Exercise sheet 5

To hand in: Wednesday 21.11.18 in the homework box of Simon Pepin Lehalleur, outside the Hörsaal 001 in Arnimallee 3.

1 (Cup products on group cohomology) Let G be a group and M, N be G-modules. To simplify the notation, we write $\otimes = \otimes_{\mathbb{Z}}$.

(a) Recall that $M \otimes N$ has a natural structure of G-module. Prove that, if M and N are projective G-modules, then $M \otimes N$ is also projective. (Hint: use the relationship between this tensor product and the tensor product of $\mathbb{Z}[G]$-modules and the fact that projective modules are direct summands of free modules).

(b) More generally, let $P_\bullet \to M \to 0$ and $Q_\bullet \to N \to 0$ be two projective resolutions of M and N. The *tensor product complex* $P_\bullet \otimes Q_\bullet$ is the double complex $P_\bullet \otimes Q_\bullet$ of G-modules (in homological convention) with horizontal differential $d_P \otimes 1$ and vertical differential $(-1)^p \cdot 1 \otimes d_Q$ at $P_p \otimes Q_q$ (check that it does form a double complex). Using the acyclic assembly lemma from Sheet 4 Exercise 1.(d), show that the total complex $\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$ is a projective resolution of $M \otimes N$.

(c) Let $A^\bullet, B^\bullet$ be complexes of R-modules. Check that the formula $(a, b) \mapsto a \otimes b$ induces a map $H^a(A^\bullet) \otimes H^b(B^\bullet) \to H^{a+b}(\mathrm{Tot}(A^\bullet \otimes B^\bullet))$.

(d) This time, let $P_\bullet$, $Q_\bullet$ be projective resolutions of the trivial G-module $\mathbb{Z}$. We define the *cup-product* as the map of chain complexes

$$\cup : \mathrm{Tot}(\mathrm{Hom}_G(P_\bullet, M) \otimes \mathrm{Hom}_G(Q_\bullet, N)) \to \mathrm{Tot}\,\mathrm{Hom}_G(P_\bullet \otimes Q_\bullet, M \otimes N)$$
$$u \otimes v \mapsto ((x \otimes y \mapsto u(x) \otimes v(y)).$$

Check that this indeed defines a map of chain complexes, and in particular it passes to cohomology. Using the two previous questions, show that we obtain a cup product map on group cohomology: for all $m, n \geq 0$, we have

$$\cup : H^m(G, M) \otimes H^n(G, N) \to H^{m+n}(G, M \otimes N).$$

(e) We want explicit formulas for this construction. Let $F_\bullet \to \mathbb{Z} \to 0$ be the standard free resolution of the trivial G-module $\mathbb{Z}$ which leads to the description in terms of inhomogeneous cocycles. Concretely, F_r is a free $\mathbb{Z}[G]$ module with basis $(g_1, g_2, \ldots, g_r)$ for $g_1, \ldots g_r \in G$, and the differential of the complex is given by the formula

$$d_i(g_1, \ldots, g_r) = \begin{cases} g_1(g_2, \ldots, g_r), \ i = 0 \\ (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_r), \ 0 < i < r \\ (g_1, \ldots, g_{r-1}), \ i = r \end{cases}$$

Given the above, we need to understand the tensor product resolution $F_* \otimes F_*$. Prove that the *Alexander-Whitney map* (or *shuffle*) map

$$\Delta : F_* \to F_* \otimes F_*, (g_1, \ldots, g_r) \mapsto \sum_{p=0}^{r} (g_1, \ldots, g_p) \otimes g_1 \ldots g_p(g_{p+1}, \ldots, g_r)$$

is a well-defined morphism of projective resolutions of the trivial G-module $\mathbb{Z}$. Deduce without any further computation that Δ is a quasi-isomorphism (and even a chain homotopy equivalence).

(f) Using the Alexander-Whitney map, deduce the the following concrete formula for the cup product. Let $c \in C^m(G, M)$, $d \in C^n(G, N)$. Then $[c] \cup [d]$ is represented by the inhomogeneous cocycle

$$c \cup d : G^{m+n} \to M \otimes N, (g_1, \dots, g_{m+n}) \mapsto c(g_1, \dots, g_p) \otimes g_1 \dots g_m d(g_{m+1}, \dots, g_{m+n}).$$

2 (More about low-dimensional cohomology) We collect some simple results on group cohomology around the interpretation of $H^1(G, M)$ with crossed homomorphisms and $H^2(G, M)$ with extensions. Let G be a group and M be a G-module. We write $\mathrm{XHom}(G, M)$ for the crossed homomorphisms from G to M and $\mathcal{E}(G, M)$ for the set of isomorphism classes of extensions of G by M.

(a) Let $f : H \to G$ be a group homomorphism. Check that the construction given in class of restriction homomorphisms

$$\mathrm{Res} : H^*(G, M) \to H^*(H, M)$$

in the case where H is a subgroup and f is the inclusion extends directly to the case where f is a general homomorphism.

(b) Let $\phi \in \mathrm{XHom}(G, M)$. Check that $\phi \circ f \in \mathrm{XHom}(H, N)$, and that if ϕ is principal then $\phi \circ f$ is principal. Show that, if we describe $H^1(G, M)$ in terms of crossed homomorphisms, then the map induced by $\phi \mapsto \phi \circ$ coincides with the restriction map Res.

(c) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of G-modules. Let $\phi : G \to M''$ be a crossed homomorphism. Show there is a extension E of G by M' which fits in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow \phi & & \\ 0 & \longrightarrow & M' & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Show that E is unique up to isomorphism of extensions. This defines a map $\mathrm{XHom}(G, M'') \to \mathcal{E}(G, M)$.

(d) Prove that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{XHom}(G, M'') & \longrightarrow & \mathcal{E}(G, M') \\ \downarrow & & \downarrow \sim \\ H^1(G, M'') & \xrightarrow{\delta} & H^2(G, M') \end{array}$$

where the top horizontal map is defined in (2c), the horizontal maps are given by the results of Sheet 3 Exercise 1 and Sheet 4 Exercise 3, and the bottom vertical map is the boundary morphism in group cohomology coming from the exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

(e) Let G be a finite group. For any homomorphism $G \to \mathbb{Q}/\mathbb{Z}$, we can construct a central extension of G by $\mathbb{Z}$ by pulling back the canonical extension:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

This defines a map $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \to \mathcal{E}(G, \mathbb{Z})$. Prove that this map is an isomorphism by using the previous results to identify it with the boundary map $H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z})$ and using Sheet 3 Exercise 1.(h).