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Number Theory I (Commutative Algebra)

Hélène Esnault, Exercises: Shane Kelly

Exercise sheet 2

As in the textbook, all rings are commutative with a unit. Recall that an element $a \in R$ of a ring is *irreducible* if it is not a unit and $a = bc$ implies that b or c is a unit. Recall that an element $a \in R$ is called *prime* if whenever $ad = bc$ for some $d \in R$, we have $ad' = b$ or $ad' = c$ for some $d' \in R$. (Equivalently, $a \in R$ is a prime element if and only if $\langle a \rangle$ is a prime ideal.)

Exercise 1 ([AK 2017, Paragraph 2.17)**.** Let R be a ring. We will show that if R is a principal ideal domain (PID), then it is a unique factorisation domain (UFD).

(1) Suppose R is a domain. Show that $\langle a \rangle = \langle b \rangle$ if and only if $a = bc$ for some unit c.

(2) Suppose R is a domain. Suppose every ascending chain of principal ideals $\langle x_1 \rangle \subset \langle x_2 \rangle \subset \dots$ of R stabilises, that is, there exists $N \in \mathbb{N}$ such that $\langle x_i \rangle = \langle x_{i+1} \rangle$ for all $i \geq N$.

   (a) Show that every nonzero nonunit a factors into a product $a = bc$ such that b is irreducible.

   (b) Deduce that every nonzero nonunit a factors into a product $a = b_1 b_2 \dots b_n$ such that each b_i is irreducible.

(3) Suppose that R is a PID. Show that every ascending chain of ideals stabilises.

(4) Suppose R is a PID. Show a nonzero element $0 \neq a \in R$ is irreducible if and only if $\langle a \rangle$ is maximal.

(5') Show that every prime element is irreducible.

(5) Suppose R is a PID. Show that every irreducible element is prime. (Recall that every maximal ideal is a prime ideal).

(6) Suppose R is a PID, and suppose that we have irreducible elements $p_1, \dots, p_r, q_1, \dots, q_s \in R$ (with $r \leq s$) such that $p_1 \dots p_r = q_1 \dots q_s$. Use part (5) to find units $u_1, \dots, u_r$ such that $q_i = u_i p_i$ for each $1 \leq i \leq r$ (possibly relabelling the q_j), and show that $r = s$.

(7) Deduce from the previous parts that every PID is a UFD.

Remark 1. Note that above we have shown that in a PID R, a nonzero element a is prime iff $\langle a \rangle$ is prime iff $\langle a \rangle$ is maximal iff a is irreducible. So we have classified the poset of prime ideals $Spec(R)$: There is (0), contained in every other prime ideal, and the set of nonzero prime ideals is in bijection with the set of irreducible (=prime) elements modulo the action of the unit group R^*, with no inclusion relations between the nonzero prime ideals.

Remark 2. Originally I forget "not a unit" in the definition of irreducible. I forgot to assume R is a domain in 1(2). I wrote $r < s$ instead of $r \leq s$, and I forgot to also ask that $r = s$ is proved. Part (5') was added. Exercises 2 and 3 were moved to the next exercise sheet.