \mapsto n$$

defines a normalized discrete valuation.

(5) Show that $\mathcal{O}_v = k[[t]]$, with maximal ideal $\mathfrak{m} = tk[[t]]$ and residue field $\mathcal{O}_v/\mathfrak{m} = k$.

(6) Show that the inclusion $k[t] \hookrightarrow k[[t]]$ extends to a homomorphism $k(t) \hookrightarrow k((t))$ and that $v_{|k(t)} = v_t$, where $v_t : k(t)^\times \to \mathbb{Z}$ is the discrete valuation associated to the polynomial t (see lecture).

Facts about the norm. Let L/K be a finite field extension. Then we can view L as a finite dimensional vector space over K and any element $x \in L$ thus defines a K-linear homomorphism $\mu_x : L \to L$, $y \mapsto xy$. We define the norm of x to be

$$\mathrm{Nm}_{L/K}(x) := \det(\mu_x) \in K.$$

We will prove later in the course that $\mathrm{Nm}_{L/K}$ has the following properties (some are easy)

(i) $\mathrm{Nm}_{L/K}(xy) = \mathrm{Nm}_{L/K}(x)\mathrm{Nm}_{L/K}(y)$.

(ii) $\mathrm{Nm}_{L/K}(x) = x^{[L:K]}$, for all $x \in K$.

(iii) If E is an intermediate field of L/K, then $\mathrm{Nm}_{L/K}(x) = \mathrm{Nm}_{E/K}(\mathrm{Nm}_{L/E}(x))$.

(iv) If $K = k((t))$ (see Exercise **??**) and B is the integral closure of $k[[t]]$ in L, then $\mathrm{Nm}_{L/K}(x) \in k[[t]] \iff x \in B$.

You can use these properties in the next exercise!

Exercise 3. (8 Points) Let $K = k((t))$ be the field of formal Laurent series with its normalized discrete valuation v from Exercise **??**, (4) and let $\bar{K}$ be a fixed algebraic closure of K.

(1) For $x \in \bar{K}$ pick a subextension L of $\bar{K}/K$ which is finite over K and contains x. Show that the value

$$\frac{1}{[L : K]} v(\mathrm{Nm}_{L/K}(x)) \in \mathbb{Q}$$

is independent of the choice of L. (*Hint:* Use that any two finite subextensions L and L' are contained in a finite subextension L'' and the properties (ii) and (iii) above.)

Define

$$w : \bar{K}^\times \to \mathbb{Q}, \quad x \mapsto \frac{1}{[L:K]} v(\mathrm{Nm}_{L/K}(x)),$$

with L as in (1).

(2) Show $w(xy) = w(x) + w(y)$.
(3) Pick $x \in \bar{K}$ and L/K finite with $x \in L$ and let B be the integral closure of $k[[t]]$ in L. Show that $w(x) \geq 0 \iff x \in B$. (*Hint:* Use (iv) above.)
(4) In the situation of (3) show that $w(x) \geq 0$ implies $w(x+1) \geq 0$.
(5) Conclude from (4) that $w(x+y) \geq \min\{w(x), w(y)\}$, for all $x, y \in \bar{K}^\times$. Thus w is a valuation.
(6) Show that $w_{|K} = v$, i.e. w extends v.
(7) Show that $w(\bar{K}) = \mathbb{Q}$, in particular w is not discrete. (*Hint:* Let $\alpha_n \in \bar{K}$ be a solution of $X^n - t$. Show that $w(\alpha_n) = \frac{1}{n}$.)
(8) Since w is not discrete we know from the lecture that $\mathcal{O}_w$ is not noetherian. Give an explicit ascending sequence of ideals $I^1 \subset I^2 \subset \ldots$, which does not terminate.