November 29, 2017

Number Theory I (Commutative Algebra)

Hélène Esnault, Exercises: Shane Kelly

Exercise sheet 6

As in the textbook, all rings are commutative with a unit.

Exercise 1. *Nakayama's Lemma.*
Suppose that $A = [a_{ij}]_{1 \leq i,j \leq n}$ is a matrix with $a_{ij} \in I$ for all i, j for some ideal $I \subset R$ of a ring R. Notice that $\det(\mathrm{id} - A) = 1 + a$ for some $a \in I$. Recall also that for any matrix $B = [b_{ij}]_{1 \leq i,j \leq n}$ there is a matrix $C = [c_{ij}]_{1 \leq i,j \leq n}$ such that $CB = (\det B)\,\mathrm{id}$ (just choose c_{ij} to be $(-1)^{i+j}$ times the determinant of the matrix obtained by removing the jth row and ith column from B; the transpose of C is sometimes called the *cofactor matrix* of B).

(1) Let M be a finitely generated R-module M such that $IM = M$ for some ideal I. By choosing generators $m_1, \dots, m_n \in M$ and elements $a_{ij} \in I$ such that $m_i = \sum a_{ij} m_j$, find an $a \in I$ such that $(1 + a)M = 0$.

(2) Let R be a local ring with maximal ideal $\mathfrak{m}$. Show that $x \in R$ is invertible if and only if $x \notin \mathfrak{m}$. In particular, if $x \in \mathfrak{m}$, then $1 + x$ is a unit.

(3) Deduce the following:

Nakayama's Lemma. *If M is a finitely generated module over a local ring R with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}M = M$, then $M = 0$.*