November 11, 2017

Number Theory I (Commutative Algebra)

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Exercise sheet 4

As in the textbook, all rings are commutative with a unit.

Let R be a ring and M an R-module. Recall that the nth exterior power $\wedge^n M$ is the quotient of $\bigotimes^n M = \underbrace{M \otimes_R \cdots \otimes_R M}_{n \text{ copies}}$ by the submodule generated by expressions of the form $m_1 \otimes \cdots \otimes m_i \otimes m_i \otimes \cdots \otimes m_{n-1}$ for $i \in \{1, \dots, n-1\}$ and $m_1, \dots, m_{n-1} \in M$. The image of $m_1 \otimes \cdots \otimes m_n$ in $\bigwedge^n M$ is denoted $m_1 \wedge \cdots \wedge m_n$.

(1) By considering $(m+n) \otimes (m+n)$ in $M \otimes M$, show that $m \wedge n = -n \wedge m$ in $\wedge^2 M$. More generally, if $\sigma : \{1, \dots, n\} \xrightarrow{\sim} \{1, \dots, n\}$ is a permutation, then $m_1 \wedge \cdots \wedge m_n = (\operatorname{sgn} \sigma) m_{\sigma(1)} \wedge \cdots \wedge m_{\sigma(n)}$. (The sign $\operatorname{sgn}(\sigma)$ of a permutation σ is -1 if it can be written as a composition of an odd number of transpositions, and 1 if it can be written as a composition of an even number of transpositions; a transposition is a permutation which swaps exactly two elements).

(2) Let e_i be the standard basis elements of R^d. That is, more explicitly we have $e_i = (0, 0, \dots, 0, \underbrace{1}_{i\text{th position}}, 0, \dots, 0)$. So for every element m of R^d, there are unique $a_1, \dots, a_d \in R$ such that $m = a_1 e_1 + \cdots + a_d e_d$. Show that

$$\{e_{i_1} \otimes \cdots \otimes e_{i_n} : i_1, \dots, i_n \in \{1, \dots, d\}\}$$

is a free basis for $\bigotimes^n R^d$. Similarly, show that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_n} : i_1 < \cdots < i_n \in \{1, \dots, d\}\}$$

is a free basis for $\bigwedge^n R^d$.

In particular, if $n, d > 0$ note that $\bigwedge^n R^d \cong 0$ if and only if $n > d$.

(3) Show that if $M \cong N$, then $\bigwedge^n M \cong \bigwedge^n N$.

(4) Show that if $M \cong R^d$ and $M \cong R^{d'}$ then $d = d'$. Deduce that if M is a free R-module, any free basis has the same number of elements, so the rank is well-defined.

Hint for 2: consider multilinear maps to R.