Freie Universität Berlin **Wintersemester 2018**
Zahlentheorie III **Prof. Esnault**

Exercise sheet 6

To hand in: Wednesday 28.11.18 in the homework box of Simon Pepin Lehalleur, outside the Hörsaal 001 in Arnimallee 3.

1 (Properties of cup products) Let G be a group, M, N, P be G-modules, $i, j, k \geq 0$. In Sheet 5 Exercise 1 we defined the cup-product homomorphisms

$$\cup : H^i(G, M) \otimes H^j(G, N) \to H^{i+j}(G, M \otimes N).$$

We prove some of their basic properties and compute some examples. See Milne Proposition 1.38-1.39 for more. For each of the following questions, it is sometimes possible to use either the conceptual definition (Sheet 5 Exercise 1.(d)) or the explicit formula (Sheet 5 Exercise 1.(f)); try to alternate.

(a) Prove that for $i = j = 0$, the cup product corresponds to the natural map of abelian groups

$$M^G \otimes N^G \to (M \otimes N)^G,\ m \otimes n \mapsto m \otimes n.$$

(b) Let $x \in H^i(G, M)$, $y \in H^j(G, N)$, $z \in H^k(G, P)$. Prove that

$$(x \cup y) \cup z = x \cup (y \cup z) \text{ in } H^{i+j+k}(G, M \otimes N \otimes P)$$

and that

$$x \cup y = (-1)^{ij} y \cup x \text{ in } H^{i+j}(G, M \otimes N)$$

Deduce that, if R is a commutative ring, seen as a G-module with trivial action, then $H^*(G, R)$ has a natural structure of $\mathbb{N}$-graded R-algebra which is graded-commutative, i.e., that satisfies $xy = (-1)^{ij} yx$ for $x \in H^i(G, R)$ and $y \in H^j(G, R)$.

(c) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of G-modules. Assume that the sequence

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is exact (e.g. if the underlying abelian group of N is torsion-free). Let $x \in H^i(G, M'')$ and $y \in H^j(G, N)$. Prove the following relation between cup products and connecting homomorphisms

$$\delta(x) \cup y = \delta(x \cup y) \in H^{i+j+1}(G, M' \otimes N).$$

and

$$y \cup \delta(x) = (-1)^j \delta(y \cup x) \in H^{i+j+1}(G, N \otimes M')$$

(d) Let G be a finite group and $H \subset G$ a subgroup. Let Q be an H-module. Let $x \in H^i(G, M)$ and $y \in H^j(H, Q)$. Prove that

$$\mathrm{Cor}(\mathrm{Res}(x) \cup y) = x \cup \mathrm{Cor}(y) \in H^{i+j+1}(G, M \otimes Q).$$

2 (Cup products and cohomology of finite cyclic groups) Let G be a cyclic group of order $n \geq 1$. Let M be a G-module. We have seen in class that for all $i > 0$, we have $H^i(G, M) \simeq H^{i+2}(G, M)$. The goal is to give another construction of these isomorphisms in terms of cup products. Let σ be a generator of G.

(a) Prove that the trivial G-module $\mathbb{Z}$ has a free resolution $P_\bullet \to \mathbb{Z} \to 0$ given by

$$\ldots \to \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where N is the norm map (multiplication by the norm element $\sum_{g \in G} g$). Note that this is different from the standard resolution (constructed for all G) and specific to finite cyclic groups.

(b) Notice that the periodicity of the complex $P_\bullet$ results in a map of complexes $\tau : P_\bullet \to P_\bullet[2]$ which is the identity in homological degrees ≥ 2. Deduce for $i > 0$ an isomorphism $\tau^* : H^i(G, M) \simeq H^{i+2}(G, M)$. This is the isomorphism which we want to explain in terms of cup products, and generalise to all $i \in \mathbb{Z}$ with Tate cohomology.

(c) Prove that $H^2(G, \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$, with $1 + n\mathbb{Z}$ corresponding to a generator γ represented in terms of the resolution $P_\bullet$ by the cocycle $\gamma : P_2 = \mathbb{Z}[G] \to \mathbb{Z}, \sigma \mapsto 1$.

(d) Let $\Delta : P_\bullet \to P_\bullet \otimes P_\bullet$ be the map whose (p, q)-component is defined as the map of $\mathbb{Z}[G]$-modules induced by

$$\Delta_{p,q} : P_{p+q} \to P_p \otimes P_q, 1 \mapsto \begin{cases} 1 \otimes 1, \ p \text{ even} \\ 1 \otimes \sigma, \ p \text{ odd}, \ q \text{ even} \\ \sum_{0 \leq i < j \leq n} \sigma^i \otimes \sigma^j, \ p, q \text{ odd} \end{cases}$$

Check that Δ is a map of resolutions of $\mathbb{Z}$, and hence, as in Sheet 5 Exercise 1.(e), a chain homotopy equivalence of free resolutions.

(e) Show that the cocycle γ induces a morphism of complexes $\gamma : P_\bullet \to \mathbb{Z}[2]$. Show that the composite

$$P_\bullet \xrightarrow{\Delta} P_\bullet \otimes P_\bullet \xrightarrow{\gamma \otimes \mathrm{id}} P_\bullet[2]$$

is equal to τ.

(f) Deduce from the previous considerations and the definition of Sheet 5 Exercise 1.(d) of the cup product in terms of tensor products of resolutions, that the cup-product map

$$\gamma \cup - : H^i(G, M) \to H^{i+2}(G, M), x \mapsto \gamma \cup x$$

is equal to τ^*, hence an isomorphism for $i > 0$.

(g) Prove that $\gamma = \delta(\chi)$, with $\delta : H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z})$ the coboundary map associated to the standard exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and $\chi : G \to \mathbb{Q}/\mathbb{Z}, \sigma \mapsto \frac{1}{n}$.

(h) What is the problem in the following argument? Let M be a finite G-module and $x \in H^i(G, M)$ for some $i > 0$. Then by Exercise 1.(c), we have $\gamma \cup x = \delta(\chi \cup x)$. But $\chi \cup x \in H^{i+1}(G, M \otimes \mathbb{Q}/\mathbb{Z})$ and $M \otimes \mathbb{Q}/\mathbb{Z} \simeq 0$ because M is finite; since $x \mapsto \gamma \cup x$ is an isomorphism, we conclude that $H^i(G, M) = 0$ for all $i > 0$ and all finite G-modules M - which is completely false!

3 (Homology and cohomology of free groups) Let G be a group. Consider the augmentation ideal $I_G = \mathrm{Ker}(\mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z})$ as G-module. Let S be a set. We denote by $F(S)$ the free group generated by S.

(a) Show that I_G is a free $\mathbb{Z}$-module with basis $\{g-1|g \in G, g \neq 1\}$.

(b) Show that $I_{F(S)}$ is a free $\mathbb{Z}[F(S)]$-module with basis $\{s-1|s \in S\}$. (Hint: This is equivalent to showing that $\{g(s-1)|g \in F(S), s \in S\}$ is also a $\mathbb{Z}$-basis of $I_{F(S)}$. Write $F(S) \setminus \{1\}$ as disjoint union of the sets of reduced words with a given s or s^{-1} ending, and do an induction on word length).

(c) Show that the trivial $F(S)$-module $\mathbb{Z}$ has a free resolution

$$0 \to I_{F(S)} \to \mathbb{Z}[F(S)] \to \mathbb{Z} \to 0$$

and use it to compute $H_*(F(S), \mathbb{Z})$ and $H^*(F(S), \mathbb{Z})$.