Freie Universität Berlin | **Wintersemester 2018**
Zahlentheorie III | **Prof. Esnault**

Exercise sheet 1

To hand in: Wednesday 24.10.18 in the homework box of Simon Pepin Lehalleur, outside the Hörsaal 001 in Arnimallee 3.

1 (Refresher on exact sequences and exact functors.) Let R, S be rings.

(a) Prove that an additive functor between categories of modules is exact if and only if it is left and right exact.

(b) Let $f : R \to S$ be a morphism of rings. Prove that the tensor product functor

$$- \otimes_R S : R - \mathrm{Mod} \to S - \mathrm{Mod}$$

is right exact in two ways: by hand, and by proving that its right adjoint, the restriction functor $S - \mathrm{Mod} \to R - \mathrm{Mod}$ is exact.

(c) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of R-modules. Prove that the following conditions are equivalent:

- The map f has a retraction $r : B \to A$, i.e. with $r \circ f = \mathrm{id}_A$.
- The map g has a section $s : C \to B$, i.e. with $g \circ s = \mathrm{id}_C$.
- The exact sequence is isomorphic to the canonical sequence

$$0 \to A \to A \oplus C \to C \to 0.$$

Such a short exact sequence is called split exact. Prove that an additive functor preserves split exact sequences.

2 (Group rings of some abelian groups.)

(a) Show that $\mathbb{Z}[\mathbb{Z}]$ is isomorphic to the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials with integral coefficients (that is, the localisation of $\mathbb{Z}[t]$ at (t)). Is the ring $\mathbb{Z}[t]$ isomorphic to the group ring of some group?

(b) Let $n \geq 0$. Show that $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ is isomorphic to the ring $\mathbb{Z}[t]/t^n$.

(c) Show that the category of $\mathbb{Z}[\mathbb{Z}]$-modules (resp. $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-modules) is equivalent to the category of pairs $(M, f : M \to M)$ with $f \in \mathrm{Aut}(M)$ (resp. with $f^{\circ n} = \mathrm{id}_M$).

(d) Let G, H be two groups. Construct an isomorphism

$$\mathbb{Z}[G \times H] \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[H].$$

(e) Using the previous questions, describe the group ring and the category of G-modules for a finitely generated abelian group.

3 (Group rings, subgroups, units.)

(a) Let G be a group and H be a subgroup of G. Explain how $\mathbb{Z}[H]$ is naturally a subring of $\mathbb{Z}[G]$; write $i : \mathbb{Z}[H] \to \mathbb{Z}[G]$ for the inclusion map.

(b) Prove that $\mathbb{Z}[G]$ is a free left $\mathbb{Z}[H]$-module for the $\mathbb{Z}[H]$-module structure defined by i.

(c) Define a group homomorphism $\pi_H : \mathbb{Z}[G] \to \mathbb{Z}[H]$ such that $\pi_H \circ i = \mathrm{id}_{\mathbb{Z}[H]}$. Deduce that $x \in \mathbb{Z}[H]$ is invertible (resp. a left or right zero-divisor) in $\mathbb{Z}[H]$ if and only if $i(x)$ is invertible (resp. a left or right zero-divisor) in $\mathbb{Z}[G]$.

(d) We denote by Groups the category of groups and by Rings the category of (unital, non-necessarily commutative) rings. Consider the two functors

$$\text{Groups} \to \text{Rings}, G \mapsto \mathbb{Z}[G]$$

and

$$\text{Rings} \to \text{Groups}, R \mapsto R^{\times}.$$

Prove that the first is a left adjoint of the second.

4 (Tensor products of G-modules.) As explained in class, there is a natural tensor structure on the category of G-modules; $M \otimes N$ is the tensor products of the underlying abelian groups, together with the G-module structure extended by linearity from the rule

$$g \cdot (m \otimes n) = g \cdot m \otimes g \cdot n.$$

Explain the relationship between this tensor structure and the one on the category of $\mathbb{Z}[G]$-modules obtained by using the anti-involution $\mathbb{Z}[G] \to \mathbb{Z}[G], [g] \mapsto [g^{-1}]$ to change a left module into a right module.