PROBEKLAUSUR

Problem 1. Give an example showing that if P, Q are two prime ideals of a ring A, then $P \cap Q$ may not be a prime ideal.

Problem 2. Let A be a commutative ring. Let

$$(*) \qquad 0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be a short exact sequence of A-modules. We call the sequence $(*)$ split if there exists an isomorphism $h : M \cong M' \oplus M''$ making the following diagram commuative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{\quad f \quad} & M & \xrightarrow{\quad g \quad} & M'' & \longrightarrow & 0 \\
 & & \| & & \downarrow h & & \| & & \\
0 & \longrightarrow & M' & \xrightarrow{\ r\ } & M' \oplus M'' & \xrightarrow{\ s\ } & M'' & \longrightarrow & 0
\end{array}$$

where r is the embedding $m' \mapsto m' \oplus 0$ and s is the second projection $m' \oplus m'' \mapsto m''$. Show that the following statements are equivalent.

(1) $(*)$ splits;
(2) there is an A-module homomorphism $\phi : M \to M'$ such that $\phi \circ f = id_{M'}$;
(3) there is an A-module homomorphism $\varphi : M'' \to M$ such that $g \circ \varphi = id_{M''}$.

Problem 3. Let $\mathbb{R}$ be the field of real numbers, $\mathbb{C}$ be the field of complex numbers. Consider $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ as a $\mathbb{C}$-vector space via $a \cdot (b \otimes c) := ab \otimes c$, for $a, b, c \in \mathbb{C}$. Compute the $\mathbb{C}$-dimension of this vector space.

Problem 4. Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, $\mathfrak{q}$ a $\mathfrak{m}$-primary ideal. Show that $A/\mathfrak{q}$ is an Artin local ring.

Problem 5. A ring A is called reduced if there are no nilpotent elements. Show that an Artin ring A is a reduced ring if and only if it is isomorphic to a finite product of fields, i.e. isomorphic to a ring of the form $\prod_{i=1}^{n} k_i$, where the k_i's are fields and $n \in \mathbb{N}^+$.

Problem 6. (1) Let A be a Noetherian ring with finite Krull dimension and suppose that an ideal $I \subseteq A$ has a minimal primary decomposition

$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$$

for some $n \in \mathbb{N}^+$. Set $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$. Then

$$\dim(A/I) = \max\{\dim(A/\mathfrak{p}_1), \dim(A/\mathfrak{p}_2), \cdots, \dim(A/\mathfrak{p}_n)\}.$$

(2) Write out a minimal primary decomposition of the ideal

$$I = (XY, YZ, XZ) \subseteq \mathbb{C}[X, Y, Z],$$

and determine the primes belonging to I. Which of them are isolated and which are embedded?

(3) Determine the dimension of the ring

$$\mathbb{C}[X, Y, Z]/(XY, YZ, XZ).$$

Problem 7. Let $A = k[X_1, \cdots, X_n]$ be the polynomial ring in n variables over a field k. Let $\mathfrak{m} := (X_1, X_2, \cdots, X_n) \subseteq A$. Show that $\mathfrak{m}$ is a maximal ideal of A. Let $\mathfrak{q} \subseteq A$ be a $\mathfrak{m}$-primary ideal of A. Set

$$V(\mathfrak{q}) := \{(a_1, a_2, \cdots, a_n) \in k^{\oplus n} \mid f(a_1, a_2, \cdots, a_n) = 0 \text{ for all } f \in \mathfrak{q}\}.$$

Show that $V(\mathfrak{q}) = \{(0, 0, \cdots, 0)\}$.

Problem 8. Recall that if $A = \oplus_{i=0}^{\infty} A_i$ is a noetherian graded ring with A_0 Artinian, $M = \oplus_{i=0}^{\infty} M_i$ is a finitely generated graded A-module, and if λ denotes the length function defined on the set of isomorphism classes of finitely generated A_0-modules, then the Poincaré series of M is defined to be

$$P(M, t) := \sum_{i=0}^{\infty} \lambda(M_i) t^i \in \mathbb{Z}[[t]].$$

We have proved that there exists a polynomial $f(X) \in \mathbb{Q}[X]$ such that $f(n) = \lambda(M_n)$ for $n \in \mathbb{N}^+$ sufficiently large, and this polynomial is called the Hilbert function of M. Recall also that if A is a Noetherian local ring with maximal ideal $\mathfrak{m}$, $\mathfrak{q}$ is $\mathfrak{m}$-primary then there exists a polynomial $\chi_{\mathfrak{q}}^{A}(X) \in \mathbb{Q}[X]$ such that the length of $A/\mathfrak{q}^n$ (as an A-module) is equal to $\chi_{\mathfrak{q}}^{A}(n)$ for n sufficiently large, and $\chi_{\mathfrak{q}}^{A}(X)$ is called the characteristic polynomial.

Now let $A = A_0[X_1, \cdots, X_s]$ be a polynomial ring in s variables over an Artin ring A_0. This is a graded ring, and can be regarded as a graded module over itself.

(1) Calculate the Poincaré series $P(A, t)$.
(2) Write out the Hilbert function of A.
(3) Let $\mathfrak{m} \subseteq A$ be a maximal ideal containing $\mathfrak{q} = (X_1, \cdots, X_s)$. Calculate the characteristic polynomial $\chi_{\mathfrak{q}}^{A_{\mathfrak{m}}}(X)$.

Problem 9. Let A be a local ring with maximal ideal $\mathfrak{m}$. Let $f : M \to N$ be a morphism of A-modules, where N is finitely generated. Show that if the map

$$\bar{f} : M \otimes_A A/\mathfrak{m} \to N \otimes_A A/\mathfrak{m}$$
$$m \otimes 1 \mapsto f(m) \otimes 1$$

is surjective then f is surjective as well.