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Number Theory I (Commutative Algebra)

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Exercise sheet 3

As in the textbook, all rings are commutative with a unit.

Exercise 1. Let R be a PID. Recall that every prime element is irreducible, and recall that in a PID, $a \neq 0 \in R$ is irreducible if and only if $\langle a \rangle$ is maximal. Show that if $\mathfrak{p}$ is a nonzero prime ideal, then $R/\mathfrak{p}$ is a field.

Recall that two elements $x, y \in R$ of a ring are said to be *coprime* if there exist $a, b \in R$ such that $ax + by = 1$ (equivalently, $\langle x, y \rangle = R$). Two elements $x, y \in R$ are said to be *relatively prime* if there is does not exist any prime z with $z|x$ and $z|y$ (equivalently, for any prime $\langle z \rangle$, either $x \notin \langle z \rangle$ or $y \notin \langle z \rangle$).

Exercise 2 ([AK2017, Theorem.2.20])**.** Let R be a PID. Let $P = R[X]$ be the polynomial ring in one variable X, and $\mathfrak{p}$ a nonzero prime ideal of P. Assuming that P is a UFD (because it is [AK2017, 2.5]), we will show that there are three classes of prime ideals of $R[X]$:

- The zero ideal (0).
- Principal ideals $\langle F \rangle$ with F a prime element of $R[X]$.
- Ideals of the form $\langle p, G \rangle$ where $p \in R$ is prime, $pR = \mathfrak{p} \cap R$, and $G \in R[X]$ is prime with image $G' \in (R/pR)[X]$ also prime.

and that all nonprincipal prime ideals are maximal.

(1) [AK2017, Exercise 2.18] Show that in a PID, nonzero elements x and y are *relatively prime* if and only if they're coprime.

(2) Let $K = Frac(R)$ be the fraction field of R. Recall that Gauss's Lemma says that "if $F \in R[X]$ is prime in $R[X]$, then it is also prime in $K[X]$". Show that given prime elements $F_1, F_2 \in R[X]$ with $F_2 \notin \langle F_1 \rangle$, the elements F_1, F_2 are coprime in $K[X]$.

(3) Given a nonprincipal prime ideal $\mathfrak{p} \subset R[X]$, use the previous two parts to find a nonzero element $c \in \mathfrak{p} \cap R$. Conclude that under the canonical map $\pi : Spec(R[X]) \to Spec(R), \mathfrak{q} \mapsto \mathfrak{q} \cap R$ from the poset of prime ideals of $R[X]$ to the poset of prime ideals of R, our nonprincipal prime $\mathfrak{p}$ is sent to a maximal ideal, say pR of R.

(4) Define $k = R/pR$. Note that the primes of $R[X]$ sent to pR by $\pi : Spec(R[X]) \to Spec(R)$, are precisely those in the image of $\iota : Spec(k[X]) \to Spec(R[X]); \mathfrak{q} \mapsto \phi^{-1}\mathfrak{q}$ where $\phi : R[X] \to k[X]$ is the canonical map. Define $\mathfrak{q} = \mathfrak{p}/pR \subset k[X]$. Show that $\mathfrak{q}$ is a maximal ideal of $k[X]$, that $\mathfrak{p} = \phi^{-1}\mathfrak{q}$, and deduce that $\mathfrak{p}$ is therefore a maximal ideal of $R[X]$.

(5) Using the fact that $k[X]$ is a PID, $\mathfrak{q}$ is prime, and $R[X]$ is a UFD, find a prime $G \in R[X]$, whose image $G' \in k[X]$ is also prime, and such that $\mathfrak{p} = \langle p, G \rangle$.

Question. What are the prime ideals of $\mathbb{C}[x, y]$? What about $\mathbb{Z}[x]$?

Question. Does there exist an example of a ring such that every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \dots$ stabilises, but such that there exists a noncountable ascending chain which does not stabilise, i.e., there is a totally ordered set S and a set of ideals $\{I_i : i \in S\}$ with $I_i \subsetneq I_j$ for every $i \lneq j$ in S.