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Number Theory I (Commutative Algebra)

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Exercise sheet 10

As in the textbook, all rings are commutative with a unit.

If you submit solutions, please choose only a few exercises.

Exercise 1 ([AK2017, Exercise 24.4])**.** Let R be a domain, S a multiplicative subset.

(1) Assume $\dim(R) = 1$. Prove $\dim(S^{-1}R) = 1$ if and only if there is a nonzero prime $\mathfrak{p}$ of R with $\mathfrak{p} \cap S = \varnothing$.

(2) Assume $\dim(R) \geq 1$. Prove $\dim(R) = 1$ if and only if $\dim(R_\mathfrak{p}) = 1$ for every nonzero prime $\mathfrak{p}$ of R.

Exercise 2 ([AK2017, Exercise 24.5])**.** Let R be a Dedekind domain, S a multiplicative subset. Show that $S^{-1}R$ is a Dedekind domain if there is a nonzero prime $\mathfrak{p}$ of R with $\mathfrak{p} \cap S = \varnothing$, and that $S^{-1}R = \mathrm{Frac}(R)$ if not.

Exercise 3 ([AK2017, Exercise 25.2])**.** Let R be a domain, M and N nonzero fractional ideals. Prove that M is principal if and only if there exists some isomorphism $R \cong M$ of abstract R-modules. Construct the following canonical maps

$$M \otimes N \to MN, \qquad (M : N) \to \hom(N, M)$$

and show that the first one is surjective, and the second one is an isomorphism.

Exercise 4 ([AK2017, Exercise 25.6])**.** Let R be a domain, M and N fractional ideals. Prove that the canonical map $M \otimes N \to MN$ is an isomorphism if M is locally principal.

Exercise 5 ([AK2017, Exercise 25.9])**.** Let R be a domain, M and N fractional ideals. Prove this:

(1) Assume N is invertible. Then $(M : N) = M \cdot N^{-1}$.

(2) Both M and N are invertible if and only if their product MN is. If so, then $(MN)^{-1} = N^{-1}M^{-1}$.

Exercise 6 ([AK2017, Exercise 25.12])**.** Let R be a UFD. Show that a fractional ideal M is invertible if and only if M is principal and nonzero.

Exercise 7 ([AK2017, Exercise 25.15])**.** Show that a ring R is PID if and only if it a 1-dimensional UFD if and only if it is a UFD and a Dedekind domain.

Exercise 8 ([AK2017, Exercise 25.17])**.** Let R be a ring, M an invertible module. Prove that M is finitely generated, and that, if R is local, then M is free of rank 1.

Exercise 9 ([AK2017, Exercise 25.18])**.** Show these conditions on an R-module M are equivalent:

(1) M is invertible.
(2) M is finitely generated, and $M_\mathfrak{m} \cong R_\mathfrak{m}$ at each maximal ideal $\mathfrak{m}$.
(3) M is locally free of rank 1.

Assuming these conditions hold, show that the canonical R-module homomorphism $M \otimes \hom(M, R) \to R$ sending $m \otimes \phi$ to $\phi(m)$ is an isomorphism.

Exercise 10 (cf. [AK2017, Exercise 25.22])**.** Let R be a domain, and $\mathfrak{F}(R)$ the set of invertible fractional ideals, equipped with the canonical multiplication. Show that the sequence

$$1 \to R^* \to K^* \stackrel{\alpha}{\to} \mathfrak{F}(R) \to Pic(R) \to 1$$

is exact, where $Pic(R)$ is the group of invertible R-modules, $\alpha(x)$ is the ideal generated by $x \in K^*$, and $\mathfrak{F}(R) \to Pic(R)$ sends an invertible fractional ideal to itself, but considered as an abstract R-module (i.e., forgetting the embedding in K^*).