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Étale Cohomology

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Exercise sheet 9[1]

Exercise 1. Let $f : Y \to X$ be a faithfully flat locally of finite presentation morphism of schemes. Let G be a finite subgroup of $\mathrm{Aut}_X(Y)$. We call f a torsor under G over X if the following diagram

$$\begin{array}{ccc} \coprod_{i \in G} Y_i & \xrightarrow{\rho} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & X \end{array}$$

is Cartesian, where $Y_i = Y$, p is the map sending Y_i identically to Y, and ρ is the map sending Y_i to Y via the automorphism $i \in G$. Show that if f is a G-torsor then f is a finite morphism.

Exercise 2. Let X be a locally Noetherian connected scheme, and let $x : \mathrm{Spec}\,(k) \to X$ be a geometric point. Let $f : Y \to X \in \mathrm{\acute{E}t}(X)$ be an object. Show that the following statements are equivalent.

(1) The object $f : Y \to X \in \mathrm{\acute{E}t}(X)$ is Galois, i.e. $\mathrm{Aut}_X(Y)$ acts transitively on $F_x(Y)$.
(2) The object $f : Y \to X \in \mathrm{\acute{E}t}(X)$ is connected and $\#(\mathrm{Aut}_X(Y))$ is equal to the degree of f.
(3) The object $f : Y \to X \in \mathrm{\acute{E}t}(X)$ is connected and f is a torsor under the abstract group $G := \mathrm{Aut}_X(Y)$.

In this case we call Y a *Galois cover* of X.

Exercise 3. Let $X = \mathrm{Spec}\,(k)$ be a spectrum of a field. Show that $Y \to X$ is Galois if and only if $Y = \mathrm{Spec}\,(l)$ where l/k is a finite Galois field extension.

Exercise 4. Let $(\mathcal{C}, F)$ be a Galois category, and let (P, ξ) be a pair, where $P \in \mathcal{C}$ is a Galois object and $\xi, \eta \in F(P)$ be two points. Show that there is always a unique isomorphism $(P, \xi) \cong (P, \eta)$. In particular

[1] If you want your solutions to be corrected, please hand them in just before the lecture on January 4, 2017. If you have any questions concerning these exercises you can contact Shane Kelly via shanekelly64@gmail.com or Lei Zhang via l.zhang@fu-berlin.de.

if Y is a Galois cover of a locally Noetherian connected scheme X, then for any two geometrical points $x_1, x_2 \in X(k)$ there is an automorphism of Y/X sending $x_1 \mapsto x_2$.