Freie Universität Berlin | **Wintersemester 2018**
Zahlentheorie III | **Prof. Esnault**

Exercise sheet 3

To hand in: Wednesday 7.11.18 in the homework box of Simon Pepin Lehalleur, outside the Hörsaal 001 in Arnimallee 3.

1 (Computation of $H^1(G,M)$) Let G be a group and M a G-module. We explain how to compute $H^1(G,M)$ using injective resolutions and "dimension shifting"; a different proof will be given in class. Write M_0 for the underlying abelian group of M.

(a) Consider the injective map $M \to M_* := \mathrm{Ind}^G(M_0), m \mapsto (\sigma \mapsto \sigma \cdot m)$ and the short exact sequence of G-modules

$$0 \to M \to M_* \to M_*/M \to 0.$$

Deduce that there exists an exact sequence of abelian groups

$$0 \to M^G \to M_*^G \to (M_*/M)^G \to H^1(G,M) \to 0.$$

(Hint: a consequence of Shapiro's lemma, which will be discussed next week, is that induced G-modules have no cohomology in positive degrees).

(b) Prove that $M_*^G \simeq M_0$.

(c) Prove that every equivalence class in M_*/M contains exactly one map $\phi : G \mapsto M$ with $\phi(\mathrm{id}) = 0$.

(d) Let $\phi : G \to M$ with $\phi(\mathrm{id}) = 0$. Prove that $[\phi] \in (M_*/M)^G$ if and only if ϕ is a *crossed homomorphism*, i.e., if for all $\sigma, \tau \in G$, we have

$$\phi(\sigma\tau) = \sigma\phi(\tau) + \phi(\sigma).$$

Prove that crossed homomorphisms form a subgroup of $\mathrm{Hom}(G,M)$.

(e) For $m \in M$, prove that $\sigma \mapsto \sigma \cdot m - m$ is a crossed homomorphism. Such crossed homomorphisms are called *principal.* Prove that principal crossed homomorphisms form a subgroup of $\mathrm{Hom}(G,M)$ as well.

(f) Deduce that

$$H^1(G,M) \simeq \frac{\{\text{crossed homomorphisms}\}}{\{\text{principal crossed homomorphisms}\}}$$

and that if M is a trivial G-module, then

$$H^1(G,M) \simeq \mathrm{Hom}_{\mathrm{grp}}(G,M).$$

(g) Conclude that, if G is finite and M_0 is a torsion-free abelian group seen as a G-module with trivial G-action, then $H^1(G,M_0) = 0$.

(h) Using the exact sequence of abelian groups (endowed with the trivial G-action)

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

deduce that for any finite group G, we have $H^2(G, \mathbb{Z}) \simeq \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. Prove that one can also write $H^2(G, \mathbb{Z}) \simeq \mathrm{Hom}(G, \mathbb{C}^\times)$.

2 (Norm element and cohomology of group rings) Let G be a finite group.

(a) Write $N = \sum_{g \in G} [g] \in \mathbb{Z}[G]$. Prove that N is a central element of G and that $N^2 = |G| \cdot N$.

(b) The group G acts on $\mathbb{Z}[G]$ by left multiplication. Prove that $\mathbb{Z}[G]^G$ is a cyclic group generated by N. In particular $\mathbb{Z}[G]^G$ is commutative.

(c) Show that a ring homomorphism $\mathbb{Z}[G] \to \mathbb{Z}$ is a morphism of $\mathbb{Z}$-algebras and that the restriction of such an homomorphism to G takes values in $\{-1, 1\}$. The *augmentation* of $\mathbb{Z}[G]$ is the morphism of rings $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ defined by $\epsilon([g]) = 1$ for every $g \in G$. The *augmentation ideal* is $I = \mathrm{Ker}(\epsilon)$. Prove that $N\alpha = \epsilon(\alpha)\alpha$ for all $\alpha \in \mathbb{Z}[G]$, and in particular that I is also the kernel of the morphism $\mathbb{Z}[G] \to \mathbb{Z}[G], \alpha \mapsto N\alpha$.

(d) Prove that $\mathbb{Z}[G] \simeq \mathrm{Ind}^G(\mathbb{Z})$ as G-modules (Hint: this is a special case of Milne's remark 1.3.(a); check that the map he defines there works).

(e) Prove that $H^r(G, \mathbb{Z}[G]) = 0$ for $r > 0$ (Hint: induced G-modules have no cohomology in positive degree).

(f) Prove that if G is infinite, then $\mathbb{Z}[G]^G = 0$, but that $H^r(G, \mathbb{Z}[G])$ is non-zero for $r > 0$ in general, for instance for $G = \mathbb{Z}$ and $r = 1$ (Hint: you can use the result of exercise 1, or the fact that $\mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[t, t^{-1}]$ and the exact sequence $0 \to \mathbb{Z}[t, t^{-1}] \overset{(t-1)\cdot}{\to} \mathbb{Z}[t, t^{-1}] \to \mathbb{Z} \to 0$).

3 (Torsion subgroup) For an abelian group A, write $T(A)$ for its torsion subgroup, i.e., the subgroup of elements $x \in A$ for which $nx = 0$ for some $n > 0$. We want to understand the functor $T(-) : \mathrm{Ab} \to \mathrm{Ab}$ and its homological algebra.

(a) Show that $T(-)$ is a left exact functor, and that $T(A) \simeq \mathrm{Ker}(A \to A \otimes_{\mathbb{Z}} \mathbb{Q})$.

(b) Show that a quotient of an injective abelian group is always an injective abelian group (remark: a ring R is called *hereditary* if a quotient of an injective R-module is an injective R-module), and deduce that any abelian group A admits a resolution

$$0 \to A \to I^0 \to I^1 \to 0$$

with I^0, I^1 injective. (Hint: an abelian group is injective iff it is divisible).

(c) Show that $\mathbb{Z}/4\mathbb{Z}$ is not hereditary, by proving that $\mathbb{Z}/4\mathbb{Z}$ is injective as a module over itself (hint: use the criterion with ideals seen in class), and that the quotient $\mathbb{Z}/2\mathbb{Z}$ is not an injective $\mathbb{Z}/4\mathbb{Z}$-module. Write an explicit resolution of $\mathbb{Z}/2\mathbb{Z}$ by injective $\mathbb{Z}/4\mathbb{Z}$-modules.

(d) Let $F : \mathrm{Ab} \to \mathcal{A}$ be a left-exact functor into an abelian category. Since Ab has enough injectives, F admits right-derived functors R^iF. Show that $R^iF = 0$ for $i \geq 2$. This applies in particular to $T(-)$.

(e) Compute the right-derived functors $R^0T(-)$ and $R^1T(-)$ (Hint: use an injective resolution as in (b), the observation of (a), the fact that $\mathbb{Q}$ is flat over $\mathbb{Z}$ and the Snake lemma). If you know about the Tor functor, the answer to this question is one motivation for the name "Tor" !