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Étale Cohomology

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Exercise sheet 6[1]

Exercise 1. Let $f : X \to Y$ be a map of schemes locally of finite presentation. We say that f is smooth if it is flat and if $\Omega^1_{X/Y}$ is a locally free sheaf of finite type. If $\Omega^1_{X/Y}$ is of rank $n \in \mathbb{N}$ we call f smooth of relative dimension n. Show that f is étale if and only if it is smooth of relative dimension 0. Show that $\mathbb{P}^n_X$, $\mathbb{A}^n_X$ are smooth of relative dimension n over X.

Exercise 2. Let $f : X \to Y$ be an étale morphism of schemes. Let $x \in X$ be a point. Show that $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,f(x)})$. Recall that the dimension of a commutative ring is the maximal length of chains of prime ideals.

Remark 1. If X, Y are algebraic varieties, and if f is étale surjective, then we have the relation $\dim(f^{-1}(y)) = \dim(X) - \dim(Y)$. Thus we have shown that for such a morphism $\dim(f^{-1}(y)) = 0$. Since the rank of $\Omega^1_{X/Y}$ is the $\kappa(y)$ dimension of $\Omega^1_{f^{-1}(y)/\kappa(y)}$ which is equal to the dimension of $f^{-1}(y)$, this exercise is another confirmation that étale morphisms are smooth of relative dimension 0.

Exercise 3. Let K be a number field, i.e. a finite extension of $\mathbb{Q}$. Let $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}$ in K. Recall that $\mathcal{O}_K$ is a Dedekind domain and a prime ideal $\mathfrak{p} \in \mathcal{O}_K$ is called unramified iff $p(\mathcal{O}_K)_\mathfrak{p} = \mathfrak{p}$ where $p = \mathfrak{p} \bigcap \mathbb{Z}$. Show that $\mathfrak{p}$ is unramified if and only if it is étale under $\operatorname{Spec}(\mathcal{O}_K) \to \operatorname{Spec}(\mathbb{Z})$.

[1] If you want your solutions to be corrected, please hand them in just before the lecture on November 29, 2016. If you have any questions concerning these exercises you can contact Shane Kelly via shanekelly64@gmail.com or Lei Zhang via l.zhang@fu-berlin.de.