Exercise sheet 4

To hand in: Wednesday 14.11.18 in the homework box of Simon Pepin Lehalleur, outside the Hörsaal 001 in Arnimallee 3.

1 (Projective modules) Let R be a ring. We say that a R-module P is projective if the functor $\mathrm{Hom}_R(P, -)$ is exact, or more concretely if for any surjective morphism $M \twoheadrightarrow N$ of R-modules and any morphism $P \to N$, there exists a morphism $P \to M$ which makes the following diagram commute:

$$\begin{array}{ccc} & & M \\ & \nearrow & \big\downarrow\!\!\downarrow \\ P & \longrightarrow & N. \end{array}$$

(a) Prove that any free R-module is projective.

(b) Prove that a R-module P is projective if and only if any exact sequence $0 \to M \to M' \to P \to 0$ splits.

(c) Prove that a R-module is projective if and only if it is a direct summand of a free module. (Hint: any R-module is a quotient of a free module).

(d) Prove that $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module which is not free. If k is a field and $n > 1$, prove that k^n, seen as a left $M_n(k)$-module via left multiplication, is a projective module which is not free.

(e) Remark: any projective module over a principal ideal domain is free. For finitely generated modules, here is the proof: projective implies torsion-free, and the classification of finitely generated modules over a PID implies that (torsion-free + f.g.) implies free. The result in general is more difficult, see for instance Corollary 3.59 in these notes `http://math.uga.edu/~pete/integral2015.pdf`.

(f) Prove that, if G is a finite group and k a field with $\mathrm{char}(k) \nmid |G|$, then any $k[G]$-module is projective.

(g) Show that the $\mathbb{Z}$-modules $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are not projective.

(h) Prove that a projective R-module P is flat, i.e. that the functor $- \otimes_R P$ from the category of right R-modules to abelian groups is exact. Give an example of a flat R-module which is not projective.

2 (Group cohomology via projective resolutions) Let G be a group. We check that group cohomology can be computed using any projective resolution of the trivial G-module $\mathbb{Z}$, for instance the one presented in class which leads to the bar complex. Let $P_\bullet \to \mathbb{Z}$ be such a resolution. Note that it is natural, for projective resolutions, to use homological indexing (i.e. the differential of the complex $P_\bullet$ goes from P_i to P_{i-1}). The goal is to show that for all $M \in G-\mathrm{Mod}$ and $r \in \mathbb{N}$, we have

$$H^r(G, M) \simeq H^r(\mathrm{Hom}_G(P_\bullet, M)). \quad (1)$$

The proof requires some basic homological algebra of double complexes.

(a) Let M be a G-module and $0 \to M \to I^\bullet$ be a resolution by injective G-modules. Form the double complex $\mathrm{Hom}_G(P_\bullet, I^\bullet)$ (with cohomological indexing) with horizontal differential

$$d_h^{p,q} : \mathrm{Hom}_G(P_p, I^q) \to \mathrm{Hom}_G(P_{p+1}, I^q), f \mapsto f \circ d_{P_\bullet, p+1}$$

and vertical differential

$$d_v^{p,q} : \mathrm{Hom}_G(P_p, I^q) \to \mathrm{Hom}_G(P_p, I^{q+1}), g \mapsto (-1)^{p+q+1} \cdot d_{I^\bullet}^q \circ g.$$

Check that this indeed forms a double complex, that is, we have

$$d_v^{p+1,q} \circ d_h^{p,q} + d_h^{p,q+1} \circ d_v^{p,q} = 0.$$

(b) Let $\mathrm{Tot}\,\mathrm{Hom}_G(P_\bullet, I^\bullet)$ be the associated total complex of this double complex, that is

$$\mathrm{Tot}\,\mathrm{Hom}_G(P_\bullet, I^\bullet) = (\oplus_{p+q=n} \mathrm{Hom}_G(P_p, I^q), (d_h^p + d_v^q)_{p+q=n})$$

Check that this indeed defines a complex.

(c) Define morphisms $\mathrm{Hom}_G(P_\bullet, M) \to \mathrm{Tot}\,\mathrm{Hom}_G(P_\bullet, I^\bullet)$ and $\mathrm{Hom}_G(\mathbb{Z}, I^\bullet) \to \mathrm{Tot}\,\mathrm{Hom}_G(P_\bullet, I^\bullet)$. Show that, to prove (1), it suffices to prove that these morphisms are both quasi-isomorphisms.

(d) Let $C^{\bullet,\bullet}$ be a double complex with $C^{a,b} = 0$ unless $a \geq 0$ and $b \geq 0$. Assume that the either the rows or the columns of $C^{\bullet,\bullet}$ are exact. Show that the total complex $\mathrm{Tot}C^{\bullet,\bullet}$ is exact. (Hint: give a cycle $(c^{p,q})_{p+q=n}$ with $d(c^{p,q}) = 0$, construct a preimage $(b^{p,q})_{p+q=n-1}$ by induction on either p or q). The same statement and essentially the same proof works for complexes with $C^{a,b} = 0$ unless $a \leq 0$ and $b \leq 0$. Both statements, and some generalisations, are called the *acyclic assembly lemma*.

(e) Apply the previous result to the complex obtained from $\mathrm{Hom}_G(P_\bullet, I^\bullet)$ by shifting up and adding a line $\mathrm{Hom}_G(P_\bullet, M)$. Deduce that the morphism $\mathrm{Hom}_G(P_\bullet, M) \to \mathrm{Tot}\,\mathrm{Hom}_G(P_\bullet, I^\bullet)$ is a quasi-isomorphism. By the same argument, deduce that $\mathrm{Hom}_G(\mathbb{Z}, I^\bullet) \to \mathrm{Tot}\,\mathrm{Hom}_G(P_\bullet, I^\bullet)$ is a quasi-isomorphism and conclude.

(f) As an application, deduce without using Exercise 1 of Sheet 2 that the cohomology groups $H^*(G, M)$ can be computed using any injective resolution of M, or using any projective resolution of the trivial G-module $\mathbb{Z}$.

3 (Explicit formulas for low-degree cohomology) Let G be a group and M be a G-module.

(a) Recall the definition of (principal) crossed homomorphisms from Exercise sheet 3. Using the complex of inhomogeneous cochains $C^\bullet(G, M)$ described in class, prove that

$$H^1(G, M) \simeq \frac{\{\text{crossed homomorphisms}\}}{\{\text{principal crossed homomorphisms}\}}.$$

(b) An extension of G by M is by definition an exact sequence of groups

$$1 \to M \to E \to G \to 1$$

such that the action of $G = E/M$ by conjugation on the normal commutative subgroup M coincides with the fixed G-module structure on M (check that this action is well-defined). An isomorphism of extensions $f : E \to E'$ is a isomorphism of groups which fits in a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & {\scriptstyle f}\downarrow{\scriptstyle \sim} & & \| & & \\ 1 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

Choose any set-theoretic section $s : G \to E$ of $E \to G$. Prove that, for all $\sigma, \tau \in G$, there exists a unique $c(\sigma, \tau) \in M$ such that $s(\sigma)s(\tau) = c_s(\sigma, \tau)s(\sigma\tau)$.

(c) Prove that the function $c_s : G \times G \to M$ is an inhomogeneous 2-cocycle (Hint: we have $E = M \times G$ as a set, write down explicitely the associativity of the group law of E). Prove that if one replaces s by another section s', so that $s' - s$ factors through M, we have that $c_{s'} - c_s = d(s' - s)$ and hence $c_{s'} - c_s$ is an inhomogeneous 2-coboundary. Deduce that there is a well-defined map $\{$ extensions of G by $M\}$/isomorphisms$\to$ $H^2(G, M), E \mapsto [c_s]$.

(d) Check that every class in $H^2(G, M)$ can be represented by an inhomogeneous cocycle c with $c(e, e) = 0$.

(e) Given an inhomogeneous 2-cocycle $c : G \times G \to M$ with $c(e, e) = 0$, construct an extension of G by M with $E = M \times G$ equipped with the operation

$$(m, \sigma) \cdot (n, \tau) = (m + \sigma n + c(\sigma, \tau), \sigma\tau)$$

and the maps $M \to E, m \mapsto (m, e)$ and $G \to E, \sigma \mapsto (0, \sigma)$. Finally, check that if c and c' are cohomologous, the corresponding extensions are isomorphic.

(f) Conclude that $H^2(G, M) \simeq \{$ extensions of G by $M\}$/isomorphisms.