Serre's Criterion for Normality

1. Associated primes

Discussion 1. Consider $Z_a = \{(0,b) \mid b \in \mathbb{C}\} \cup \{(a,0)\}$, the union of the y-axis and a point $(a,0)$ in the plane $\mathbb{C}^2$, for some nonzero $a \in \mathbb{C}$. The set of polynomials which vanish on Z is $I_a = \langle xy, x(x-a)\rangle$ and so the ring of polynomial functions on Z is $\mathbb{C}[x,y]/\langle xy, x(x-a)\rangle$. As a moves to zero, this ideal becomes $I_0 = \langle xy, x^2\rangle$. However, if we put $a = 0$ into the above set, we get $Z_0 = \{(0,b) \mid b \in \mathbb{C}\}$, and the set of polynomials vanishing on this set is $\langle x\rangle \neq \langle xy, x^2\rangle$. So the ideal $\langle xy, x^2\rangle$ somehow remembers that it has an extra point at the origin.

As further evidence for this idea, consider the map $Z_a \subset \mathbb{C}^2 \to \mathbb{C}; (a,b) \mapsto b$. This induces the ring morphisms $\mathbb{C}[y] \subset \mathbb{C}[x,y] \to \mathbb{C}[x,y]/I_a$. The fibre over $y = b$ is

$$(\mathbb{C}[x,y]/\langle xy, x(x-a)\rangle) \otimes_{\mathbb{C}[y]} \mathbb{C}[y]/(y-b) \cong \begin{cases} \mathbb{C} & b \neq 0 \\ \mathbb{C}[x]/\langle x(x-a)\rangle & b = 0 \end{cases}$$

even when $a = 0$.

Now recall that the irreducible components of an algebraic variety correspond to the minimal primes of its associated ring. Associated primes encode the idea that there might be more "irreducible components" hiding inside the bigger irredicuble components.

More generally, associated primes of a module correspond to associated primes of its support (when support is considered as a scheme, not just a set).

Definition 2 ([AK2017, 17.1])**.** A prime $\mathfrak{p}$ of a ring R is said to be an *associated prime* of an R-module M if it is of the form $\mathfrak{p} = \mathrm{Ann}(m) = \{x \in R \mid xm = 0\}$ for some nonzero $m \in M$. The set of associated primes is denoted $\mathrm{Ass}(M)$. A prime $\mathfrak{p} \in \mathrm{Ass}(M)$ is said to be *embedded* if $\mathfrak{p}_0 \subsetneq \mathfrak{p}$ for some $\mathfrak{p}_0 \in \mathrm{Ass}(M)$.

Remark 3. We will see later that if R is Noetherian, a prime $\mathfrak{p} \in \mathrm{Spec}\,(R)$ contains a prime of $\mathrm{Ass}(M)$ if and only if it contains a prime of $\mathrm{Supp}(M)$. That is, we have $\overline{\mathrm{Ass}(M)} = \mathrm{Supp}(M)$ where the closure is taken in the topological space $\mathrm{Spec}\,(R)$.

Lemma 4 ([AK2017, Lem.17.5])**.** *Let R be a ring, $\mathfrak{p}$ a prime, and $m \in R/\mathfrak{p}$ nonzero. Then* $\mathrm{Ann}(m) = \mathfrak{p}$ *and* $\mathrm{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$.

Proof. Let $y \in R$ be an element such that $y \mapsto m$ under $R \to R/\mathfrak{p}$. Note that $y \neq \mathfrak{p}$ because $m \neq 0$. Then $x \in \mathrm{Ann}(m) \iff xm = 0 \iff xy \in \mathfrak{p} \iff x \in \mathfrak{p}$ since $y \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime. □

Proposition 5 ([AK2017, Lem.17.6], [Liu, Cor.7.1.5])**.** *Let R be a ring, and $N \subset M$ an inclusion of R-modules. Then*

$$\mathrm{Ass}(N) \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N).$$

Proof. For any $m \in N$, the ideal $\mathrm{Ann}(m)$ is the same whether we regard m as an element of N or M. So $\mathrm{Ass}(N) \subset \mathrm{Ass}(M)$. Let $\mathfrak{p} \in \mathrm{Ass}(M)$, and choose $m \in M$ with $\mathfrak{p} = \mathrm{Ann}(m)$. The map $R \to M, a \mapsto am$ induces an injection $R/\mathfrak{p} \cong \langle m\rangle \subset M$. If $\langle m\rangle \cap N = 0$, then the composition $\langle m\rangle \to M \xrightarrow{\pi} M/N$ is injective, implying $\mathfrak{p} = \mathrm{Ann}(\pi(m))$ so $\mathfrak{p} \in \mathrm{Ass}(M/N)$. If there is some nonzero $m' \in \langle m\rangle \cap N \neq 0$, then under the isomorphism $\alpha : R/\mathfrak{p} \cong \langle m\rangle$ we get a nonzero $a' = \alpha^{-1}m' \in R/\mathfrak{p}$, and by [AK2017, Lem.17.5], $\mathfrak{p} = \mathrm{Ann}(a')$ so $\mathfrak{p} \in \mathrm{Ass}(N)$ since $\mathrm{Ann}(a') = \mathrm{Ann}(m')$. □

Recall that we denote the set of zero divisors of an R-module M by

$$\mathrm{z.div}(M) = \{x \in R \mid xm = 0 \text{ for some nonzero } m \in M\}.$$

Lemma 6 ([AK2017, Lem.17.9])**.** *Let R be a ring, M an R-module. Suppose $\mathfrak{p}$ is a maximal element in the poset of ideals $\{\mathrm{Ann}(m) \mid m \in M, m \neq 0\}$. Then $\mathfrak{p}$ is prime. So $\mathfrak{p} \in \mathrm{Ass}(M)$.*

Proof. Let $m \in M$ be a nonzero element with $\mathfrak{p} = \mathrm{Ann}(m)$. Note $1 \notin \mathfrak{p}$, so $\mathfrak{p} \neq R$. Consider $b, c \in R$ with $bc \in \mathfrak{p}$ and $c \notin \mathfrak{p}$. We wish to show $b \in \mathfrak{p}$. Note $cm \neq 0$. For any $a \in \mathfrak{p} = \mathrm{Ann}(m)$, we have $acm = c(am) = 0$ so we have $\mathfrak{p} \subset \mathrm{Ann}(cm)$. By maximality, we deduce that this is actually an equality $\mathfrak{p} = \mathrm{Ann}(cm)$, so it suffices to show that $b \in \mathrm{Ann}(cm)$. But we have $bcm = 0$ since we have assumed $bc \in \mathfrak{p} = \mathrm{Ann}(m)$, so $b \in \mathrm{Ann}(cm)$ as desired. □

Proposition 7 ([AK2017, Prop.17.10], [Liu, 7.1.2])**.** *Let R be a Noetherian ring and M an R-module. Then $M = 0$ if and only if $\mathrm{Ass}(M) = \varnothing$.*

Proof. Consider the poset of ideals $\{\mathrm{Ann}(m) \mid m \in M, m \neq 0\}$, and note that it contains $\mathrm{Ass}(M)$. If $M = 0$, then this set is empty. If not, then since R is Noetherian, it must contain some maximal element, which is in $\mathrm{Ass}(M)$ by the above lemma. □

We will write $\mathrm{Ass}_R(M)$ and $\mathrm{Ann}_R(M)$ for $\mathrm{Ass}(M)$ and $\mathrm{Ann}(M)$ if we want to make the ring more explicit.

Proposition 8 ([Liu, 7.1.2])**.** *Let R be a ring, M an R-module, S a multiplicative set. Then $\mathrm{Ass}_{S^{-1}R}(S^{-1}M) = \mathrm{Ass}_R(M) \cap \mathrm{Spec}\,(S^{-1}R)$.*

Remark 9. Recall that

$$\begin{aligned} \mathrm{Spec}\,(S^{-1}R) &\cong \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid S \cap \mathfrak{p} = \varnothing\} \\ \mathfrak{p} &\mapsto \phi^{-1}\mathfrak{p} \\ S^{-1}\mathfrak{q} &\leftarrow\!\shortmid \mathfrak{q} \end{aligned}$$

where $\phi : R \to S^{-1}R$ is the canonical map.

Remark 10. Geometrically, $S^{-1}R$ corresponds to an intersection U of open subsets of $\mathrm{Spec}\,(R)$. So this proposition is saying that the the associated primes of the restriction of M to U is just the set of those associated primes of M contained in U.

Proof. Suppose that $\mathfrak{p} \in \mathrm{Ass}_R(M)$ and $S^{-1}\mathfrak{p} \in \mathrm{Spec}\,(S^{-1}R)$. That is, $\mathfrak{p} = \mathrm{Ann}_R(x)$ for some $x \in M$, and $S^{-1}\mathfrak{p}$ is not $S^{-1}R$ and therefore prime. Then $0 \to \mathfrak{p} \to R \to M$ is exact, where the second map is $a \mapsto ax$. Since the localisation of modules functor S^{-1} is exact, the sequence $0 \to S^{-1}\mathfrak{p} \to S^{-1}R \to S^{-1}M$ is also exact. So we have $S^{-1}\mathfrak{p} = \mathrm{Ann}_{S^{-1}R}(x)$ and $S^{-1}\mathfrak{p} \in \mathrm{Ass}_{S^{-1}R}(S^{-1}M)$.

Conversely, let $S^{-1}\mathfrak{p} \in \mathrm{Ass}_{S^{-1}R}(S^{-1}M)$ and choose $x/s \in S^{-1}M$ such that $S^{-1}\mathfrak{p} = \mathrm{Ann}_{S^{-1}R}(s^{-1}x)$. Since s is invertible in $S^{-1}R$ we have $\mathrm{Ann}_{S^{-1}R}(s^{-1}x) = \mathrm{Ann}_{S^{-1}R}(x)$. Hence, $\mathfrak{p} = \mathrm{Ann}_R(x)$. □

Lemma 11 (Lemma A)**.** *Let R be a Noetherian ring and M an R-module. Then the canonical map $M \to \prod_{\mathfrak{p} \in \mathrm{Ass}(M)} M_{\mathfrak{p}}$ is injective.*

Proof. Suppose that m is a nonzero element in the kernel. Then $m = 0$ in each $M_\mathfrak{p}$, so for each $\mathfrak{p} \in \mathrm{Ass}(M)$ there exists some $a_\mathfrak{p} \in\notin \mathfrak{p}$ such that $a_\mathfrak{p} m = 0$ in M. In particular, $\langle a_\mathfrak{p} \rangle \subset \mathrm{Ann}(m)$. But R is Noetherian so $\mathrm{Ann}(m)$ is contained in some prime $\mathfrak{p} \in \mathrm{Ass}(M)$ [AK2017, 17.9], contradicting the assumption $a_\mathfrak{p} \notin \mathfrak{p}$. □

Proposition 12 ([AK2017, 17.12], [Liu, 7.1.3]). *Let R be a Noetherian ring and M an R-module. Then* $\mathrm{z.div}(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$. *As a special case, we have* $\mathrm{z.div}(R) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathfrak{p}$.

Remark 13. Geometrically, this proposition says that a function on $\mathrm{Spec}\,(R)$ is a zero divisor of M if and only if it vanishes on some irreducible (or "embedded") component of $\mathrm{Supp}(M)$.

Proof. The inclusion $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p} \subset \mathrm{z.div}(M)$ follows directly from the definitions, so lets prove $\mathrm{z.div}(M) \subset \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$. By definition, if $x \in \mathrm{z.div}(M)$, then there is some nonzero $m \in M$ with $xm = 0$. So the submodule $Rm \subset M$ is nonzero, and therefore there is some $\mathfrak{p} \in \mathrm{Ass}(Rm)$ [AK2017, Prop.17.10]. In other words, there is a prime with $\mathfrak{p} = \mathrm{Ann}(ym)$ for some $y \in R$ with $ym \neq 0$. Now $xm = 0$, so $xym = 0$, so $x \in \mathfrak{p}$. Since $\mathrm{Ass}(Rm) \subset \mathrm{Ass}(M)$, $\mathfrak{p} \in \mathrm{Ass}(M)$, so $x \in \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$. □

Lemma 14 ([AK2017, Lem.17.16], [Liu, Lem.7.1.4]). *Let R be a Noetherian ring and M a finitely generated R-module. Then there is a finite chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Spec}\,(R)$ for $i = 1, \ldots, n$.

Proof. If $M = 0$ there is nothing to show. If not, then $\mathrm{Ass}(M)$ is nonempty (by [AK2017, Prop.17.10]) so there exists some $m \in M$ such $\mathfrak{p}_1 = \mathrm{Ann}(m)$ is prime. But this ideal is precisely the kernel of the map $R \to M; a \mapsto am$, and so there is an induced injection of R-modules $R/\mathfrak{p}_1 \hookrightarrow M; \overline{a} \mapsto am$. Replacing M with $M/\langle m \rangle$ and repeating, we obtain a sequence of quotients $M = N_0 \twoheadrightarrow N_1 \twoheadrightarrow N_2 \twoheadrightarrow \ldots$ with the property that $\ker(N_{i-1} \to N_i) \cong R/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i \in \mathrm{Ass}(N_{i-1})$. Defining $M_i = \ker(M \twoheadrightarrow N_i)$ we convert this into a sequence of submodules $0 = M_0 \hookrightarrow M_1 \hookrightarrow \ldots$, with the property that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ since $R/\mathfrak{p}_i \cong \ker(N_{i-1} \twoheadrightarrow N_i) = \ker(M/M_{i-1} \twoheadrightarrow M/M_i) = M_i/M_{i-1}$. The sequence $(M_i)_{i \geq 0}$ stabilises because M is finitely generated. □

Corollary 15 ([AK2017, Thm.17.17], [Liu, Cor.7.1.5]). *Let R be a Noetherian ring and M a finitely generated R-module. Then* $\mathrm{Ass}(M)$ *is finite.*

Proof. By [AK2017, Lem.17.16], there is a sequence of submodules $0 = M_0 \subset \cdots \subset M_n = M$ with $M_{i+1}/M_i \cong R/\mathfrak{p}_i$ for some primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Then by [AK2017, Lem.17.6] we have $\mathrm{Ass}(M) \subset \mathrm{Ass}(R_{\mathfrak{p}_n}) \cup \mathrm{Ass}(M_{n-1}) \subset \mathrm{Ass}(R_{\mathfrak{p}_n}) \cup \mathrm{Ass}(R_{n-1}) \cup \mathrm{Ass}(M_{n-2}) \subset \cdots \subset \bigcup \mathrm{Ass}(R_{\mathfrak{p}_i})$, and by [AK2017, Lem.17.5], $\mathrm{Ass}(R/\mathfrak{p}_i) = \{\mathfrak{p}_i\}$, so we deduce that $\mathrm{Ass}(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. □

The Prime Avoidance Lemma has already appeared previously in this course.

Lemma 16 (Prime Avoidance Lemma [AK2017, 3.12]). *Let R be a ring, $\mathfrak{a}$ a subset of R that is closed under addition and multiplication, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ prime ideals. If $\mathfrak{a} \subset \bigcup_j \mathfrak{p}_j$ then $\mathfrak{a} \subset \mathfrak{p}_j$ for some j.*

Exercise 1. Let $(R, \mathfrak{m})$ be a Noetherian local ring, and M a finitely generated R-module. Show that $\mathfrak{m} \in \mathrm{Ass}(M)$ if and only if $\mathfrak{m} = \mathrm{z.div}(M)$.

Lemma 17 ([Liu, 8.2.12(a,b)]). *Let M be a finitely generated module over a Noetherian ring A. Consider* $\operatorname{Supp} M = \{\mathfrak{p} \mid M_{\mathfrak{p}} \neq 0\}$. *The following are true.*

(1) *Any prime that is minimal in* $\operatorname{Supp}(M)$ *belongs to* $\operatorname{Ass}(M)$.
(2) *We have* $\operatorname{Supp} M = \{\mathfrak{q} \mid \mathfrak{p} \subset \mathfrak{q}$ *for some* $\mathfrak{p} \in \operatorname{Ass}(M)\}$.

Remark 18. Geometrically, this says $\overline{\operatorname{Ass}(M)} = \operatorname{Supp}(M)$.

Proof. First we show

$$\operatorname{Supp} M \supset \{\mathfrak{q} \mid \mathfrak{p} \subset \mathfrak{q} \text{ for some } \mathfrak{p} \in \operatorname{Ass}(M)\}.$$

The contrapositive is: given a prime $\mathfrak{q}$, if $M_{\mathfrak{q}} = 0$ then $\mathfrak{q}$ cannot contain any associated prime. Indeed, saying $\mathfrak{q}$ contains an associated prime is to say it contains a set of the form $\operatorname{Ann}(m)$ for some nonzero $m \in M$. But if $M_{\mathfrak{q}} = 0$, then there is some $s \notin \mathfrak{q}$ such that $sm = 0$, contradicting the assumption $\operatorname{Ann}(m) \subset \mathfrak{q}$.

Now let us show (1). Let $\mathfrak{q}$ be a prime which is minimal in $\operatorname{Supp}(M)$. As $\mathfrak{q} \in \operatorname{Supp}(M)$, the localisation $M_{\mathfrak{q}}$ is nonzero. So $\operatorname{Ass}(M_{\mathfrak{q}})$ is nonempty [AK2017, 17.10]. But the elements of $\operatorname{Ass}(M_{\mathfrak{q}})$ correspond to primes of $\operatorname{Ass}(M)$ contained in $\mathfrak{q}$ [Liu, 7.1.2], so there is some prime $\mathfrak{p} \in \operatorname{Ass}(M)$ with $\mathfrak{p} \subset \mathfrak{q}$. As $\mathfrak{p}$ is an associated prime, it trivially contains an associated prime (itself) so $\mathfrak{p} \in \operatorname{Supp}(M)$. But we assumed $\mathfrak{q}$ was minimal, so $\mathfrak{p} = \mathfrak{q}$, and therefore $\mathfrak{q} \in \operatorname{Ass}(M)$.

Now let us show (2) by showing

$$\operatorname{Supp} M \subset \{\mathfrak{q} \mid \mathfrak{p} \subset \mathfrak{q} \text{ for some } \mathfrak{p} \in \operatorname{Ass}(M)\}.$$

and combining it with the opposite inclusion, proven above. Indeed, let $\mathfrak{q} \in \operatorname{Supp}(M)$, and choose a minimal element $\mathfrak{p} \in \operatorname{Supp}(M)$ with $\mathfrak{p} \subset \mathfrak{q}$. By (1), $\mathfrak{p} \in \operatorname{Ass}(M)$, and therefore, $\mathfrak{q}$ is in the set on the RHS. □

2. Depth

Discussion 19. We have defined the dimension of a ring A to be the length of a maximal chain of prime ideals. Geometrically the idea is that if we have a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ then $\operatorname{Spec}(A/\mathfrak{p}_d)$ is a point, $\operatorname{Spec}(A/\mathfrak{p}_{d-1})$ should be like a curve containing the point, $\operatorname{Spec}(A/\mathfrak{p}_{d-2})$ should be like a surface containing the curve, etc.

$$\operatorname{Spec}(A/\mathfrak{p}_d) \hookrightarrow \operatorname{Spec}(A/\mathfrak{p}_{d-1}) \hookrightarrow \operatorname{Spec}(A/\mathfrak{p}_{d-2}) \hookrightarrow \dots$$

However, not every chain of prime ideals can necessarily be completed to a chain of length $\dim A$, e.g., $k[x,y,z]/(xy,xz)$. A different approach would be to consider sequences of elements $a_1, a_2, \dots \in A$ such that $\operatorname{Spec}(A/a_1)$ is like a proper hypersurface in $\operatorname{Spec}(A)$, such that $\operatorname{Spec}(A/\langle a_1, a_2\rangle)$ is like a proper hypersurface in $\operatorname{Spec}(A/a_1)$, etc. This leads to the notion of depth. This notion is a bit more robust than dimension, cf. [Liu, 8.2.11] below.

Definition 20 ([Liu, 8.2.9]). Let A be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and M an A-module. We say that an element $a \in \mathfrak{m}$ is *M-regular* if the map $M \to M; m \mapsto am$ is injective. A sequence of elements $a_1, \dots, a_n$ of $\mathfrak{m}$ is called *M-regular*, if a_i is $M/(a_1M + \cdots + a_{i-1}M)$-regular for each $i = 1, \dots, n$. The *depth* of M, denoted $\operatorname{depth} M$ is the maximal number of elements of an M-regular sequence.

Remark 21 ([Liu, 8.2.11])**.** Let $(A, \mathfrak{m})$ be a Noetherian local ring, and let M be a finitely generated A-module of depth $d = \operatorname{depth} M$. Then any M-regular sequence can be completed to an M-regular sequence with d elements. In particular, if $a \in \mathfrak{m}$ is M-regular, then

$$\operatorname{depth}(M/aM) = \operatorname{depth} M - 1.$$

Lemma 22 ([Liu, 8.2.12(c)])**.** *Let A be a local Noetherian ring and M a nonzero finitely generated module. Then* $\operatorname{depth} M \leq \dim \operatorname{Supp} M$*. As a special case, we have* $\operatorname{depth} A \leq \dim A$*.*

Remark 23. If $\operatorname{depth} M \leq \dim \operatorname{Supp} M$, then M is called *Cohen-Macaulay*. Rings that are Cohen-Macaulay have good properties. Every Noetherian Cohen-Macaulay ring is universally catenary, and in particular, every chain of primes can be completed to a chain of length the dimension. Noetherian Cohen-Macaulay rings have dualizing sheaves (as opposed to complexes); these generalise the sheaf of volume forms on smooth manifolds.

Proof. Let $a \in \mathfrak{m}$ be an M-regular element. We claim that

$$0 \leq \dim \operatorname{Supp}(M/aM) \leq \dim \operatorname{Supp}(M) - 1.$$

First note that $M/aM \neq 0$ since $aM = M$ implies $\mathfrak{m}M = M$ which implies $M = 0$ by Nakayama's Lemma. So $\operatorname{Supp}(M/aM)$ is nonempty. Let $\mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ be a maximal chain of primes in $\operatorname{Supp}(M/aM)$. Since

$$(M/aM)_{\mathfrak{p}} \cong M \otimes_A A/\langle a \rangle \otimes_A A_{\mathfrak{p}} \cong M \otimes_A A_{\mathfrak{p}} \otimes_A A/\langle a \rangle \cong M_{\mathfrak{p}}/aM_{\mathfrak{p}}$$

we see that all $\mathfrak{p}_i$ are in $\operatorname{Supp}(M)$. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1$ be a minimal element of $\operatorname{Supp}(M)$ contained in $\mathfrak{p}_1$. To finish proving the claim, it suffices to show that $\mathfrak{p}_0 \neq \mathfrak{p}_1$. Since $\mathfrak{p}_0$ is minimal, it is in $\operatorname{Ann}(M)$ [Liu, 8.2.12]. In particular, $a \notin \mathfrak{p}_0$ because a regular means it is not a zerodivisor. On the other hand $a \in \mathfrak{p}_1$, since otherwise we would have $(M/aM)_{\mathfrak{p}_1} = 0$ (as elements of $A - \mathfrak{p}_1$ are invertible in $A_{\mathfrak{p}_1}$).

Now if $a_1, \ldots, a_r$ is an M-regular sequence, then by induction

$$0 \leq \dim \operatorname{Supp}(M/(a_1M + \cdots + a_rM)) \leq \dim \operatorname{Supp} M - r,$$

so $\operatorname{depth} M \leq \dim \operatorname{Supp} M$. □

3. Serre's criterion for normality

Definition 24 ([Liu, 8.2.19])**.** We say a ring A *verifies property* (R_k) if $A_{\mathfrak{p}}$ is regular for every prime $\mathfrak{p}$ of height $\leq k$. We say a ring A *verifies property* (S_k) if for every prime $\mathfrak{p}$,

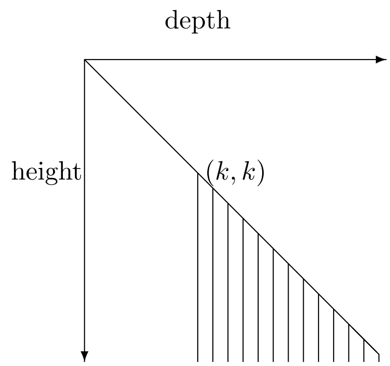
$$\operatorname{depth} A_{\mathfrak{p}} \geq \min\{k, \operatorname{ht} \mathfrak{p}\}$$

Remark 25. Note that when A is Noetherian we always have $\operatorname{depth} A_{\mathfrak{p}} \leq \dim A_{\mathfrak{p}} = \operatorname{ht}(\mathfrak{p})$ [Liu, 8.2.12] so if $\operatorname{ht}(\mathfrak{p}) \leq k$ then we are actually asking for equality $\operatorname{depth} A_{\mathfrak{p}} = \dim A_{\mathfrak{p}} = \operatorname{ht}(\mathfrak{p})$.

Remark 26. The condition for (S_k) is a bit clearer if we separated it into its two cases. A ring A satisfies (S_k) if

(1) $\operatorname{depth} A_{\mathfrak{p}} \geq \operatorname{ht} \mathfrak{p}$ for every prime $\mathfrak{p}$ of height $\leq k$,
(2) $\operatorname{depth} A_{\mathfrak{p}} \geq k$ for every prime $\mathfrak{p}$ of height $\leq k$.

If we group the primes of A according to their height and depth, then since $\operatorname{depth} A_{\mathfrak{p}} \leq \operatorname{ht} \mathfrak{p}$, the condition (S_k) asks that they lie in the region:



Lemma 27 ([Liu, 8.2.20]). *Let A be a Noetherian ring.*

1. *A always satisfies (S_0) (even without the Noetherian hypothesis).*
2. *Property (S_1) is equivalent to A having no embedding primes.*

Proof.

1. Condition (S_0) says that $\operatorname{depth} A_{\mathfrak{p}} \geq \min\{0, \operatorname{ht} \mathfrak{p}\}$. Since both height and depth are always ≥ 0, every ring satisfies (S_0).
2. The condition (S_1) says that $\operatorname{depth} A_{\mathfrak{p}} \geq \min\{1, \operatorname{ht} \mathfrak{p}\}$. Since depth is always nonnegative, the condition is automatically satisfied for primes of height zero (i.e., minimal primes). For primes of height ≥ 1, condition (S_1) asks $\operatorname{depth} A_{\mathfrak{p}} \geq 1$.

   Suppose that A has an embedded prime ideal $\mathfrak{p}$. That is, an associated prime that strictly contains another associated prime, so in particular, $\operatorname{ht} \mathfrak{p} \geq 1$. Consider $A_{\mathfrak{p}}$. Since $\operatorname{Ass}(A_{\mathfrak{p}})$ corresponds to the set of associated primes of A contained in $\mathfrak{p}$, the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ is an associated prime. But since $\operatorname{z.div}(A_{\mathfrak{p}}) = \bigcup_{\operatorname{Ass}(A_{\mathfrak{p}})} \mathfrak{q}$ [AK2017, 17.12], this implies that $\mathfrak{p}A_{\mathfrak{p}} = \operatorname{z.div}(A_{\mathfrak{p}})$. So in particular, $\operatorname{depth} A_{\mathfrak{p}} = 0$ so (S_1) fails.

   Conversely, suppose (S_1) fails, so there is a prime $\mathfrak{p}$ of height ≥ 1 (i.e., nonminimal), with $\operatorname{depth} A_{\mathfrak{p}} = 0$. From the definition of depth, this implies $\mathfrak{p}A_{\mathfrak{p}} = \operatorname{z.div} A_{\mathfrak{p}}$. But then $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}(A_{\mathfrak{p}})$ [Exercise 1]. Now the canonical bijection [Liu, 7.1.2] between $\operatorname{Ass}(A_{\mathfrak{p}})$ and primes of $\operatorname{Ass}(A)$ containing $\mathfrak{p}$ shows that $\mathfrak{p} \in \operatorname{Ass}(A)$. But $\mathfrak{p}$ was nonminimal, so it is an embedded prime since all minimal primes are associated primes [Liu, 8.2.12].

□

Definition 28. A ring is called *reduced* if it has no nonzero nilpotent elements.

Exercise 2 ([AK2017, Ex.13.57]). Let A be a ring.

1. Show that if A is reduced, then so is $S^{-1}A$ for any multiplicative set S.
2. Show that A is reduced if $A_{\mathfrak{p}}$ is reduced for all primes $\mathfrak{p}$.
3. Let $\operatorname{nil}(A)$ denote the set of nilpotent elements of a ring A. Show that $\operatorname{nil}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)} \mathfrak{p}$.

Lemma 29 ([AK2017, Ex.23.12]). *Let A be a Noetherian ring. Then A is reduced if and only if A satisfies (R_0) and (S_1).*

Proof. Suppose A is reduced. Then if $\mathfrak{p}$ is a minimal prime, $A_{\mathfrak{p}}$ is a reduced ring with exactly one prime ideal. Therefore it is a field. Since fields are regular, (R_0)

is satisfied. Let $\mathfrak{p}$ be a prime of height ≥ 1. Then $A_\mathfrak{p}$ is a reduced local ring of dimension ≥ 1. If $\mathfrak{p}A_\mathfrak{p} \in \mathrm{Ass}(A_\mathfrak{p})$, then $\mathfrak{p}A_\mathfrak{p} = \mathrm{Ann}(x)$ for some nonzero x, but since $x \in \mathfrak{p}A_\mathfrak{p}$, we have $x \cdot x = 0$, contradicting $A_\mathfrak{p}$ being reduced. So $\mathfrak{p}A_\mathfrak{p} \notin \mathrm{Ass}(A_\mathfrak{p})$. But then [Exercise 1] says that $\mathfrak{p}A_\mathfrak{p} \neq \mathrm{z.div}(A_\mathfrak{p})$, so we deduce that there is a nonzerodivisor in $\mathfrak{p}A_\mathfrak{p}$. That is, $A_\mathfrak{p}$ has depth ≥ 1. So (S_1) holds.

Conversely, suppose A satisfies (R_0) and (S_1). By Lemma A, the canonical map $A \to \prod_{\mathfrak{p}\in\mathrm{Ass}(A)} A_\mathfrak{p}$ is injective. Condition (S_1) implies there are no embedded primes [Liu, 8.2.20], that is, all associated primes are minimal in $\mathrm{Ass}(A)$. These are exactly the minimal primes of A [Liu, 8.2.12]. But (R_0) implies $A_\mathfrak{p}$ is a field for any minimal prime $\mathfrak{p}$. So $\prod_{\mathfrak{p}\in\mathrm{Ass}(A)} A_\mathfrak{p}$ is a product of fields. Therefore it has no nilpotents, and we deduce that its subring A also has no nilpotents. □

Lemma 30 ([AK2017, Lem.23.13]). *Let R be a Noetherian domain, and M a nonzero torsionfree module. Then $M = \bigcap_{\mathfrak{p} \text{ s.t. } \mathrm{depth}(M_\mathfrak{p})=1} M_\mathfrak{p}$, where the intersection is in $M_{(0)}$.*

Proof. Omitted. □

Lemma 31 ([AK2017, Thm.23.15], [Liu, Thm.8.2.23]). *Let A be a Noetherian domain satisfying (S_2) and (R_1). Then A is normal.*

Proof. Suppose that $a/b \in \mathrm{Frac}(A)$ is integral over A. Then a is integral over $A_\mathfrak{p}$ for every prime $\mathfrak{p}$. Condition (R_1) implies that $A_\mathfrak{p}$ is regular for every prime of height one, and therefore a dvr. We know that dvr's are normal [AK2017, 23.6], so we deduce that x is contained in $A_\mathfrak{p}$ for every $\mathfrak{p}$ of height one. But by (S_2), these are precisely the primes $\mathfrak{p}$ such that $A_\mathfrak{p}$ has depth one. Hence, $a \in A$ by [AK2017, 23.13]. □

Lemma 32 ([AK2017, Thm.23.15], [Liu, 8.2.21]). *Let A be a normal Noetherian domain. Then A verifies (R_1) and (S_2).*

Proof. Let $\mathfrak{p}$ be a prime of height zero. Since A is a domain, we must have $\mathfrak{p} = (0)$, and $A_{(0)} = \mathrm{Frac}(A)$. Fields are regular, and have depth zero, so (R_1) and (S_2) are satisfied for height zero primes.

Let $\mathfrak{p}$ be a prime of height one. Since A is normal, so is $A_\mathfrak{p}$. But then $A_\mathfrak{p}$ is a normal Noetherian dimension one local ring, i.e., a dvr, and therefore regular [AK2017, Thm.23.6] so (R_1) is satisfied. One shows directly that dvr's have depth one (write the elements as $u\pi^n$ where u is a unit and π generates the maximal ideal). So (S_2) is satisfied.

Let $\mathfrak{p}$ be a prime of height ≥ 2. It remains to show that $\mathrm{depth}(A_\mathfrak{p}) \geq 2$. Replacing $A_\mathfrak{p}$ with A we will assume that A is local and $\mathfrak{p}$ the maximal ideal, which we now write as $\mathfrak{m}$. We will prove the contrapositive, that $\mathrm{depth}(A) \leq 1 \implies \dim A \leq 1$. Indeed, if $\mathrm{depth}(A) \leq 1$, then $\mathrm{depth}(A) = 1$, since A is a domain. But then A is a Noetherian local domain of depth one, which implies it is a dvr [AK2017, Thm.23.6], and therefore $\dim(A) = 1$. □