December 8, 2016

Étale Cohomology

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Exercise sheet 7[1]

Exercise 1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. In the class we have shown that there exists a local ring map $R \to R^h$ with the following properties

(i) R^h is henselian,
(ii) R^h is a filtered colimit of étale R-algebras,
(iii) $\mathfrak{m}R^h$ is the maximal ideal of R^h, and $\kappa = R^h/\mathfrak{m}R^h$.

Show that a local ring with those properties is necessarily unique, i.e. if there is a local ring map $R \to R'$ such that R' satisfies all the properties $(i), (ii), (iii)$, then $R \to R'$ is $R \to R^h$. Show the same thing also for the strict Henselization.

Exercise 2. Let k be a field. Show that $k^h = k$ and $k^{sh} = k^s$, where k^s is the separable closure of k.

Exercise 3. Let R be a ring with only one prime ideal $\mathfrak{m}$. Show that R is Henselian. (Hint: Show that any finite R-algebra A is a finite product of local rings.)

Remark 1. If $\mathfrak{m}$ is finitely generated (e.g. when R is Noetherian), then R is its own completion, i.e. R is complete. Then one can use Hensel's Lemma.

Exercise 4. Let R be a strictly Henselian local ring. Show that any finite étale cover of Spec (R) is just a disjoint union of Spec (R).

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[1]If you want your solutions to be corrected, please hand them in just before the lecture on December 7, 2016. If you have any questions concerning these exercises you can contact Shane Kelly via shanekelly64@gmail.com or Lei Zhang via l.zhang@fu-berlin.de.