December 22, 2017

Number Theory I (Commutative Algebra)

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Exercise sheet 9

As in the textbook, all rings are commutative with a unit.

If you submit solutions, please choose only one or two exercises.

Exercise 1 ([AK2017, Exercise 15.24]). Let R be a domain of (finite) dimension r, and $\mathfrak{p}$ a nonzero prime. Prove that $\dim(R/\mathfrak{p}) < r$.

Exercise 2 ([AK2017, Exercise 15.25]). Given an integral extension of rings R'/R, show $\dim(R) = \dim(R')$.

Exercise 3 ([AK2017, Exercise 15.26]). Let R'/R be an integral extension of domains with R normal, $\mathfrak{m}$ a maximal ideal of R', and set $\mathfrak{n} = \mathfrak{m} \cap R$. Show $\mathfrak{n}$ is maximal, and $\dim(R'_{\mathfrak{m}}) = \dim(R_{\mathfrak{n}})$.

Exercise 4 ([AK2017, Exercise 15.27]).

(1) Given a product of rings $R = R' \times R''$, show

$$\dim(R) = \max\{\dim(R'), \dim(R'')\}$$

(2) Find a ring R with a maximal chain of primes $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$, but $r < \dim(R)$.

Exercise 5 ([AK2017, Exercise 15.28]). Let k be a field, R_1 and R_2 domains which are finitely generated k-algebras, and $\mathfrak{p}$ a minimal prime of $R_1 \otimes_k R_2$. Use Noether Normalisation and [AK2017, Exercise 14.20(3)] to prove this:

$$\dim((R_1 \otimes_k R_2)/\mathfrak{p}) = \dim(R_1) + \dim(R_2).$$

Exercise 6 ([AK2017, Exercise 15.31]). Let R be a ring, $R[X]$ the polynomial ring. Prove

$$1 + \dim(R) \le \dim(R[X]) \le 1 + 2\dim(R).$$