Freie Universität Berlin **Wintersemester 2018**
Zahlentheorie III **Prof. Esnault**

Exercise sheet 7

To hand in: Wednesday 12.12.18 in the homework box of Simon Pepin Lehalleur, outside the Hörsaal 001 in Arnimallee 3.

1 (Herbrand quotients) Let G be a finite cyclic group and M be a G-module. Assume that $H^0_T(G,M)$ and $H^1_T(G,M)$ are both finite groups; this occurs for instance if M is finite (but also in some cases where M is infinite) and implies by periodicity that $H^i_T(G,M)$ is finite for all $i \in \mathbb{Z}$. We then define the *Herbrand quotient*

$$h(M) = \frac{|H^0_T(G,M)|}{|H^1_T(G,M)|}.$$

(a) We admit that cup-products extend to Tate cohomology for G-modules in the following sense; for any M, N G-modules and $i, j \in \mathbb{Z}$, there are bilinear morphisms

$$\cup : H^i_T(G,M) \times H^j_T(G,N) \to H^{i+j}_T(G, M \otimes N)$$

which satisfy the same properties as in Sheet 6 Exercise 1. The key to this extension is that Tate cohomology can be computed using a *complete resolution* of $\mathbb{Z}$, that is, a kind of projective resolution in both directions; you can read this in Brown's book "Cohomology of groups", Chap VI.3-5. Let $\gamma \in H^2_T(G,\mathbb{Z})$ be the class defined in Sheet 6 Exercise 2. Prove that for all $i \in \mathbb{Z}$, the cup-product map

$$\gamma \cup - : H^i_T(G,M) \to H^{i+2}_T(G,M), x \mapsto \gamma \cup x$$

is an isomorphism. (Hint: starting from Sheet 6 Exercise 2.(f), use dimension-shifting and the formula of Sheet 6 Exercise 1.(c).)

(b) Show that for all $i \in \mathbb{Z}$,

$$h(M) = \left(\frac{|H^i_T(G,M)|}{|H^{i+1}_T(G,M)|} \right)^{(-1)^i}.$$

(c) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of G-modules. Assume that the Herbrand quotient is defined for two of the three modules. Then show that it is defined for the third and

$$h(M) = h(M')h(M'').$$

(Hint: for this and the next two questions, you can look at Milne p.79-80.)

(d) Suppose that M is finite. Prove that $h(M) = 1$.

(e) Let $f : M \to N$ be an homomorphism of G-modules with finite kernel and cokernel. Show that $h(M)$ is defined iff $h(N)$ is, and if it is the case then $h(M) = h(N)$.

(f) Let $M = \mathbb{Z}$ with trivial G-action. Show that $h(\mathbb{Z})$ is well defined and equals $|G|$.

(g) Let k be a finite field of characteristic p and l/k a finite extension. Recall that the Galois group $\mathrm{Gal}(l/k)$ is a cyclic group (generated by a certain power of the Frobenius automorphism $x \mapsto x^p$) and that it acts on $l^\times$. Prove that $H^*_T(\mathrm{Gal}(l/k), l^\times) = 0$. (Hint: combine Hilbert's Theorem 90 with the previous results).

2 (Group (co)homology as a part of algebraic topology) This is an introduction to a different, topological, point of view on group homology and group cohomology. You should read and try to understand it. Note that it requires some prerequisites from algebraic topology: fundamental groups and homotopy groups, covering spaces, singular chain complexes, singular cohomology, CW-complexes. This topological approach is not necessary for the applications of group cohomology to Galois cohomology and number theory, but is very important in algebraic topology and for the computation of group cohomology of finite groups.

(a) Let G be a group and X a topological space equipped with an action of G. Assume that the action of G on X is continuous, i.e., that for all $g \in X$, the map $X \to X, x \mapsto g \cdot x$ is continuous (or equivalently, and more conceptually, the action map $G \times X \to X$ is continuous if we equip G with the discrete topology). Assume furthermore that the action is free and proper, i.e., that every $x \in X$ has an open neighbourhood $x \in U \subset X$ such that, for all $g \neq h \in H$, $gU \cap hU = \emptyset$. Equip X/G with the quotient topology. Prove that the quotient map $X \to X/G$ is continuous, and is a covering space (in the sense .

(b) Let X be a connected, locally simply connected space (e.g a connected CW-complex), and $x \in X$. Let $p : \tilde{X} \to X$ be the universal covering of X based at x, which exists and is simply connected under these assumptions. Prove that the group $\pi_1(X, x)$ has a natural action on $\tilde{X}$ which is continuous, free and proper, that π is invariant with respect to this action, and that $\tilde{X}/\pi_1(X, x) \simeq X$ via p.

(c) Let X be a space equipped with a continuous, free and proper action of G. Prove that the singular chain complex $S_*(X)$ has a natural structure of chain complex of free $\mathbb{Z}[G]$-modules, and that the chain complex $S_*(X)_G$ of coinvariants is isomorphic to the singular chain complex of the quotient X/G.

(d) A classifying space of the group G is a pointed connected CW-complex $(BG, \star)$ such that

$$\forall i \geq 1,\ \pi_i(BG, \star) = \begin{cases} G, & i = 1 \\ 0, & i \geq 2 \end{cases}.$$

Equivalently, $\pi_1(BG, \star) = G$ and the universal covering of BG is contractible (use this definition if you are not familiar with higher homotopy groups $\pi_i(-)$). Construct a pointed CW-complex $(EG, \star)$ as follows: the n-cells of $(EG, \star)$ are in bijection with the sets $\{(g_0, \ldots, g_n) | g_i \in G\}$, and the attaching maps are obtained by gluing the cell $(g_0, \ldots, g_n)$ along the $n+1$ $(n-1)$-cells $(g_0, \ldots, \hat{g}_i, \ldots g_n)$ for $0 \leq i \leq n$. The complex $(EG, \star)$ has a G-action where $g \in G$ identifies the cells $(g_0, \ldots, g_n)$ and $(gg_0, \ldots, gg_n)$. Show that this action is free and proper. Show that EG is contractible. Show that the quotient space EG/G is a classifying space BG.

(e) Any two classifying spaces of a group G are homotopy equivalent, so that it makes sense to talk about *the* homotopy type BG. This is not so easy, see Hatcher's Algebraic Topology Theorem 1B.8, Proposition 1B.9.

(f) Show that the singular chain complex $S_*(EG)$, as a complex of $\mathbb{Z}[G]$-modules, is isomorphic to the standard free P_* resolution of the trivial module $\mathbb{Z}$ introduced in class

(Milne p.61). Deduce that

$$H_*(G,\mathbb{Z}) \simeq H_*(BG,\mathbb{Z}) \text{ and } H^*(G,\mathbb{Z}) \simeq H^*(BG,\mathbb{Z})$$

where for a topological space X the group $H_*(X,\mathbb{Z})$ (resp. $H^*(X,\mathbb{Z})$ denotes singular homology (resp. cohomology) with integer coefficients.

There is a similar formula for more general G-modules, which goes as follows: one can associate to a G-module M a local coefficient system $\tilde{M}$ on BG, and one has

$$H_*(G,M) \simeq H_*(BG,\tilde{M}) \text{ and } H^*(G,M) \simeq H^*(BG,\tilde{M}).$$

See Hatcher Appendix 3H for a discussion of local coefficients.

(g) Show that the spaces S^1 and $\mathbb{C}^\times$ are both classifying spaces for $G = \mathbb{Z}$.

(h) Show that the infinite real projective space $\mathbb{RP}^\infty$ (obtained as the colimit of inclusions $\ldots \hookrightarrow \mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1} \hookrightarrow \ldots$) is a classifying space for $G = \mathbb{Z}/2\mathbb{Z}$. (Hint: prove that the infinite sphere $\mathbb{S}^\infty$, obtained as the colimit of the standard inclusions of spheres S^n, is contractible)

(i) Give a purely topological proof of the fact that $H_1(G,\mathbb{Z}) \simeq G/[G,G]$ (Hint: for any connected pointed topological space (X,x), one has $\pi_1(X,x)^{\text{ab}} \simeq H_1(X)$ via the Hurewicz map).

(j) Let G be a group which has torsion. We will prove that any CW-complex model of the homotopy type BG has to be infinite-dimensional (i.e., has cells of arbitrary large dimension). Assume by contradiction that there is a finite-dimensional BG. Since G has torsion, it admits a finite cyclic subgroup H. Prove that the covering space Y of BG corresponding to $H \leq \pi_1(BG,\star)$ is a finite dimensional CW-complex which is a BH. Using the fact that $H^*(H,\mathbb{Z})$ is non-zero in infinitely many degrees, obtain a contradiction. For finite groups, a much stronger fact is true: there exists an integer $n \geq 1$ such that for all $i \geq n$, we have $H^i(G,\mathbb{Z}) \neq 0$. See

`https://mathoverflow.net/questions/64688/`

(k) Try to find topological analogues and proofs of as many constructions and facts about group homology and cohomology as possible!