Exercise sheet 8

To hand in: Wednesday 19.12.18 in the homework box of Simon Pepin Lehalleur, outside the Hörsaal 001 in Arnimallee 3.

Let K be a complete discrete valued field with valuation v_K, ring of integers $\mathcal{O}_K$ and residue field k. Recall that $\mathcal{O}_K$ is by assumption a discrete valuation ring, with ideal $\mathfrak{m}_K$. A generator of $\mathfrak{m}_K$ will be called a uniformizer of K. We also have, for every $\alpha > 1$ real number, an associated non-archimedean norm $|-|_K : K \to \mathbb{R}_{\geq 0}, x \mapsto \alpha^{-v_K(x)}$.

Recall that, if L/K is a finite extension, then the field L_K admits a unique non-archimedean norm $|-|_L$ extending $|-|_K$, which is also discrete and thus makes L into a complete discrete valued field with valuation v_L, ...

Note that the valuation v_L does NOT in general extend v_K; rather, there is an integer $e = e(L/K)$, the ramification index of L/K, such that

$$(v_L)_{|K} = e \cdot v_K.$$

Concretely, that means that if π_K is a uniformizer of K and π_L is a uniformizer of L, we have $\frac{\pi_K}{\pi_L^e} \in \mathcal{O}_L^\times$.

We say that the extension L/K is unramified if it is separable and $e = 1$. This last condition is equivalent to π_K being a uniformizer of L.

There is also an induced extension of residue fields l/k with degree $f = f(L/K)$. We say that L/K is totally ramified if $l = k$. The fundamental basic fact is that

$$e(L/K)f(L/K) = [L : K]$$

One can then show that L/K is unramified if and only if l/k is separable[1] and $[L : K] = [l : k]$. Also, we see that L/K is totally ramified if and only if $e(L/K) = [L : K]$.

1 Let K be a complete discrete valued field and K^s be a fixed separable closure.

(a) Let $K' \subset K^s$ and $L \subset K^s$ be finite extensions of K. Show that, if L/K is unramified, then LK'/K' is unramified (Hint: use the same method as in Exercise **??**).

(b) Show that a subextension of an unramified extension is unramified (Hint: question (1a), together with the fact that separability is transitive and field extension degrees are multiplicative in towers).

(c) Let $K', K'' \subset K^s$ be unramified extensions. Prove that the composite extension $K'K''/K$ is unramified (same hint as the previous question).

[1] For local fields, the residue fields will be finite fields, thus perfect, and this separability condition is automatic. Also most of the time we consider Galois extensions anyway, so that separability of L/K is assumed from the start.

(d) We say that an algebraic, not necessarily finite, extension L/K is unramified if it can be written as an increasing union of finite unramified extensions of K. Show that L/K is unramified if and only if every finite subextension of L is unramified.

(e) Deduce that, for any algebraic extension L/K, the maximal unramified extension of K in L is well-defined.

(f) Let k be a field. The field $k((t))$, equipped with the t-adic valuation, is a complete discrete valued field. Show that any finite unramified extension K of $k((t))$ is of the form $k'((t))$ for k'/k finite separable extension. (Hint: compare K, $k'((t))$, and the composite $Kk'((t))$ inside a fixed separable closure of K).

(g) Let K be a p-adic field with residue field k. Let K' be a finite unramified extension of K with residue field k'. Show that one has $K' = K \otimes_{W(k)} W(k')$.

2 Let K be a local field with residue field k, and K^s be a separable closure of K. Write K^{ur} for the maximal unramified extension of K in K^s. As with every algebraic extension of K, there is a unique extension of the norm on K to K^s and to K^{ur}. Let $\{K_i\}_{i\in\mathbb{N}^\times}$ be an infinite tower of distinct finite extensions of K in K^{ur}, that is $K_i \subsetneq K_{i+1}$. For instance, K_i could be the unique unramified extension of K of degree i, but this is not necessary. Show that $\mathcal{O}_{K_i} \neq \mathcal{O}_{K_{i+1}}$. Fix $\theta_i \in \mathcal{O}_{K_i} \setminus \mathcal{O}_{K_{i+1}}$ for each $i \in \mathbb{N}$ and form the series

$$\alpha_n = \sum_{i=1}^{n} \theta_i \pi^i.$$

Prove that $(\alpha_n)_{n\geq 1}$ is a Cauchy sequence for the norm on K^{ur} but does not converge. This means that the normed field K^{ur} is not complete. Prove that the completion $\widehat{K^{\mathrm{ur}}}$ has same residue field as K^{ur} but is not algebraic, hence it is not unramified; so there is no "complete maximal unramified extension" of K. On the other hand, one can show that $\widehat{K^s}$ is still separably closed, so there is (at least) one "complete separably closed extension" of K.

3 Let K be a complete discrete valued field and L/K a separable totally ramified field extension of degree n. Recall that any uniformizer π of L generates L/K and its minimal polynomial over F is an Eisenstein polynomial of degree n. Prove that this Eisenstein polynomial is in fact $N_{L/K}(X - \pi)$.