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Number Theory I (Commutative Algebra)

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Exercise sheet 8

As in the textbook, all rings are commutative with a unit.

If you submit solutions, please choose only **one** exercise.

Exercise 1. In the lecture, we saw that a ring is a DVR if and only if it is a local dimension one normal (resp. regular) noetherian ring.

(1) Give an example of a noetherian dimension 1 normal ring which is not local.
(2) Give an example of a local regular noetherian ring of dimension > 1, and one of dimension < 1.
(3) Give an example of a local dimension 1 noetherian ring which is not regular.
(4) Give an example of a local dimension 1 normal ring which is not noetherian (cf. Exercise 2).

Exercise 2. Recall that $\mathbb{C}[[t]] = \{\sum_{i\geq 0} a_i t^i \mid a_i \in \mathbb{C}\}$ is a discrete valuation ring.

(1) For $m \in \mathbb{N}$, show that $\mathbb{C}[[t]][s]/\langle s^m - t\rangle$ (in other words, the ring $\mathbb{C}[[t^{1/m}]] = \{\sum_{i\geq 0} a_i t^{i/m} \mid a_i \in \mathbb{C}\}$) is also a discrete valuation ring.
(2) If $v : \mathbb{C}((t))^\times \to \mathbb{Z}$ is the valuation of $\mathbb{C}[[t]]$, and $v' : \mathbb{C}((t^{1/m}))^\times \to \mathbb{Z}$ is the valuation of $\mathbb{C}[[t^{1/m}]]$, and $\iota : \mathbb{C}((t)) \to \mathbb{C}((t^{1/m}))$ is the canonical inclusion, show that $v' \circ \iota = mv$.
(3) Show that $A = \cup_{m>0}\mathbb{C}[[t^{1/m}]]$ is a valuation ring, but that it's not a discrete valuation ring (by showing that $(A - \{0\})/A^\times \not\cong \mathbb{N}$ as an abelian monoid for example).

Exercise 3. Let $p \in \mathbb{Z}$ be a prime. Recall that one can define the ring of p-adic integers $\mathbb{Z}_p$ as the ring of compatible sequences

$$\mathbb{Z}_p = \{(\dots, a_2, a_1, a_0) \in \dots \times \mathbb{Z}/p^3 \times \mathbb{Z}/p^2 \times \mathbb{Z}/p \mid a_{i+1} \equiv a_i \mod p^{i+1} \text{ for all } i\}$$

with addition and multiplication defined termwise.

(1) Describe the group of units $(\mathbb{Z}/p^n)^\times$ of $\mathbb{Z}/p^n$.
(2) Show that $(a_i) \in \mathbb{Z}_p$ is a unit if and only if $a_i \neq 0$ for all i if and only if $a_0 \neq 0$.
(3) Given $a = (\dots, a_2, a_1, a_0) \in \mathbb{Z}_p$, define $v_p(a) = \min\{i \mid a_i \neq 0\}$. Show that multiplication by $p = (\dots, p, p, p, 0) \in \mathbb{Z}_p$ induces a bijection

$$v_p^{-1}(n) \xrightarrow{\sim} v_p^{-1}(n+1); \qquad a \mapsto pa.$$

Since $\mathbb{Z}_p^\times = v_p^{-1}(0)$, deduce that every element of $\mathbb{Z}_p$ can be written uniquely as up^n for some $n \in \mathbb{N}$ and some $u \in \mathbb{Z}_p^\times$.
(4) Show that $\mathbb{Z}_p$ is a discrete valuation ring.
(5) Show that the canonical map $\mathbb{Z} \to \mathbb{Z}_p; a \mapsto (\dots, a, a, a)$, extends to a map $\mathbb{Z}_{(p)} \to \mathbb{Z}_p$ where $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \text{ such that } p \nmid b\}$.