**Freie Universität Berlin** **Wintersemester 2018**
**Zahlentheorie III** **Prof. Esnault**

# Exercise sheet 2

**To hand in:** Wednesday 31.10.18 in the homework box of Simon Pepin Lehalleur, outside the Hörsaal 001 in Arnimallee 3.

**1** (Injective resolutions) The goal of this exercise is to prove some points which were admitted in the lecture. Let $R$ be a ring. We assume known that $R-\mathrm{Mod}$ has enough injectives. This is true in general, and was proven directly for $R = \mathbb{Z}[G]$ in class (in fact, the argument we gave can be adapted to a general ring: the forgetful functor $R - \mathrm{Mod} \to \mathbb{Z} - \mathrm{Mod}$ admits a right adjoint, which is defined in a similar way to $\mathrm{Ind}^G$, and which can be used to construct injective $R$-modules from injective abelian groups).

(a) Let $M$ be an $R$-module. Prove that $M$ admits an injective resolution, i.e., an exact complex $0 \to M \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \dots$ with $I^i$ injective for all $i \in \mathbb{N}$. (Hint: embed $M$ into an injective module $I^0$, embed the cokernel $M/I^0$ into an injective module $I^1$, and so on...)

(b) Let $a : M \to N$ be a morphism of $R$-modules and $I^\bullet$, $J^\bullet$ be injective resolutions of $M, N$. Prove that there exists a morphism $f^\bullet : I^\bullet \to J^\bullet$ (i.e., a collection of morphisms $f^i : I^i \to J^i$ compatible with the differentials) such that the square

$$\begin{array}{ccc} M & \xrightarrow{a} & N \\ \downarrow & & \downarrow \\ I^0 & \xrightarrow{f^0} & J^0 \end{array}$$

commutes. (Hint: only the injectivity of the $J^i$ plays a role. Construct the $f^i$ inductively. Start from the fact that since $J^0$ is injective, the composite morphism $M \to J^0$ extends to $I^0$. Then introduce the image of the differential $d^0$ of $I^\bullet$ which is a submodule of $I^1$, prove that $f^0$ descends to a morphism $\mathrm{Im}(d^0_{I^\bullet}) \to J^0$, and carry on).

(c) Let $f^\bullet : C^\bullet \to D^\bullet$ be a morphism of complexes. Prove that $f$ induces a map $H^i(f^\bullet) : H^i(C^\bullet) \to H^i(D^\bullet)$ for every $i \in \mathbb{N}$.

(d) Let $f^\bullet, g^\bullet : C^\bullet \to D^\bullet$ be morphisms of complexes. An *homotopy* between $f^\bullet$ and $g^\bullet$ is a collection of maps $h^i : f^i \to g^{i-1}$ such that $g^i - f^i = d^{i-1}_{D^\bullet} \circ h^i + h^{i+1} \circ d^i_{C^\bullet}$ (the terminology can be justified by an analogy with homotopies in algebraic topology). Prove that, if there exists such an homotopy, we have $H^i(f^\bullet) = H^i(g^\bullet)$ for all $i \in \mathbb{Z}$. Prove that if $F : R - \mathrm{Mod} \to S - \mathrm{Mod}$ is an additive functor, then the induced morphisms $F(f^\bullet)$ and $F(g^\bullet)$ are also homotopic.

(e) Let $a : M \to N$ be a morphism of $R$-modules and $I^\bullet$, $J^\bullet$ be injective resolutions of $M, N$. Prove that any two morphisms $f^\bullet$ and $g^\bullet$ extending $a$ as in (b) above are homotopic. (Hint: first reduce to the case where $a$ and all the $f^i$ are the zero morphism, which simplifies notation. It then remains to construct $h^i$ such that $g^i : d^{i-1}_{D^\bullet} \circ h^i + h^{i+1} \circ d^i_{C^\bullet}$. Do this inductively, using the same ideas as in the proof of (b)).

(f) Conclude that, for $M$ a $G$-module, the choice of two resolutions of $M$ by injective $G$-modules induces an isomorphism on group cohomology, and that hence group cohomology is well-defined up to isomorphism. (Hint: apply the previous results for $a$ the identity of $M$).

**2** (Maschke's theorem and Group cohomology)

(a) Assume that $G$ is a finite group and that $K$ is a field of characteristic 0. A $K$-linear $G$-module is a $K$-vector space together with a $K$-linear $G$-action (i.e., for all $g \in G$, the map $g \cdot - : M \to M$ is $K$-linear; equivalently, a $K[G]$-module). Prove that the functor $(-)^G$ from the category of $K$-linear $G$-modules to the category of $K$-vector spaces is exact. (Hint: by assumption, $|G|$ is invertible in $K$, so it is possible to average over $G$.)

(b) Under the assumptions of (a), let $M$ a $K$-linear $G$-module. Prove that $H^r(G, M) = 0$. (Hint: prove that $K$ vector spaces are injective as abelian groups. Use this fact and the induced $G$-module construction to build a resolution of $M$ by $K$-linear $G$-modules which are injective as $\mathbb{Z}[G]$-modules, and apply the previous question).

(c) (Maschke's theorem) An $R$-module $M$ is called semi-simple if every submodule $N \subset M$ admits a supplement, i.e., a submodule $P \subset M$ such that $M \simeq M \oplus P$. A ring $R$ is called semi-simple if every $R$-module is semi-simple. Under the same assumptions as before, prove that $K[G]$ is semi-simple. (Hint: vector spaces over a field are semi-simples. The idea is then to take a supplement of $N$ as a $K$-vector space, consider the projection map $\pi : M \to N$, modify it by averaging over $G$ to get a map $\tilde{\pi} : M \to N$ which is a map of $G$-modules, and use the results of Exercise 1.(c) of Sheet 1 to prove that $M \simeq N \oplus \mathrm{Ker}(\tilde{\pi})$).

(d) Remarks: questions (b) and (c) are closely related. In fact, one could deduce (b) from (c), because one can show that in the category of $R$-modules over a semi-simple ring, Ext-groups $\mathrm{Ext}^r$ for $r > 0$ are always 0, and in the situation of ($b$) one can prove that $H^r(G, M) = \mathrm{Ext}^r_{K[G]}(K, M)$. Also, both (b) and (c) hold under the weaker assumption that $|G|$ is invertible in $K$; the proof in (c) goes through directly, but the proof in (b) does not work because a field of positive characteristic is not injective as abelian group. To repair the argument, one needs to know that group cohomology can be computed with resolutions by induced modules: see next week's lecture.

(e) Write $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ considered as a field. Prove that $H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2) \simeq \mathbb{Z}/2\mathbb{Z}$ (where $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $\mathbb{F}_2$); this can be done by an explicit injective resolution. Deduce that the assumption on the characteristic in (b) is necessary. One can also prove that the group ring $\mathbb{F}_2[\mathbb{Z}/2\mathbb{Z}]$ is not semi-simple.